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On the Lie Algebras, Generalized Symmetries and Darboux Transformations of the Fifth-Order Evolution Equations in Shallow Water*

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Abstract By considering the one-dimensional model for describing long, small amplitude waves in shallow water, a generalized fifth-order evolution equation named the Olver water wave (OWW) equation is investigated by virtue of some new pseudo-potential systems. By introducing the corresponding pseudo-potential systems, the authors systematically construct some generalized symmetries that consider some new smooth functions $\{X_{i\beta}\}_{\beta=1,2,\cdots,N}^{i=1,2,\cdots,n}$ depending on a finite number of partial derivatives of the nonlocal variables v^β and a restriction $\sum\limits_{i,\alpha,\beta}\big(\frac{\partial\xi^i}{\partial v^\beta}\big)^2+\big(\frac{\partial\eta^\alpha}{\partial v^\beta}\big)^2\neq 0$, i.e., $\sum\limits_{i,\alpha,\beta}\big(\frac{\partial G^\alpha}{\partial v^\beta}\big)^2\neq 0$. Furthermore, the authors investigate some structures associated with the Olver water wave (AOWW) equations including Lie algebra and Darboux transformation. The results are also extended to AOWW equations such as Lax, Sawada-Kotera, Kaup-Kupershmidt, Itô and Caudrey-Dodd-Gibbon-Sawada-Kotera equations, et al. Finally, the symmetries are applied to investigate the initial value problems and Darboux transformations.

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1 Introduction

Group-theory methods are useful for finding symmetry reductions and corresponding group-invariant solutions of a partial differential equations (PDEs) system (see [2, 14, 23]). The local theory of symmetries of differential equations has been well-established since the days of Sophus Lie. Generalized, or higher-order symmetries can be traced back to the original paper of Noether [22], and has acquired received added importance after the discovery that they play a critical

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role in integrable (soliton) partial differential equations (see [2, 14, 23]). While the local theory is very well developed, the theory of nonlocal symmetries of nonlocal differential equations remains incomplete. However, little importance is attached to the existence and applications of nonlocal symmetries [2–3, 23]. Recently, there is some outstanding literature on research of the generalized symmetries. Internal, external and generalized symmetries are investigated in [1]. A loop algebra of nonlocal isovectors of the Korteweg-de Vries (KdV) equation is introduced in [10]. By inverse recursion operators, infinitely many nonlocal symmetries and the conformal invariant forms (Schwartz forms) are researched in [6, 20–21]. Based on the geometric heat flows, symmetries, invariant solutions and reduced equations for the affine case are investigated in [11, 19, 25–26, 38]. Complex $\mathcal{PT}$-symmetric extensions of the non-$\mathcal{PT}$-symmetric Burger's equation are researched in [37]. Furthermore, the introduction of potential-type symmetries (see [2]) and pseudopotential-type symmetries (see [36]) is proposed, which admits close prolongation extending the applicability of symmetry methods to obtain analytic solutions of evolution equations. In that context, the original given system can be embedded in some prolonged systems. Hence, these nonlocal symmetries with close prolongation are anticipated in [9, 27].

Recently, together with Bluman et al., we use the nonclassical method to construct "nonclassical symmetries" and time-dependent exact solutions for the dimensional nonlinear Kompaneets equation (see [4]). Interestingly, each of these solutions is expressed in terms of elementary functions. Three of the classes exhibit quiescent behavior, and the other two classes exhibit blow-up behavior in finite time. As a consequence, it is shown that the corresponding nontrivial stationary solutions are unstable. By virtue of the Riemann theta function, we obtain some periodic wave solutions of nonlinear evolution equations, discrete evolution equations and supersymmetric evolution equations (see [12, 29–33]). In particular, we investigate the integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation and a forced Korteweg-de Vries equation in fluids (see [34–35]), respectively. In this paper, we present a method to research generalized symmetries and their applications of a generalized fifth-order evolution equation by virtue of some new pseudo-potential systems. The method is applied to obtain the generalized symmetries of a generalized fifth-order Korteweg-de Vries equation, named the OWW equation. As applications of the symmetries, we further investigate Lie algebras, initial value problems and Darboux transformations of these fifth-order evolution equations.

In 1984, Olver [24] derived a one-dimensional model for describing long, small amplitude waves in shallow water. The model can take the wave velocity or, alternatively, the surface elevation as the principal variable. Exact solutions for the first case were obtained in the work [17]. The second case leads to the equation

$$u_t + u_x + q_1 u_{xxxxx} + q_2 u^2 u_x + q_3 u u_{xxx} + q_4 u_x u_{xx} + q_5 u_{xxx} + q_6 u u_x = 0, \tag{1.1}$$

where $u = u(x,t)$ provides a surface elevation, x is the horizontal coordinate, and real constants q_i $(i = 1, \cdots, 6)$ depend on the surface tension, which are given by

$$\begin{aligned} &q_1 = \Big(\frac{19}{360} - \frac{\mu}{12} - \frac{\mu^2}{8}\Big)\nu^2, \quad q_2 = -\frac{3}{8}\chi^2, \quad q_3 = \Big(\frac{5}{12} - \frac{\mu}{4}\Big)\chi\nu, \\ &q_4 = \Big(\frac{23}{24} + \frac{5}{8}\mu\Big)\chi\nu, \quad q_5 = \Big(\frac{1}{6} - \frac{\mu}{2}\Big)\nu, \quad q_6 = \frac{3}{2}\chi, \end{aligned} \tag{1.2}$$

where μ denotes a dimensionless surface tension coefficient, ν is the square of the ratio of fluid depth to wave length, and χ is the ratio of wave amplitude to undisturbed fluid depth. The small parameters ν and χ are assumed to be the same order of smallness. For no surface tension, all these coefficients are nonzero, otherwise some of them can be taken as zero values.

The purpose of this paper is to present the method of obtaining generalized symmetries, Lie algebras, initial value problems and Darboux transformations. Taking the well-known fifth-order evolution equation, the OWW equation (1.1), as a special example, we present some structures associated with the Olver water wave (AOWW) equations, such as a Lie algebra of the generalized symmetries, which consider some new smooth functions: $\{X_{i\beta}\}_{\beta=1,2,\cdots,N}^{i=1,2,\cdots,n}$ and a restriction: $\sum\limits_{i,\alpha,\beta}\left(\frac{\partial\xi^i}{\partial v^\beta}\right)^2+\left(\frac{\partial\eta^\alpha}{\partial v^\beta}\right)^2\neq 0$, i.e., $\sum\limits_{i,\alpha,\beta}\left(\frac{\partial G^\alpha}{\partial v^\beta}\right)^2\neq 0$. Moreover, the generalized symmetries are applied to investigate initial value problems and Darboux transformations. The results are also extended to AOWW equations, such as Lax, Sawada-Kotera, Kaup-Kupershmidt, Itô and Caudrey-Dodd-Gibbon-Sawada-Kotera equations, et al.

The rest of the paper is organized as follows. In Section 2, for nonlinear partial differential equations, we propose a detailed description of the method to construct the generalized symmetries by considering the nonlocal condition. By virtue of the method, in Section 3, we introduce the corresponding pseudo-potential systems to obtain some generalized symmetries and Lie algebras of the OWW and AOWW equations, respectively. In Section 4, we present some applications for the generalized symmetries, such as initial value problems and Darboux transformations. Finally, conclusions and discussions are presented in Section 5.

2 Generalized Symmetries of Partial Differential Equations

In this section, based on [1–28], we mainly present a detailed description of the method to construct the generalized symmetries by considering the nonlocal condition.

Notational conventions Throughout this paper, we suppose the following notations. Independent variables are denoted by x^i, $i=1,2,\cdots,n$, $\mathbf{x}=(x^1,x^2,\cdots,x^n)$, and dependent variables are denoted by u^α, $\alpha=1,2,\cdots,m$, $\mathbf{u}=(u^1,u^2,\cdots,u^m)$. Partial derivatives with respect to x^i are indicated by sub-indices, and D_i implies the total derivatives with respect to some independent variable x^i,

$$D_i=\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{m}\sum_{\#J\geq 0}u^\alpha_{Ji}\frac{\partial}{\partial u^\alpha_J}, \tag{2.1}$$

where the unordered k-tuple $J=(j_1,j_2,\cdots,j_k)$, $0\leq j_1,j_2,\cdots,j_k\leq n$ stands for a multi-index of order $\#J=k$, $u^\alpha_{Ji}=\frac{\partial u^\alpha_J}{\partial x^i}$, and D_J implies the composition $D_J=D_{j_1}D_{j_2}\cdots D_{j_k}$.

Definition 2.1 *Suppose that N is a non-zero integer. An N-dimensional covering π of a system of partial differential equation(s) $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}]=0$, $\sigma=1,2,\cdots,k$, is a triplet*

$$\begin{aligned}\pi=(&\{v^\beta:\beta=1,2,\cdots,N\};\{X_{i\beta}:i=1,2,\cdots,n;\beta=1,2,\cdots,N\};\\&\{\widetilde{D}_i:i=1,2,\cdots n\})\end{aligned} \tag{2.2}$$

of variables v^β: $v=(v^1,v^2,\cdots,v^N)$, where smooth functions $X_{i\beta}$ depend on x^i, u^α, v^β and a *finite number of partial derivatives of* u^α, v^β, *and linear operators*

$$\widetilde{D}_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{\#J \geq 0} u^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J} + \sum_{\beta=1}^{N} X_{i\beta} \frac{\partial}{\partial v^\beta}, \tag{2.3}$$

satisfy the following system:

$$\widetilde{D}_i(X_{j\beta}) = \widetilde{D}_j(X_{i\beta}), \quad i, j = 1, 2, \cdots, n, \ \beta = 1, 2, \cdots, N, \tag{2.4}$$

whenever $u^\alpha(x^i)$ *is a solution of* $\triangle^\sigma[\boldsymbol{x}, \boldsymbol{u}] = 0$.

As we understand hereafter that the index i runs from 1 to n and that the index β runs from 1 to N, we replace (2.2) by writing $\pi = (v^\beta; X_{i\beta}; \widetilde{D}_i)$. The variables v^β are considered as new dependent variables (the "nonlocal variables" of the theory) and the operators $\widetilde{D}_i$ satisfying system (2.4) are new total derivatives which are used to consider the nonlocal variables v^β. Interestingly, the operators $\widetilde{D}_i$ satisfy

$$\widetilde{D}_i(v^\beta) = X_{i\beta}, \tag{2.5}$$

and these equations are compatible because the system (2.4) holds. Owing to the solutions of system $\triangle^\sigma[\mathbf{x}, \mathbf{u}] = 0$, the total derivatives $\widetilde{D}_i$ become ordinary partial derivatives. The system

$$\frac{\partial v^\beta}{\partial x^i} = X_{i\beta} \tag{2.6}$$

holds for each index β and each index i whenever $u^\alpha(x^i)$ is a solution of $\triangle^\sigma[\mathbf{x}, \mathbf{u}] = 0$. These compatible equations specify the relations between the dependent variables u^α and the nonlocal variables v^β.

The nonlocal version of the formal linearization of the system $\triangle^\sigma[\mathbf{x}, \mathbf{u}] = 0$ is the matrix

$$\widetilde{\triangle}_* = \Big(\sum_l \frac{\partial \triangle^\sigma}{\partial u^\alpha_l} \widetilde{D}_l\Big) = \begin{pmatrix} \dfrac{\partial \triangle^1}{\partial u^1} + \cdots + \dfrac{\partial \triangle^1}{\partial u^1_I} \widetilde{D}_I & \dfrac{\partial \triangle^1}{\partial u^2} + \cdots + \dfrac{\partial \triangle^1}{\partial u^2_J} \widetilde{D}_J & \cdots \\ \dfrac{\partial \triangle^2}{\partial u^1} + \cdots + \dfrac{\partial \triangle^2}{\partial u^1_K} \widetilde{D}_K & \cdots & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{2.7}$$

The following is our definition of nonlocal symmetries by considering the nonlocal condition.

Definition 2.2 *Suppose that* $\triangle^\sigma[\boldsymbol{x}, \boldsymbol{u}] = 0$, $\sigma = 1, 2, \cdots, k$, *is a system of differential equations, and assume that* $\pi = (v^\beta; X_{i\beta}; \widetilde{D}_i)$ *is a covering of* $\triangle^\sigma[\boldsymbol{x}, \boldsymbol{u}] = 0$. *A nonlocal* π-*symmetry of* $\triangle^\sigma[\boldsymbol{x}, \boldsymbol{u}] = 0$ *given by a generalized vector field*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \phi^\beta \frac{\partial}{\partial v^\beta} \tag{2.8}$$

of the augmented system

$$\triangle^\sigma[\boldsymbol{x}, \boldsymbol{u}] = 0, \quad \frac{\partial v^\beta}{\partial x^i} = X_{i\beta} \tag{2.9}$$

is a nonlocal symmetry if and only if X satisfies

$$\mathrm{pr}X(\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}])=0, \tag{2.10}$$

$$\mathrm{pr}X\Big(\frac{\partial v^\beta}{\partial x^i}-X_{i\beta}\Big)=0, \tag{2.11}$$

$$\sum_{i,\alpha,\beta}\Big(\frac{\partial \xi^i}{\partial v^\beta}\Big)^2+\Big(\frac{\partial \eta^\alpha}{\partial v^\beta}\Big)^2\neq 0, \tag{2.12}$$

where $\mathrm{pr}X$ *is given by*

$$\mathrm{pr}X=\sum_i \xi^i\frac{\partial}{\partial x^i}+\sum_{\alpha,J}\eta_J^\alpha\frac{\partial}{\partial u_J^\alpha}+\sum_{\beta,J}\phi_J^\beta\frac{\partial}{\partial v_J^\beta} \tag{2.13}$$

and

$$\eta_J^\alpha=D_J\Big(\eta^\alpha-\sum_i\xi^i u_i^\alpha\Big)+\sum_i\xi^i u_{Ji}^\alpha,\quad \phi_J^\beta=D_J\Big(\phi^\beta-\sum_i\xi^i v_i^\beta\Big)+\sum_i\xi^i v_{Ji}^\beta. \tag{2.14}$$

Otherwise, X is a local symmetry of the system $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}]=0$.

Now, in order to capture all possible generalized symmetries of the augmented system (2.9), as explained in [23], it is enough to consider evolutionary generalized vector fields. Thus, hereafter we shall only consider the generalized vector field X of the form

$$X=\sum_{\alpha=1}^m G^\alpha\frac{\partial}{\partial u^\alpha}+\sum_{\beta=1}^N H^\beta\frac{\partial}{\partial v^\beta}, \tag{2.15}$$

where G^α and H^β are smooth differential functions. As we know, in this case the generalized symmetry conditions (2.10) and (2.11) imply that the infinitesimal deformation $u^\alpha\mapsto u^\alpha+\varepsilon G^\alpha$ satisfies the system $\triangle^\sigma[\mathbf{x},\mathbf{u}]=0$ with first order for the deformation parameter ε, and that the infinitesimal deformation $v^\beta\mapsto v^\beta+\varepsilon H^\beta$ satisfies the compatible system (2.9) with first order for ε. Then we have the following proposition.

Proposition 2.1 *Suppose that $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}]=0$, $\sigma=1,2,\cdots,k$, is a system of partial differential equations, and assume that $\pi=(v^\beta;X_{i\beta};\widetilde{D}_i)$ is a covering of $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}]=0$. A nonlocal π-symmetry of $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}]=0$ is a generalized vector field generated by an ordered $(m+N)$-tuple of functions (G^α,H^β) depending on x^i, u^α, v^β and a finite number of x^i-derivatives of u, v^β*

$$X=\sum_\alpha G^\alpha\frac{\partial}{\partial u^\alpha}+\sum_\beta H^\beta\frac{\partial}{\partial v^\beta} \tag{2.16}$$

of the augmented system

$$\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}]=0,\quad \widetilde{D}_i(X_{j\beta})=\widetilde{D}_j(X_{i\beta}), \tag{2.17}$$

where G^α and H^β are differential functions, and

$$G=(G^1,G^2,\cdots,G^m)^t, \tag{2.18}$$

if and only if X *satisfies*

$$\widetilde{\triangle}_*(G) = 0, \tag{2.19}$$

$$\widetilde{D}_i(H^\beta) = \widetilde{D}_\varepsilon(X_{i\beta}), \tag{2.20}$$

$$\sum_{i,\alpha,\beta}\Big(\frac{\partial G^\alpha}{\partial v^\beta}\Big)^2 \neq 0, \tag{2.21}$$

whenever $u^\alpha(x^i)$ *is a solution of* $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}] = 0$*, where the operator* $\widetilde{D}_\varepsilon$ *appearing in* (2.20) *is given by*

$$\widetilde{D}_\varepsilon = \sum_{\alpha=1}^{m}\sum_{\#K\geq 0}\widetilde{D}_K(G^\alpha)\frac{\partial}{\partial u_K^\alpha} + \sum_{\beta=1}^{N} H^\beta\frac{\partial}{\partial v^\beta}. \tag{2.22}$$

Otherwise, X *is a local symmetry of the system* $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}] = 0$*.*

It is worth emphasizing that (2.19) depends only on the vector G and the system $\triangle^\sigma[\mathbf{x},\mathbf{u}]$. From the view point of Krasil'shchik and Vinogradov [36], one can see that the vector G is the π-shadow of the nonlocal π-symmetry (G^α, H^β). Also important to note is that the differential operator $\widetilde{D}_\varepsilon$ defined in (2.22) is the nonlocal version of the infinite prolongation of the vector field

$$G^\alpha\frac{\partial}{\partial u^\alpha} + H^\beta\frac{\partial}{\partial v^\beta},$$

by considering the fact that the derivatives of the new nonlocal variables v^β can be written in terms of the variables x^i, u^α, u_J^α, v^β and v_J^β.

We always call "nonlocal symmetry" instead of "nonlocal π-symmetry", and suppose that a covering (2.2) of the system $\triangle^\sigma[\mathbf{x},\mathbf{u}] = 0$ has been fixed. The fact that this method of nonlocal symmetries depends essentially on coverings indicates that one should perhaps consider nonlocal symmetries to be properly generalizing the class of intrinsic symmetries studied in [1].

Proposition 2.2 *Let* (G^α, H^β) *be a nonlocal* π*-symmetry of the system* $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}] = 0$*, where the covering* π *is given by* (2.2)*, and then the vector field*

$$G^\alpha\frac{\partial}{\partial u^\alpha} + H^\beta\frac{\partial}{\partial v^\beta} \tag{2.23}$$

is a generalized symmetry of the augmented system (2.9)*. That is to say, if* (G^α, H^β) *is a nonlocal* π*-symmetry of the system* $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}] = 0$*, the linearized system*

$$\triangle_\varepsilon^\sigma[\boldsymbol{x},\boldsymbol{u}] = 0, \quad \frac{\partial}{\partial\varepsilon}\Big(\frac{\partial v^\beta}{x^i}\Big) = X_{i\beta,\varepsilon} \tag{2.24}$$

is satisfied with $u_\varepsilon^\alpha = G^\alpha$ *and* $v_\varepsilon^\beta = H^\beta$ *whenever* $u^\alpha(x^i)$ *and* $v^\beta(x^i)$ *satisfy the augmented system* (2.9)*. On the other hand, if the vector field* (2.23) *is a generalized symmetry of the augmented system* (2.9)*, then* (G^α, H^β) *is a nonlocal* π*-symmetry of* $\triangle^\sigma[\boldsymbol{x},\boldsymbol{u}] = 0$*, where* $\pi = (v^\beta; X_{i\beta}; \widetilde{D}_i)$ *and* $\widetilde{D}_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m}\sum_{\#J\geq 0} u_{Ji}^\alpha\frac{\partial}{\partial u_J^\alpha} + \sum_{\beta=1}^{N} X_{i\beta}\frac{\partial}{\partial v^\beta}$*.*

Because the generalized symmetry transforms a solution into another solution (see [23]), Proposition 2.2 indicates the following corollary.

Corollary 2.1 *Let $u_0^\alpha(x^i)$ and $v_0^\beta(x^i)$ be solutions of the augmented system* (2.9), *and the solution of the Cauchy problem*

$$\begin{aligned} &\frac{\partial u^\alpha}{\partial \varepsilon} = G^\alpha, \quad \frac{\partial v^\beta}{\partial \varepsilon} = H^\beta; \\ &u^\alpha(x^i, 0) = u_0^\alpha(x^i), \quad v^\beta(x^i, 0) = v_0^\beta(x^i) \end{aligned} \tag{2.25}$$

is a one-parameter family of solutions to the augmented system (2.9). *In particular, nonlocal symmetries transform solutions of the system $\triangle^\sigma[\boldsymbol{x}, \boldsymbol{u}] = 0$ to solutions for the same one.*

We end this section by providing some examples.

Example 2.1 A nonlinear telegraph (NLT) system reads

$$\begin{aligned} &u_x^1 - u_t^2 = 0, \\ &u_t^1 - (u^1)^2 u_x^2 - u^1(1 - u^1) = 0. \end{aligned} \tag{2.26}$$

The functions

$$\xi^1 = f^1(X^1, X^2), \quad \xi^2 = f^2(X^1, X^2)\exp(-t), \quad \eta = u^1 f^2(X^1, X^2)\exp(-t), \tag{2.27}$$

where $X^1 = x - u^2$, $X^2 = t - \log u^1$, and f^1, f^2 satisfy the following linear PDE system:

$$\frac{\partial f^2}{\partial X^2} - \exp(X^2)\frac{\partial f^1}{\partial X^1} = 0, \quad \frac{\partial f^2}{\partial X^1} - \exp(X^2)\frac{\partial f^1}{\partial X^2} = 0, \tag{2.28}$$

which admit the following condition: Let u^1, u^2 be solutions of the NLT system (2.26), and the "deformation" $u + \varepsilon\eta$ with $x + \varepsilon\xi^1$, $t + \varepsilon\xi^2$ is also a solution of the first order in parameter ε, namely, the functions ξ^1, ξ^2 and η satisfy the linearized NLT equation. On the other hand, in order to make this observation rigorous, we should take into account an extra nonlocal variable v^1 by satisfying

$$v_t^1 = u^1, \quad v_x^1 = u^2. \tag{2.29}$$

Therefore, the function η in (2.27) can be rewritten as $\eta = v_t^1 f^2(X^1, X^2)\exp(-t)$, and then η is a "local" and could perhaps be considered as the characteristic of a local symmetry for the following augmented system:

$$v_t^1 = u^1, \quad v_x^1 = u^2, \quad u_t^1 - (u^1)^2 u_x^2 - u^1(1 - u^1) = 0. \tag{2.30}$$

Example 2.2 The Burger's equation reads

$$u_t = u_{xx} + u u_x. \tag{2.31}$$

The function

$$G = (2f_x - uf)\exp\Big(-\frac{1}{2}\int u\mathrm{d}x\Big), \tag{2.32}$$

where $f = f(x,t)$ satisfying $f_t - f_{xx} = 0$, admits the following condition: Let u be a solution of the Burger's equation (2.31), the "deformation" $u + \varepsilon G$ is also a solution of the first order in parameter ε, namely, the function G satisfies the linearized Burger's equation. On the other hand, in order to make this observation rigorous, we would take into account an extra nonloal variable v^1 by satisfying

$$v_x^1 = u, \quad v_t^1 = u_x + \frac{1}{2}u^2. \tag{2.33}$$

Therefore, the function (2.32) can be rewritten as $G = (2f_x - uf)\exp\big(-\frac{1}{2}v^1\big)$, and then G is a "local" and could perhaps be considered as the characteristic of a local symmetry for the following augmented system:

$$u_t - u_{xx} - uu_x = 0, \quad v_x^1 - u = 0, \quad v_t^1 - u_x - \frac{1}{2}u^2 = 0. \tag{2.34}$$

Remark 2.1 In Example 2.1, setting $x^1 = x$ and $x^2 = t$, one has $N = 1$, $X_{11} = u^2$, $X_{21} = u^2$, and the first two equations of (2.30) correspond to the system (2.6). Similarly, in Example 2.2, setting $x^1 = x$ and $x^2 = t$, one has $N = 1$, $X_{11} = u$, and $X_{21} = u_x + \frac{1}{2}u^2$, and the last two equations of (2.34) correspond to the system (2.6).

Remark 2.2 Since we are allowed to replace all derivatives of v^β appearing in the equation (2.16) by virtue of system (2.6), it is worth emphasizing that the coefficients G^α and H^β of the vector field (2.16) are supposed to depend not only on x^i, u^α, finite numbers of derivatives of u^α, and the new variables v^β, but also finite numbers of derivatives of v^β. This simplification is crucial to obtain the classification results.

3 Lie Algebras and Generalized Symmetries of the OWW and AOWW Equations

Let's begin this part with some classical reductions of physical and mechanical interests, recent examples of the OWW equation (1.1), among others, include the following:

(1) The Lax equation reads (see [18])

$$u_t + 10uu_{3x} + 20u_xu_{2x} + 30u^2u_x + u_{5x} = 0. \tag{3.1}$$

(2) The integrable Sawada-Kotera (SK) equation reads (see [29])

$$u_t - 5uu_{3x} - 5u_xu_{2x} - 5u^2u_x - u_{5x} = 0. \tag{3.2}$$

(3) The Kaup-Kupershmidt (KK) equation reads (see [15–16])

$$u_t + 10uu_{3x} + 25u_xu_{2x} + 20u^2u_x + u_{5x} = 0. \tag{3.3}$$

(4) The Itô equation reads (see [13])

$$u_t + 3uu_{3x} + 6u_xu_{2x} + 2u^2u_x + u_{5x} = 0. \tag{3.4}$$

(5) The Caudrey-Dodd-Gibbon-Sawada-Kotera equation (CDGSK) reads (see [7–8, 29–30])

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u_{xx}^2u_x = 0. \tag{3.5}$$

In what follows, we investigate some pseudo-potentials, generalized symmetries and Lie algebras for the OWW and AOWW equations, respectively.

Theorem 3.1 *The system of equations*

$$\begin{aligned} v_x &= \alpha(v)(\lambda_1 u^2 + \lambda_2 u + \lambda_3), \\ v_t &= \Theta_1(u,v)u_{xxxx} + \Theta_2(u,v)u_x u_{xxx} + \Theta_3(u,v)u_{xx}^2 \\ &\quad + \Theta_4(u,v)u_{xx} + \Theta_5(u,v)u_x^2 + \Theta_6(u,v), \end{aligned} \tag{3.6}$$

with

$$\Theta_1(u,v) = -q_1(2\lambda_1 u + \lambda_2)\alpha(v), \quad \Theta_2(u,v) = 2q_1\lambda_1\alpha(v), \quad \Theta_3(u,v) = -\lambda_1 q_1 \alpha(v),$$
$$\Theta_4(u,v) = -\left(\lambda_2 q_5 + 2\lambda_1 q_3 u^2 + \lambda_2 q_3 u + 2\lambda_1 q_5 u\right)\alpha(v), \quad \Theta_5(u,v) = -\frac{1}{2}\lambda_2 q_3\alpha(v) + \lambda_1 q_5 \alpha(v),$$
$$\Theta_6(u,v) = -\Big(\lambda_2 u + \frac{2}{3}\lambda_1 q_6 u^3 + \frac{1}{3}\lambda_2 q_2 u^3 + \frac{1}{2}\lambda_2 q_6 u^2 - \frac{1}{2}\lambda_1 q_2 u^4 + \lambda_1 u^2 - 1\Big)\alpha(v),$$

is completely integrable on solutions to the OWW equation for $q_4 = 2q_3$, where λ_1, λ_2, λ_3 are some nonzero real parameters and $\alpha(v)$ is an arbitrary function of v. Hence it determines a pseudo-potential v for the OWW equation. On the other hand, the following system of equations are compatible whenever $u(x,t)$ is a solution of the OWW equation (1.1) *and hence another potential w for* (*nonlocal*) *conservation laws of the OWW equation:*

$$\begin{aligned} w_x &= u + \lambda_4, \\ w_t &= -u - q_1 u_{xxxx} - \frac{1}{3}q_2 u^3 - q_3 u u_{xx} \\ &\quad - \frac{1}{2}(q_4 - q_3)u_x^2 - q_5 u_{xx} - \frac{1}{2}q_6 u^2 + \lambda_5, \end{aligned} \tag{3.7}$$

where λ_4 and λ_5 are also nonzero real parameters.

According to the idea of [5], the pseudo-potential v can be understood geometrically in terms of geodesics of the pseudo-spherical surfaces associated with the OWW equation (1.1). It is important to us that the compatible system of systems (3.6)–(3.7) yields a three-dimensional covering π of the Olver water wave equation with nonlocal variables v and w.

Remark 3.1 By taking $\alpha(v) = \sum_{i=0}^{n} v_i v^i$, here v_i $(i = 1, 2, \cdots, n)$ are arbitrary constants. Theorem 3.1 is the "Riccati form" of the linear problem associated with the OWW equation (1.1) (see the classical paper by Chern et al [5]).

In what follows, we classify all first-order nonlocal π-symmetries of the OWW equation (1.1). In order to do this, one can see that it is necessary to assume that the parameter λ_i appearing in the systems (3.6)–(3.7) is also affected by the symmetry transformation. Namely, the augmented system includes (1.1), (3.6)–(3.7) with the parameter λ_i.

The evolutionary vector field is of the form

$$X = G\frac{\partial}{\partial u} + H^1\frac{\partial}{\partial v} + H^2\frac{\partial}{\partial w}, \tag{3.8}$$

where G, H^1 and H^2 are functions of x, t, u, v, w, λ_i and the derivatives of u, v and w.

Theorem 3.2 *The first-order generalized symmetries of the augmented Olver water wave* (*AOWW*) *systems* (1.1), (3.6) *and* (3.7), *represented by vector fields* (3.8), *are linear combina-*

tions of

$$X_1 = u_x\frac{\partial}{\partial u}, \tag{3.9}$$

$$X_2 = u_t\frac{\partial}{\partial u}, \tag{3.10}$$

$$X_3 = \frac{\partial}{\partial w}, \tag{3.11}$$

$$X_4 = tu_t\frac{\partial}{\partial u}, \tag{3.12}$$

$$X_5 = \Big[q_3u + q_5 - q_3u_x\Big(-\frac{1}{2}q_3x + q_5q_6t - 2q_3t\Big) + \frac{5}{2}q_3^2tu_t\Big]\frac{\partial}{\partial u}, \tag{3.13}$$

$$X_6 = q_3^2tu_t\frac{\partial}{\partial u} + \left(-4\lambda_4q_2 - 6\lambda_5q_2 - q_6 + \lambda_4q_6^2 + q_2w_t\right)t\frac{\partial}{\partial w}, \tag{3.14}$$

$$\begin{aligned} X_7 = {}& \Big[q_3u + q_5 - q_3u_x\Big(-\frac{1}{2}q_3x + q_5q_6t - 2q_3t\Big)\Big]\frac{\partial}{\partial u} \\ & + \frac{1}{2q_2}[-2\lambda_4q_2x + q_6x - 4\lambda_4q_2t - 6\lambda_5q_2t - q_6t + \lambda_4q_6^2t + 2q_2q_3w \\ & - (-2\lambda_4q_2 + q_6 + q_2w_x)(-q_2x - 4q_2t + q_6^2t)]\frac{\partial}{\partial w}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} X_8 = {}& \Big[2q_2q_3u + 2q_2q_5 - 2q_2q_3u_x\Big(-\frac{1}{2}q_3x + q_5q_6t - 2q_3t\Big) + 5q_2q_3^2tu_t\Big]\frac{\partial}{\partial u} \\ & + [-2\lambda_4q_2x + q_6x - 4\lambda_4q_2t - 6\lambda_5q_2t - q_6t + \lambda_4q_6^2t + 2q_2q_3w \\ & - (-2\lambda_4q_2 + q_6 + q_2w_x)(-q_2x - 4q_2t + q_6^2t) \\ & + 5q_2(-4\lambda_4q_2 - 6\lambda_5q_2 - q_6 + \lambda_4q_6^2 + q_2w_t)t]\frac{\partial}{\partial w}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} X_9 = {}& \Big[-(q_3u + q_5) + q_3u_x\Big(1 - \frac{1}{2}q_3x + q_5q_6t - 2q_3t\Big) - q_3u_t\Big(1 - \frac{5}{2}q_3t\Big)\Big]\frac{\partial}{\partial u} \\ & + \Big\{-\alpha(v)(\lambda_2q_5 + 2\lambda_3q_3 + 4q_3)t + \beta(v)\Big(x + \int\frac{3}{2}\frac{q_1q_3}{\beta(v)}\mathrm{d}v + 1\Big) \\ & - \Big[-\alpha'(v)v_x(\lambda_2q_5 + 2\lambda_3q_3 + 4q_3)t + \beta'(v)v_x\Big(x + \int\frac{3}{2}\frac{q_1q_3}{\beta(v)}\mathrm{d}v + 1\Big) \\ & + \beta(v)\Big(x + \int\frac{3}{2}\frac{q_1q_3}{\beta(v)}\mathrm{d}v + 1\Big)_x\Big]\Big(1 - \frac{1}{2}x - 2t\Big) \\ & - \Big[-\alpha'(v)v_t(\lambda_2q_5 + 2\lambda_3q_3 + 4q_3)t - \alpha(v)(\lambda_2q_5 + 2\lambda_3q_3 + 4q_3) \\ & + \beta'(v)v_t\Big(x + \int\frac{3}{2}\frac{q_1q_3}{\beta(v)}\mathrm{d}v + 1\Big) + \beta(v)\Big(x + \int\frac{3}{2}\frac{q_1q_3}{\beta(v)}\mathrm{d}v + 1\Big)_t\Big]\Big(1 - \frac{5}{2}t\Big)\Big\}\frac{\partial}{\partial v}, \end{aligned} \tag{3.17}$$

where $\beta(v) = (\lambda_2q_5 - 2\lambda_3q_3)\alpha(v)$. *Consequently, the vector field* X_9 *is a nonlocal* π*-symmetry of the OWW equation.*

It is remarkable that the function $\beta(v)$ is included in X_9 since it affects the way in which λ varies with the infinitesimal symmetry transformation (3.17). This function is also of importance to our observations on the Lie algebra structure of nonlocal π-symmetries; see Corollary 3.1 below. We also note that X_1 and X_2 are simply the generators of shifts with respect to the independent variables: They are equivalent to $-\frac{\partial}{\partial x}$ and $-\frac{\partial}{\partial t}$ for the OWW equation, respectively.

Corollary 3.1 *The nine symmetries* (3.9)–(3.17) *of the OWW equation generate a Lie algebra with the commutation Table* 3.1, *whenever u, v and w satisfy the augmented OWW system including* (1.1), (3.6) *and* (3.7).

Table 3.1 The commutation table of the symmetry algebra for the OWW equation with $X_0 = X_2 - X_1$ and $X_\triangle = q_3^2 X_0 - \frac{5}{2} q_3^3 X_4$.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1					q_3X_1		q_3X_1	$2q_2q_3X_1$	$-q_3X_1$
X_2					q_3X_2		q_3X_2	$2q_2q_3X_2$	$-q_3X_2$
X_3							q_3X_3	$2q_2q_3X_3$	
X_4					q_3X_4		q_3X_4	$2q_2q_3X_4$	$-q_3X_4$
X_5	$-q_3X_1$	$-q_3X_2$		$-q_3X_4$			$\frac{5}{2}q_3^3X_4$		$-q_3^2X_0$
X_6							q_3X_6	$2q_2q_3X_6$	$-q_3^3X_4$
X_7	$-q_3X_1$	$-q_3X_2$	$-q_3X_3$	$-q_3X_4$	$-\frac{5}{2}q_3^3X_4$	$-q_3X_6$		$-5q_2q_3X_6$	$X_\triangle$
X_8	$-2q_2q_3X_1$	$-2q_2q_3X_2$	$-2q_2q_3X_3$	$-2q_2q_3X_4$		$-2q_2q_3X_6$	$5q_2q_3X_6$		$2q_2q_3^2X_0$
X_9	q_3X_1	q_3X_2		q_3X_4	$-q_3^2X_0$	$q_3^3X_4$	$-X_\triangle$	$-2q_2q_3^2X_0$	

Corollary 3.1 implies that the symmetries (3.9)–(3.17) generate a nine-dimensional Lie algebra $\mathcal{G}_9$. We stress the fact that this Lie algebra exists because we work on a fixed covering of the OWW equation: we cannot expect the "space of all nonlocal symmetries" of a given equation to possess a Lie algebra structure (see [36]).

It is worth emphasizing that the results of Theorems 3.1–3.2 and Corollary 3.1 depend on the condition $q_4 = 2q_3$. In what follows, replacing the condition by $q_3 = q_4$, we consider the pseudo-potentials and nonlocal π-symmetries AOWW equations by taking the SK equation (3.2) for example, the results of which are quite different from those of Theorem 3.1 since the condition is changed.

We can obtain the following conclusions analogous to those obtained for the OWW equation.

Theorem 3.3 *The SK* (*AOWW*) *equation admits a pseudo-potential v determined by the compatible equations*

$$\begin{aligned} v_x &= \lambda v^2 u + \frac{1}{\lambda}, \\ v_t &= \lambda v^2 u_{xxxx} - 2v u_{xxx} + \Big(3\lambda v^2 u + \frac{2}{\lambda}\Big) u_{xx} + \lambda v^2 u_x^2 \\ &\quad - 2v u u_x + \lambda v^2 u^3 + \frac{u^2}{\lambda}. \end{aligned} \tag{3.18}$$

Moreover, the SK equation admits a potential w determined by the following two systems of equations, which are compatible whenever $u(x,t)$ satisfies (3.2)*:*

$$\begin{aligned} w_x &= \lambda u + \lambda, \\ w_t &= \lambda u_{xxxx} + 5\lambda u u_{xx} + \frac{5}{3}\lambda u^3. \end{aligned} \tag{3.19}$$

Theorem 3.4 *The first-order generalized symmetries of the augmented associated SK system* (3.2), (3.18) *and* (3.19), *represented by vector fields* (3.8), *with G, H^1 and H^2 being functions of the variables u, v, w, and the derivatives of u, v and w only, are linear combinations* *of*

$$T_1 = u_x \frac{\partial}{\partial u}, \tag{3.20}$$

$$T_2 = u_t \frac{\partial}{\partial u}, \tag{3.21}$$

$$T_3 = \frac{\partial}{\partial w}, \tag{3.22}$$

$$T_4 = \Big(u + \frac{1}{2}xu_x + \frac{5}{2}tu_t\Big)\frac{\partial}{\partial u}, \tag{3.23}$$

$$T_5 = u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}, \tag{3.24}$$

$$T_6 = u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v}, \tag{3.25}$$

$$T_7 = u_x \frac{\partial}{\partial u} + w_x \frac{\partial}{\partial w}, \tag{3.26}$$

$$T_8 = u_t \frac{\partial}{\partial u} + w_t \frac{\partial}{\partial w}, \tag{3.27}$$

$$T_9 = -(2u + xu_x + 5tu_t)\frac{\partial}{\partial u} + (v - xv_x - 5tv_t)\frac{\partial}{\partial v}, \tag{3.28}$$

$$T_{10} = \Big(u + \frac{1}{2}xu_x + \frac{5}{2}tu_t\Big)\frac{\partial}{\partial u} + \Big(\frac{1}{2}w - \frac{3}{2}\lambda x + \frac{1}{4}xw_x + \frac{5}{4}tw_t\Big)\frac{\partial}{\partial w}. \tag{3.29}$$

Corollary 3.2 *The ten symmetries* (3.20)–(3.29) *of the SK equation generate a Lie algebra with the commutation Table* 3.2*, whenever u, v and w satisfy the augmented SK systems* (3.2), (3.18) *and* (3.19).

Table 3.2 The commutation table of the symmetry algebra for the SK equation.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1				T_1					$-2T_1$	T_1
X_2				T_2					$-2T_2$	T_2
X_3										$\frac{1}{2}T_3$
X_4	$-T_1$	$-T_2$			$-T_1$	$-T_2$	$-T_1$	$-T_2$		
X_5				T_1					T_5-3T_1	T_1
X_6				T_2					T_6-3T_2	T_2
X_7				T_1					$-2T_1$	$2T_7-T_1$
X_8				T_2					$-2T_2$	$2T_8-T_2$
X_9	$2T_1$	$2T_2$			$3T_1-T_5$	$3T_2-T_6$	$2T_1$	$2T_2$		
X_{10}	$-T_1$	$-T_2$	$-\frac{1}{2}T_3$		$-T_1$	$-T_2$	T_1-2T_7	T_2-2T_8		

4 Applications: The Finite Symmetry Transformation and Darboux Transformation

In this section, by virtue of the flow of the vector fields obtained in Theorems 3.2 and 3.4, we use our analysis to obtain explicit solutions to the OWW and AOWW (SK) equations. By using the standard theory of generalized symmetries (see [1, 3, 14, 23–24]), we obtain a system of equations for the flow of the vector fields obtained in Theorem 3.2 and given by

$$\frac{\partial x}{\partial \varepsilon} = -\frac{1}{2}q_3 x(\varepsilon,\tau) + q_5 q_6 t(\varepsilon,\tau) - 2q_3 t(\varepsilon,\tau), \tag{4.1}$$

$$\frac{\partial t}{\partial \varepsilon} = -\frac{5}{2}q_3 t(\varepsilon,\tau), \tag{4.2}$$

$$\frac{\partial v}{\partial \varepsilon} = \beta(v(\varepsilon,\tau))t(\varepsilon,\tau) - \alpha(v(\varepsilon,\tau))(\lambda_2 q_5 - 2\lambda_3 q_3)x(\varepsilon,\tau) + \beta(v(\varepsilon,\tau))\int \frac{3q_1 q_3}{2\beta(v(\varepsilon,\tau))}\mathrm{d}v, \tag{4.3}$$

$$\begin{aligned}\frac{\partial w}{\partial \varepsilon} &= \frac{1}{2}q_3 w(\varepsilon,\tau) - 2\lambda_4 q_3 t(\varepsilon,\tau) - 3\lambda_5 q_3 t(\varepsilon,\tau) - \lambda_4 q_3 x(\varepsilon,\tau)\\ &\quad + \lambda_4 q_5 q_6 t(\varepsilon,\tau) - q_5 t(\varepsilon,\tau) + q_5 x(\varepsilon,\tau),\end{aligned} \tag{4.4}$$

where ε denotes a flow parameter. There is no need to take into account an equation for $u(\varepsilon,\tau)$ since it is proven that actually $u(\varepsilon,\tau)$ is determined by the equations (4.1)–(4.4).

Proposition 4.1 *The initial value problem* (4.1)–(4.4) *with initial conditions*

$$v_0 = v(0,\tau), \quad w_0 = w(0,\tau), \quad x_0 = x(0,\tau), \quad t_0 = t(0,\tau) = \tau, \tag{4.5}$$

admits the following solution:

$$t(\varepsilon,\tau) = \tau \exp\Big(-\frac{5}{2}q_3\varepsilon\Big), \tag{4.6}$$

$$x(\varepsilon,\tau) = \Big(\frac{1}{2q_3}(2q_3 - q_5 q_6)\tau\exp(-2q_3\varepsilon) + x_0\Big)\exp\Big(-\frac{1}{2}q_3\varepsilon\Big), \tag{4.7}$$

$$v(\varepsilon,\tau) = \frac{-q_3^2 B(\tau)v_0(3q_1-1)(3q_1-5)(4q_3+2\lambda_3 q_3+\lambda_5 q_5)\exp(\frac{3}{2}q_1 q_3\varepsilon)}{A(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3 q_3+\lambda_2 q_5)v_0 - (A(\tau)+q_3^2 B(\tau))B(\tau)}, \tag{4.8}$$

$$\begin{aligned}w(\varepsilon,\tau) &= -\frac{1}{6q_3^2}\left(6\lambda_4 q_3^2 x_0 - 6q_3 q_5 x_0 - 6q_3^2 w_0 + \gamma(\tau)\right)\exp\Big(\frac{1}{2}q_3\varepsilon\Big)\\ &\quad + \frac{1}{6q_3^2}\left(6\lambda_4 q_3^2 x_0\exp(2q_3\varepsilon) - 6q_3 q_5 x_0\exp(2q_3\varepsilon) + \gamma(\tau)\right)\exp\Big(-\frac{5}{2}q_3 x\Big),\end{aligned} \tag{4.9}$$

by taking $\alpha(v) = -\frac{v(\varepsilon,\tau)^2}{\lambda_2 q_5 + (2\lambda_3+4)q_3}$ *and* $q_2 q_5 = \frac{1}{2}q_3 q_6$, *where* $\gamma(\tau)$ *is given by*

$$\gamma(\tau) = 6\lambda_4 q_3^2\tau + 6\lambda_5 q_3^2\tau - 3\lambda_4 q_3 q_5 q_6\tau + q_5^2 q_6\tau, \tag{4.10}$$

$$A(\tau) = (3q_1-1)(\lambda_2 q_5^2 q_6 - 2\lambda_3 q_3 q_5 q_6 + 8\lambda_3 q_3^2 + 8q_3^2)t - 2q_3(3q_1-5)(\lambda_2 q_5 - 2\lambda_3 q_3)x_0, \tag{4.11}$$

$$B(\tau) = (9q_1^2 - 18q_1 + 5)(4q_3 + \lambda_2 q_5 + 2\lambda_3 q_3). \tag{4.12}$$

From this proposition, one can construct explicit families of solutions to the interesting OWW equation. In fact, it includes a Darboux transformation. Let us assume that the "old" independent variables are τ and t, and we can obtain the following results.

Theorem 4.1 *Suppose that the OWW equation* (1.1)*, understood as an equation for* $v(\tau,t)$*, is invariant under the transformations* $\tau \mapsto x$ *and* $v(\tau,t) \mapsto \overline{v}(x,t)$*. Then*

$$\begin{aligned}x = x(\tau,t) &= \frac{1}{2q_3}(2q_3 - q_5 q_6)\,t\\ &\quad + x_0\Big(\frac{A(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3 q_3+\lambda_2 q_5)v_0 - (A(\tau)+q_3^2 B(\tau))B(\tau)}{-q_3^2 B(\tau)v_0(3q_1-1)(3q_1-5)(4q_3+2\lambda_3 q_3+\lambda_5 q_5)}\Big)^{-\frac{1}{3}} v^{-\frac{1}{3}},\end{aligned} \tag{4.13}$$

and $\overline{v}(x,t)$ *is obtained by inverting* (4.13) *as follows:*

$$\overline{v}(x,t) = x_0^3\Big(x(\tau,t) - \frac{1}{2q_3}\left(2q_3 - q_5q_6\right)\Big)^{-3} \times \frac{-q_3^2B(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3q_3+\lambda_5q_5)v(x,t)}{A(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3q_3+\lambda_2q_5)v(x,t)-(A(\tau)+q_3^2B(\tau))B(\tau)}, \quad (4.14)$$

with $A(\tau)$ *and* $B(\tau)$ *given by* (4.11) *and* (4.12), *respectively.*

Theorem 4.2 *Assume that the OWW equation* (1.1), *understood as an equation for* $w(\tau,t)$, *is invariant under the transformations* $\tau \mapsto x$ *and* $w(\tau,t) \mapsto \overline{w}(x,t)$. *Then*

$$x = x(\tau,t) = x_0\Gamma(\tau,t) + \frac{1}{2q_3}\left(2q_3 - q_5q_6\right)\tau\Gamma(\tau,t)^5, \quad (4.15)$$

where $\Gamma(\tau,t)$ *satisfies the following system*:

$$q_3\gamma(\tau)\Gamma(\tau,t)^6 + (\lambda_4q_3 - q_5)x_0\Gamma(\tau,t)^2 + q_3\Gamma(\tau,t)w + \frac{1}{6q_3}(6\lambda_4q_3^2x_0 - 6q_3q_5x_0 - 6q_3^2w_0 + \gamma(\tau)) = 0, \quad (4.16)$$

and $\overline{w}(x,t)$ *is obtained by inverting* (4.15)–(4.16) *as follows*:

$$\overline{w}(x,t) = -\frac{1}{6q_3^2\Gamma(\tau,t)}\big[6q_3^6\gamma(\tau)\Gamma(\tau,t)^6 + 6q_3(\lambda_4q_3 - q_5)x_0\Gamma(\tau,t) + 6q_3(\lambda_4q_3 - q_5)x_0 - 6q_3^2w(x,t) + \gamma(\tau)\big], \quad (4.17)$$

with

$$\left(2q_3 - q_5q_6\right)\tau\Gamma(\tau,t)^5 + 2q_3x_0\Gamma(\tau,t) - 2q_3x(\tau,t) = 0. \quad (4.18)$$

Similarly, taking τ and x as the "old" independent variables, we can obtain the following results.

Theorem 4.3 *Suppose that the OWW equation* (1.1), *understood as an equation for* $v(x,\varepsilon)$, *is invariant under the transformations* $\varepsilon \mapsto t$ *and* $v(x,\varepsilon) \mapsto \widetilde{v}(x,t)$. *Then*

$$t = t(x,\varepsilon) = \tau\Big(\frac{A(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3q_3+\lambda_2q_5)v_0 - (A(\tau)+q_3^2B(\tau))B(\tau)}{-q_3^2B(\tau)v_0(3q_1-1)(3q_1-5)(4q_3+2\lambda_3q_3+\lambda_5q_5)}\Big)^{-\frac{5}{3q_1}} v^{-\frac{5}{3q_1}}, \quad (4.19)$$

and $\widetilde{v}(x,t)$ *is obtained by inverting* (4.19) *as follows*:

$$\widetilde{v}(x,t) = \frac{-q_3^2B(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3q_3+\lambda_5q_5)t(x,\varepsilon)v(x,t)}{\left[A(\tau)(3q_1-1)(3q_1-5)(4q_3+2\lambda_3q_3+\lambda_2q_5)v(x,t) - (A(\tau)+q_3^2B(\tau))B(\tau)\right]\tau}, \quad (4.20)$$

with $A(\tau)$ *and* $B(\tau)$ *given by* (4.11) *and* (4.12), *respectively.*

Theorem 4.4 *Assume that the OWW equation* (1.1), *understood as an equation for* $w(x,\varepsilon)$, *is invariant under the transformations* $\varepsilon \mapsto t$ *and* $w(x,\varepsilon) \mapsto \widetilde{w}(x,t)$. *Then*

$$t = t(x,\varepsilon) = \tau\Gamma(x,\varepsilon)^5, \quad (4.21)$$

where $\Gamma(x,\varepsilon)$ *satisfies the system* (4.16)*, and* $\widetilde{w}(x,t)$ *is obtained by inverting* (4.21), (4.16) *as follows*:

$$\begin{aligned}\widetilde{w}(x,t) = &- \frac{1}{6q_3^2\Gamma(x,\varepsilon)}\big[6q_3^6\gamma(\tau)\tau^{-1}t(x,\varepsilon)\Gamma(x,\varepsilon) + 6q_3(\lambda_4 q_3 - q_5)x_0\Gamma(x,\varepsilon)\\ &+6q_3(\lambda_4 q_3 - q_5)x_0 - 6q_3^2 w(x,t) + \gamma(\tau)\big],\end{aligned} \tag{4.22}$$

with

$$\tau\Gamma(x,\varepsilon)^5 - t(x,\varepsilon) = 0. \tag{4.23}$$

In what follows, we can investigate the SK (AOWW) equation. Following the foregoing theory, we can obtain similar Darboux transforms.

With initial conditions $v_0 = v(0,\tau)$, $w_0 = w(0,\tau)$, $x_0 = x(0,\tau)$ and $t_0 = t(0,\tau) = \tau$, we have a system for the flow of the vector fields obtained in Theorem 3.4 and given by

$$\begin{aligned}&x(\varepsilon,\tau) = x_0 \exp\Big(\frac{1}{2}\varepsilon\Big), \quad t(\varepsilon,\tau) = t_0 \exp\Big(\frac{5}{2}\varepsilon\Big),\\ &v(\varepsilon,\tau) = v_0 \exp(\varepsilon), \quad w(\varepsilon,\tau) = (w_0 - \lambda\varepsilon x_0)\exp\Big(\frac{1}{2}\varepsilon\Big).\end{aligned} \tag{4.24}$$

As in the OWW equation case, we do not calculate explicitly $u(\varepsilon,\tau)$, since this function is completely determined by $v(\varepsilon,\tau)$ and $w(\varepsilon,\tau)$.

For simplicity, we consider the SK equation (3.2) as an equation for $w(\tau,t)$ and $w(x,\varepsilon)$. In the same way as stated in the OWW equation, we have the following theorem.

Theorem 4.5 (i) *Suppose that the SK* (*AOWW*) *equation* (3.2)*, understood as an equation for* $w(\tau,t)$*, is invariant under the transformations* $\tau\mapsto x$ *and* $w(\tau,t)\mapsto\overline{w}(x,t)$*. Then*

$$x = x(\tau,t) = \frac{x_0 w}{w_0 - \lambda x_0(\ln v - \ln v_0)}, \tag{4.25}$$

and $\overline{w}(x,t)$ *is obtained by inverting* (4.25) *and replacing into*

$$\overline{w}(x,t) = \frac{x(\tau,t)}{x_0}\left[w(x,t) - \lambda x_0\left(\ln\overline{v}(x,t) - \ln v(x,t)\right)\right], \tag{4.26}$$

where $\overline{v}(x,t)$ *can be derived from* (4.24)

$$\overline{v}(x,t) = \Big(\frac{x(\tau,t)}{x_0}\Big)^2 v(x,t), \tag{4.27}$$

with $v(x,t)$ *determined by* (4.24)*.*

(ii) *Assume that the SK* (*AOWW*) *equation* (3.2)*, understood as an equation for* $w(x,\varepsilon)$*, is invariant under the transformations* $\varepsilon\mapsto t$ *and* $w(x,\varepsilon)\mapsto\widetilde{w}(x,t)$*. Then*

$$t = t(x,\varepsilon) = t_0\Big(\frac{w}{w_0 - \lambda x_0(\ln v - \ln v_0)}\Big)^5, \tag{4.28}$$

and $\widetilde{w}(x,t)$ *is obtained by inverting* (4.28) *and replacing into*

$$\widetilde{w}(x,t) = \Big(\frac{t(x,\varepsilon)}{t_0}\Big)\left[w(x,t) - \lambda x_0\left(\ln\overline{v}(x,t) - \ln v(x,t)\right)\right], \tag{4.29}$$

where $\widetilde{v}(x,t)$ *can be derived from* (4.24)

$$\widetilde{v}(x,t)=\Big(\frac{t(x,\varepsilon)}{t_0}\Big)^{\frac{2}{5}}v(x,t), \tag{4.30}$$

with $v(x,t)$ *is determined by* (4.24).

Following the foregoing method, by virtue of the pseudo-potential variables $v(x,t)$, $w(x,t)$ and generalized symmetries, we present Darboux transformations of the OWW equation (1.1) and the (SK) AOWW equation (3.2), respectively. These formulae are used to find families of non-trivial solutions to the OWW equation and SK (AOWW) equation.

5 Conclusions and Discussions

In this paper, we have shown that combining generalized symmetries with the nonlocal condition can result in a variety of applications. The main new progresses made in this paper in the general aspect of evolution equations are given as follows:

(i) The generalized symmetries can be used to investigate the initial value problems.

(ii) The generalized symmetries can be used to construct Darboux transformations.

(iii) Lie algebras and generalized symmetries can be obtained from new pseudo-potential systems and vice versa.

(iv) Different kinds of Darboux transformations may assume the same infinitesimal forms, and then new kinds of Darboux transformations may be obtained from old ones.

(v) New finite-dimensional pseudo-potential systems can be solved by generalized symmetries and related Darboux transformations, and then the original evolution equation can be investigated from lower-dimensional ones owing to the existence of generalized symmetries.

The above results are realized especially for the fifth-order evolution equation, OWW and AOWW equations. For an OWW equation, it admits a new class of pseudo-potential systems resulting in its Lie algebras, generalized symmetries, initial value problems and Darboux transformations. Since such pseudo-potential systems are of the Riccati type, more information about their bilinear forms are also researched through the Cole-Hopf transformation.

For the purpose of extending applicability of the generalized symmetry to obtain analytic solutions of the OWW and AOWW equations, we introduce two new pseudo-potential variables v and w to form a new class of pseudo-potential systems, so that the original generalized symmetries can be transformed to some local symmetries of the new systems. Based on the generalized symmetries, we can investigate Lie algebras, initial value problems and Darboux transformations, et al. By means of the Darboux transformations, the analytic solutions of OWW and AOWW equations are obtained via the transformations of trivial solutions. Considering the complete local symmetries of the pseudo-potential systems, we can further achieve rich group invariant solutions, such as rational solutions, special function solutions, solitary wave solutions and periodic function solutions, et al.

However, in this paper, it still remains unclear what kind of pseudo-potential systems can be used to construct the generalized symmetries of the original evolution equation and what kind of generalized symmetries can be applied to obtain nontrivial solutions. The integrability of the pseudo-potential systems should be further investigated. Furthermore, one may consider soliton solutions, rational solutions, peakon solutions, breather solutions and algebraic geometry solutions of the completely integrable pseudo-potential systems to achieve corresponding solutions of the OWW and AOWW equations, respectively. It is quite reasonable and meaningful that these matters merit our further study.

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