Chin. Ann. Math.
36B(4), 2015, 579–602
DOI: 10.1007/s11401-015-0936-2

Chinese Annals of Mathematics, Series B

On the Tangent Bundle of a Hypersurface in a Riemannian Manifold*

Zhonghua HOU[1] Lei SUN[2]

Abstract Let (M^n, g) and (N^{n+1}, G) be Riemannian manifolds. Let TM^n and TN^{n+1} be the associated tangent bundles. Let $f : (M^n, g) \to (N^{n+1}, G)$ be an isometrical immersion with $g = f^*G$, $F = (f, df) : (TM^n, \overline{g}) \to (TN^{n+1}, G_s)$ be the isometrical immersion with $\overline{g} = F^*G_s$ where $(df)_x : T_xM \to T_{f(x)}N$ for any $x \in M$ is the differential map, and G_s be the Sasaki metric on TN induced from G. This paper deals with the geometry of TM^n as a submanifold of TN^{n+1} by the moving frame method. The authors firstly study the extrinsic geometry of TM^n in TN^{n+1}. Then the integrability of the induced almost complex structure of TM is discussed.

Keywords Hypersurfaces, Tangent bundle, Mean curvature vector, Sasaki metric, Almost complex structure, Kählerian form
2000 MR Subject Classification 32Q60, 53C42

1 Introduction

Let (M, g) be a Riemannian manifold, and TM be the tangent bundle of M. Let G_s be the Sasaki metric on TM introduced by Sasaki [13] in terms of g.

The geometry of (TM, G_s) and the unit tangent sphere bundle $(S(TM), G_s)$ have attracted many mathematicians in the last decades. Kowalski [5] showed that if (TM, G_s) is locally symmetric, then the base metric is flat and so does G_s. Musso and Tricerri [9] proved that (TM, G_s) has constant scalar curvature if and only if (M, g) is flat. Nagano [10], Tachibana and Okumura [15] studied the almost complex structure on (TM, G_s). Nagy [11] studied the geometry of the unit tangent sphere bundle of a surface. Klingenberg and Sasaki [2] showed that $(S(TS^2(1)), G_s)$ is isometric to the elliptic space of curvature $\frac{1}{4}$. Nagy [12], Sasaki [14], Konno and Tanno [3–4] studied the geodesics and Killing vector fields on $(S(TM), G_s)$. Tashiro [16–17] studied the contact structure on $S(TM)$.

Deshmukh, Al-Odan and Shaman [1] considered an orientable hypersurface M^n of the Euclidean space R^{n+1} and observed that the tangent bundle TM of M is an immersed submanifold of the Euclidean space R^{2n+2}. They obtained expressions for the horizontal and vertical lifts of the vector fields on M and showed that the induced metric on TM is not a natural metric

Manuscript received February 24, 2013. Revised October 18, 2014.
[1]Institute of Mathematics, Dalian University of Technology, Dalian 116024, Liaoning, China.
E-mail: zhonghua@dlut.edu.cn
[2]Institute of Mathematics, Dalian University of Technology, Dalian 116024, Liaoning, China; Department of Mathematics, Northeast Forestry University, Harbin 150040, China. E-mail: mike.lei.sun@gmail.com
*This work was supported by the National Natural Science Foundation of China (No. 61473059) and the Fundamental Research Funds for the Central University (No. DUT11LK47).

in general. In the special case that the induced metric on TM becomes a natural metric, they proved that the tangent bundle TM is trivial.

In this paper, we suppose that M^n is a hypersurface of a Riemannian manifold (N^{n+1}, G). We use the moving frame method to study the geometry of the tangent bundle TM^n with the induced metric from (TN, G_s).

In Section 3, we study the extrinsic geometry of TM^n in $(TN^{n+1}(c), G_s)$ where $N^{n+1}(c)$ is a space form of constant curvature c. In Section 4, we study the integrability of the almost complex structure J on TM and the Kählerian form ω on TM induced by J.

2 Preliminaries

Suppose that (N, G) is an $(n+1)$-dimensional Riemannian manifold. Let D be the Levi-Civita connection in N, and $\pi : TN \to N$ be the natural projection. Through out this paper, we use the Einstein convention and the following ranges of indices:

$$1 \le A, B, C, \cdots \le n+1, \quad 1 \le i, j, k, \cdots, \le n. \tag{2.1}$$

Let $(\{y^A\})$ and $(\{y^A\}, \{v^B\})$ be local coordinate systems in N and TN, respectively. Denote

$$\partial_{y^A} = \frac{\partial}{\partial y^A}, \quad \partial_{v^B} = \frac{\partial}{\partial v^B}, \quad \partial^2_{y^A y^B} = \frac{\partial^2}{\partial y^A \partial y^A}, \quad \partial^2_{v^A v^B} = \frac{\partial^2}{\partial v^A \partial v^B}, \quad \text{etc.} \tag{2.2}$$

At first, we introduce the following lemmas.

Lemma 2.1 (cf. [5]) *Let (N, G) be a Riemannian manifold and $Y \in \Gamma(TN)$ be a vector field on N which is locally represented by $Y = Y^A \partial_{y^A}$. Then the vertical and horizontal lifts Y^V and Y^H of Y over TN are given by*

$$Y^V_{(y,v)} = Y^A \partial_{v^A}, \qquad Y^H_{(y,v)} = Y^A \partial_{y^A} - \Gamma^A_{BC} Y^B v^C \partial_{v^A}, \tag{2.3}$$

respectively, where $\{\Gamma^A_{BC}\}$ are the Christoffel symbols of D.

Lemma 2.2 (cf. [5]) *The Lie bracket of vector fields over TN is completely determined by*

$$[X^H, Y^H]_{(y,v)} = [X, Y]^H_{(y,v)} - (R^N_{XY} v)^V, \quad [X^H, Y^V]_{(y,v)} = (D_X Y)^V_{(y,v)}, \quad [X^V, Y^V]_{(y,v)} = 0,$$

where $X, Y \in \Gamma(TN)$ and ${R^N}_{XY}$ is the Riemannian curvature operator of N.

The Sasaki metric G_s on TN can be described as follows.

Definition 2.1 (cf. [13]) *Let (N, G) be a Riemannian manifold. The Sasaki metric G_s on TN is defined by*

$$G_{s_{(y,v)}}(X^H, Y^H) = g_y(X, Y), \quad G_{s_{(y,v)}}(X^H, Y^V) = 0, \quad G_{s_{(y,v)}}(X^V, Y^V) = g_y(X, Y) \tag{2.4}$$

for any point $(y, v) \in TN$ and vectors $X, Y \in T_y N$.

By direct computation, we have the following lemma.

Lemma 2.3 (cf. [7]) *Let (N,G) be a Riemannian manifold, and TN be the tangent bundle with the Sasaki metric G_s. Then the Levi-Civita connection $\overline{D}$ on (TN,G_s) is determined by*

$$\overline{D}_{X^H}Y^H=(D_XY)^H-\frac{1}{2}(R^N_{XY}v)^V,\quad \overline{D}_{X^V}Y^H=\frac{1}{2}(R^N_{vX}Y)^H,$$

$$\overline{D}_{X^H}Y^V=(D_XY)^V+\frac{1}{2}(R^N_{vY}X)^H,\quad \overline{D}_{X^V}Y^V=0$$

for any point $(y,v)\in TN$ and vectors $X,Y\in T_yN$.

3 Geometry of the Tangent Bundle of a Hypersurface

Suppose that $f:M^n\to N^{n+1}$ is a smooth immersion from M^n into N^{n+1}. Let

$$\mathrm{d}f(x):T_xM\to T_{f(x)}N$$

be the differential map of f at any $x\in M$. We define the smooth immersion $F:TM\to TN$ to be

$$F(x,u)=(f(x),\mathrm{d}f(x)u) \tag{3.1}$$

for any point $(x,u)\in TM$.

Lemma 3.1 *Suppose that h and ν are the second fundamental form and the unit normal vector field of M in N, respectively. Let X^h and X^v be the horizontal and the vertical lifts of $X\in\Gamma(TM)$ onto TM, respectively, with respect to $g=f^*G$. Then the differential map $\mathrm{d}F$ of F is defined by*

$$\mathrm{d}F(x,u)(X^v)=(\mathrm{d}f(x)(X))^V,\quad \mathrm{d}F(x,u)(X^h)=(\mathrm{d}f(x)(X))^H+h(X,u)\nu^V \tag{3.2}$$

for any point $(x,u)\in TM$ and the vector field $X\in\Gamma(TM)$.

Proof For any $(x,u)\in TM$, we have $F(x,u)=(f(x),\mathrm{d}f(x)u)\in TN$. Let $(\{x^i\},\{u^j\})$ and $(\{y^A\},\{v^B\})$ be the local coordinates around (x,u) and $F(x,u)$. Let $\{\gamma^k_{ij}\}$ be the Levi-Civita connection of the induced metric g. Then the local representation of $f(x)$ is of the form

$$(f^1(x^1,\cdots,x^n),\cdots,f^{n+1}(x^1,\cdots,x^n)).$$

Moreover, we have

$$(\partial_{x^i})^h=\partial_{x^i}-\gamma^k_{ij}u^j\partial_{u^k},\quad (\partial_{x^i})^v=\partial_{u^i},\quad \mathrm{d}f(x)\partial_{x^i}=(\partial_{x^i}f^A)\partial_{y^A}. \tag{3.3}$$

Therefore,

$$\mathrm{d}F(x,u)\partial_{x^i}=(\partial_{x^i}f^A)\partial_{y^A}+(\partial^2_{x^ix^j}f^B)u^j\partial_{v^B},\quad \mathrm{d}F(x,u)\partial_{u^k}=(\partial_{x^k}f^A)\partial_{v^A}. \tag{3.4}$$

It follows from (3.3)–(3.4) that

$$\mathrm{d}F(x,u)(\partial_{x^i})^h=(\partial_{x^i}f^A)\partial_{y^A}+(\partial^2_{x^ix^j}f^B)u^j\partial_{v^B}-\gamma^k_{ij}u^j(\partial_{x^k}f^B)\partial_{v^B}, \tag{3.5}$$

$$\mathrm{d}F(x,u)(\partial_{x^i})^v=\mathrm{d}F(x,u)\partial_{u^i}=(\partial_{x^i}f^A)\partial_{v^A}=\{\mathrm{d}f(x)\partial_{x^i}\}^V. \tag{3.6}$$

On the other hand, from Lemma 2.1, we have

$$(\partial_{y^A})^V_{(y,v)} = \partial_{v^A}, \qquad (\partial_{y^A})^H_{(y,v)} = \partial_{y^A} - \Gamma^C_{AB} v^B \partial_{v^C}. \tag{3.7}$$

Therefore,

$$\begin{aligned}
\mathrm{d}F(x,u)(\partial_{x^i})^h &= (\partial_{x^i} f^A)\partial_{y^A} + (\partial^2_{x^i x^j} f^B) u^j \partial_{v^B} - \gamma^k_{ij} u^j (\partial_{x^k} f^B)\partial_{v^B} \\
&= (\partial_{x^i} f^A)[(\partial_{y^A})^H_{F(x,u)} + \Gamma^C_{AB} u^k (\partial_{x^k} f^B)\partial_{v^C}] \\
&\quad + (\partial^2_{x^i x^j} f^B) u^j \partial_{v^B} - \gamma^k_{ij} u^j (\partial_{x^k} f^B)\partial_{v^B} \\
&= (\partial_{x^i} f^A \partial_{y^A})^H_{F(x,u)} + \Gamma^C_{AB}(\partial_{x^i} f^A) u^k (\partial_{x^k} f^B)\partial_{v^C} + (\partial^2_{x^i x^j} f^B) u^j \partial_{v^B} \\
&\quad - \gamma^k_{ij} u^j (\partial_{x^k} f^B)\partial_{v^B} \\
&= (\partial_{x^i} f^A \partial_{y^A})^H_{F(x,u)} + [D_{(\partial_{x^i} f^A \partial_{y^A})}\{u^j(\partial_{x^j} f^B)\} \\
&\quad - \nabla_{(\partial_{x^i} f^A \partial_{y^A})}\{u^j(\partial_{x^j} f^B)\}](\partial_{y^B})^V \\
&= (\partial_{x^i} f^A \partial_{y^A})^H_{F(x,u)} + [D_{(\partial_{x^i} f^A \partial_{y^A})}\{u^j(\partial_{x^j} f^B)\}(\partial_{y^B}) \\
&\quad - \nabla_{(\partial_{x^i} f^A \partial_{y^A})}\{u^j(\partial_{x^j} f^B)\}(\partial_{y^B})]^V \\
&= (\partial_{x^i} f^A \partial_{y^A})^H_{F(x,u)} + [D_{(\partial_{x^i} f^A \partial_{y^A})}\{u^j(\partial_{x^j} f^B)(\partial_{y^B})\} - \mathrm{d}f(x)\nabla_{\partial_{x^i}}\{u^j \partial_{x^j}\}]^V \\
&= (\partial_{x^i} f^A \partial_{y^A})^H_{F(x,u)} + h(\partial_{x^i}, u^j \partial_{x^j})\nu^V.
\end{aligned}$$

It follows from (3.3) that

$$\mathrm{d}F(x,u)(\partial_{x^i})^h = [\mathrm{d}f(x)\partial_{x^i}]^H_{F(x,u)} + h(\partial_{x^i}, u)\nu^V. \tag{3.8}$$

Lemma 3.1 follows immediately from (3.6) and (3.8).

From Lemma 3.1, we have the following lemma.

Lemma 3.2 *Denote $\overline{g} = F^* G_s$. Then at any point $(x,u) \in TM$, we have*

$$\overline{g}_{(x,u)}(X^h, Y^h) = g(X,Y) + h(X,u)h(Y,u), \tag{3.9}$$

$$\overline{g}_{(x,u)}(X^h, Y^v) = 0, \quad \overline{g}_{(x,u)}(X^v, Y^v) = g(X,Y) \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$.

Remark 3.1 From (3.9) of Lemma 3.2, we can see that $(TM, \overline{g})$ is not a natural metric.

Suppose that $(N^{n+1}(c), G)$ is a space form of constant section curvature c. The Riemannian curvature operator of $(N^{n+1}(c), G)$ is given by

$$R^N_{XY} Z = c\big\{G(Y,Z)X - G(X,Z)Y\big\} \tag{3.11}$$

for any $X, Y, Z \in \Gamma(TN)$.

Suppose that $f : (M,g) \to (N^{n+1}(c), G)$ is an isometrical immersion of M^n into $N^{n+1}(c)$. Let A be the shape operator of M in $N^{n+1}(c)$. Consider $F : (TM, \overline{g}) \to (TN^{n+1}(c), G_s)$ defined by (3.1). Then TM is a submanifold of TN with codimension 2.

In the sequel, we proceed to study the extrinsic geometry of TM in $(TN^{n+1}(c), G_s)$.

The two local orthonormal normal vector fields ν_1, ν_2 of TM in $TN^{n+1}(c)$ are given by

$$\nu_1 = (\nu)^H_{F(x,u)}, \quad \nu_2 = \frac{1}{\tau}\big\{\nu^V_{F(x,u)} - [\mathrm{d}f(x)A(u)]^H_{F(x,u)}\big\}, \tag{3.12}$$

where $\tau^2 = 1 + g(A(u), A(u))$ with $\tau > 0$ at any point $(x, u) \in TM$, so that the normal bundle $T^\perp(M)$ of TM in TN is locally spanned by ν_1 and ν_2.

From now on, we denote briefly $X := [\mathrm{d}f(x)X]_{f(x)}$ and

$$X^h := [\mathrm{d}F(x,u)X^h]_{F(x,u)}, \quad X^v := [\mathrm{d}F(x,u)X^v]_{F(x,u)}, \tag{3.13}$$
$$X^H := [\mathrm{d}f(x)X]^H_{F(x,u)}, \qquad X^V := [\mathrm{d}f(x)X]^V_{F(x,u)} \tag{3.14}$$

for any $X \in T_xM$.

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field on (M, g), and $\{\theta^1, \cdots, \theta^n\}$ be its dual frame field. We denote

$$h = h_{ij}\theta^i \otimes \theta^j, \quad A(u) = u^i h_{ij} e_j \tag{3.15}$$

for any $u = u^i e_i \in \Gamma(TM)$. It follows from (3.9) that

$$\begin{cases} \overline{g}_{ij} = G_s(e_i^h, e_j^h) = \delta_{ij} + (h_{ik}u_k)(h_{jl}u_l), \quad (\overline{g}_{ij})^{-1}_{n\times n} := (\overline{g}^{ij})_{n\times n}, \\ \overline{g}^{ij} = \delta_{ij} - \tau^{-2}(h_{ik}u_k)(h_{jl}u_l), \quad \overline{g}^{ik}(h_{kl}u_l) = \tau^{-2}(h_{ij}u_j). \end{cases} \tag{3.16}$$

Moreover, (3.2) and (3.12) turn into

$$e_i^v = e_i^V, \quad e_i^h = e_i^H + (h_{ij}u_j)\nu^V \quad \text{and} \quad \nu_1 = \nu^H, \quad \nu_2 = \tau^{-1}[\nu^V - (h_{kl}u_l)e_k^H], \tag{3.17}$$

respectively. From (3.17), we obtain

$$e_k^H = \overline{g}^{jk}e_j^h - \tau^{-1}(h_{kj}u_j)\nu_2, \quad \nu^V = \tau^{-2}(h_{il}u_l)e_i^h + \tau^{-1}\nu_2. \tag{3.18}$$

Using (3.11), from Lemma 2.3, we have

$$\begin{aligned}
\overline{D}_{e_i^v}\nu_2 &= [\overline{D}_{e_i^V}(\tau^{-1})][\nu^V - (h_{kl}u_l)e_k^H] - \tau^{-1}\overline{D}_{e_i^V}[(h_{kl}u_l)e_k]^H \\
&= -\tau^{-2}(h_{ik}h_{kl}u_l)\nu_2 - \tau^{-1}\Big[h_{ik}e_k^H + \frac{1}{2}(h_{kl}u_l)(R^N_{ue_i}e_k)^H\Big] \\
&= -\tau^{-2}(h_{ik}h_{kl}u_l)\nu_2 - \tau^{-1}\Big\{h_{ij} + \frac{c}{2}[(h_{il}u_l)u_j - (h_{kl}u_ku_l)\delta_{ij}]\Big\}e_j^H \\
&= -\tau^{-2}(h_{ik}h_{kl}u_l)\nu_2 - \tau^{-1}\Big\{h_{ij} + \frac{c}{2}[(h_{il}u_l)u_j - (h_{kl}u_ku_l)\delta_{ij}]\Big\}\overline{g}^{jk}e_k^h \\
&\quad + \tau^{-2}\Big\{h_{ij}(h_{jk}u_k) + \frac{c}{2}[(h_{il}u_l)u_j(h_{jk}u_k) - (h_{kl}u_ku_l)\delta_{ij}(h_{jk}u_k)]\Big\}\nu_2 \\
&= -\tau^{-1}\Big\{h_{ij} + \frac{c}{2}[(h_{il}u_l)u_j - (h_{kl}u_ku_l)\delta_{ij}]\Big\}\overline{g}^{jk}e_k^h,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\overline{D}_{e_i^h}\nu_2 &= [\overline{D}_{e_i^h}(\tau^{-1})](\nu^V - h_{kl}u_le_k^H) + \tau^{-1}\overline{D}_{e_i^h}[\nu^V - (h_{kl}u_l)e_k^H] \\
&= -\tau^{-2}(h_{kji}h_{jm}u_ku_m)\nu_2 + \tau^{-1}\overline{D}_{e_i^h}[\nu^V - (h_{kl}u_l)e_k^H] \\
&= \tau^{-1}\{\overline{D}_{e_i^H}\nu^V + (h_{kl}u_l)\overline{D}_{\nu^V}\nu^V - [\overline{D}_{e_i^h}(h_{kl}u_l)]e_k^H - (h_{kl}u_l)\overline{D}_{e_i^h}e_k^H\} \\
&\quad - \tau^{-2}(h_{kji}h_{jm}u_ku_m)\nu_2 \\
&= -\tau^{-2}(h_{kji}h_{jm}u_ku_m)\nu_2 - \tau^{-1}\Big\{[A(e_i)]^V - \frac{1}{2}(R^N_{u\nu}e_i)^H + [\overline{D}_{e_i^h}(h_{kl}u_l)]e_k^H \\
&\quad + (h_{kl}u_l)\Big[(D_{e_i}e_k)^H - \frac{1}{2}(R^N_{e_ie_k}u)^V + \frac{1}{2}(h_{im}u_m)(R^N_{u\nu}e_k)^H\Big]\Big\} \\
&= -\tau^{-2}(h_{kji}h_{jm}u_ku_m)\nu_2 - \tau^{-1}\Big\{h_{ij}e_j^V + \frac{c}{2}u_i\nu^H + (h_{kli}u_l)e_k^H
\end{aligned}$$

$$
\begin{aligned}
&\quad + (h_{kl}u_l)\Big[(h_{ik}\nu)^H - \frac{c}{2}(u_k e_i - u_i e_k)^V - \frac{c}{2}(h_{im}u_m)u_k\nu^H\Big]\Big\}\\
&= -\tau^{-2}(h_{kji}h_{jm}u_k u_m)\nu_2 - \tau^{-1}\Big\{\Big[h_{ij} - \frac{c}{2}(h_{kl}u_k u_l)\delta_{ij} + \frac{c}{2}u_i(h_{jl}u_l)\Big]e_j^V\\
&\quad + (h_{kli}u_l)[\overline{g}^{jk}e_j^h - \tau^{-1}(h_{kj}u_j)\nu_2] + \Big[\frac{c}{2}u_i + (h_{ik}h_{kl}u_l) - \frac{c}{2}(h_{im}u_m)(h_{kl}u_k u_l)\Big]\nu^H\Big\}\\
&= -\tau^{-1}\Big\{\Big[h_{ij} - \frac{c}{2}(h_{kl}u_k u_l)\delta_{ij} + \frac{c}{2}u_i(h_{jl}u_l)\Big]e_j^v + (h_{kli}u_l)\overline{g}^{kj}e_j^h\\
&\quad + \Big[\frac{c}{2}u_i + (h_{ik}h_{kl}u_l) - \frac{c}{2}(h_{im}u_m)(h_{kl}u_k u_l)\Big]\nu_1\Big\}.
\end{aligned}
\tag{3.20}
$$

Moreover, we also have

$$
\begin{aligned}
\overline{D}_{e_i^h}\nu_1 &= \overline{D}_{e_i^H}\nu^H + (h_{il}u_l)\overline{D}_{\nu^V}\nu^H\\
&= (D_{e_i}\nu)^H - \frac{1}{2}(R^N_{e_i\nu}u)^V + \frac{1}{2}(h_{il}u_l)(R^N_{u\nu}\nu)^H\\
&= [-A(e_i)]^H + \frac{c}{2}[G(e_i,u)\nu - G(\nu,u)e_i]^V - \frac{c}{2}(h_{il}u_l)[G(u,\nu)\nu - G(\nu,\nu)u]^H\\
&= -h_{ik}e_k^H + \frac{c}{2}u_i\nu^V + \frac{c}{2}(h_{ij}u_j u_k)e_k^H\\
&= \frac{c}{2}u_i\nu^V - \Big[h_{ik} - \frac{c}{2}(h_{ij}u_j u_k)\Big]e_k^H\\
&= -\Big[h_{ik} - \frac{c}{2}(h_{il}u_l u_k)\Big][\overline{g}^{jk}e_j^h - \tau^{-1}(h_{kj}u_j)\nu_2] + \frac{c}{2}u_i[\tau^{-2}(h_{jl}u_l)e_j^h + \tau^{-1}\nu_2]\\
&= -\Big[h_{ik}\overline{g}^{jk} - \frac{c}{2}(h_{il}u_l u_k)\overline{g}^{jk} - \frac{c}{2}\tau^{-2}u_i(h_{jl}u_l)\Big]e_j^h\\
&\quad + \tau^{-1}\Big[h_{ik}(h_{kl}u_l) - \frac{c}{2}(h_{ij}u_j)(h_{kl}u_k u_l) + \frac{c}{2}u_i\Big]\nu_2,
\end{aligned}
\tag{3.21}
$$

$$
\overline{D}_{e_i^v}\nu_1 = \overline{D}_{e_i^V}\nu^H = \frac{1}{2}(R^N_{ue_i}\nu)^H = \frac{c}{2}\big\{G(e_i,\nu)u - G(u,\nu)e_i\big\} = 0. \tag{3.22}
$$

From (3.19)–(3.20), we can see that

$$
G_s(\overline{D}_{e_i^v}\nu_2, e_j^h) = -\tau^{-1}\Big\{h_{ij} + \frac{c}{2}[(h_{ik}u_k)u_j - (h_{kl}u_k u_l)\delta_{ij}]\Big\} = G_s(\overline{D}_{e_i^h}\nu_2, e_j^v).
$$

From (3.19) to (3.21), we immediately obtain the following proposition.

Proposition 3.1 *Let $\overline{\nabla}^\perp$ be the normal connection of $(TM,\overline{g})$ in $TN^{n+1}(c)$. Then*

$$
\overline{\nabla}^\perp_{e_i^h}\nu_1 = T_i\,\nu_2, \quad \overline{\nabla}^\perp_{e_i^v}\nu_1 = 0; \quad \overline{\nabla}^\perp_{e_i^h}\nu_2 = -T_i\,\nu_1, \quad \overline{\nabla}^\perp_{e_i^v}\nu_2 = 0, \tag{3.23}
$$

where for any $1 \le i \le n$,

$$
T_i = \tau^{-1}\Big[\frac{c}{2}u_i + h_{ik}(h_{kl}u_l) - \frac{c}{2}(h_{ij}u_j)(h_{kl}u_k u_l)\Big], \quad \tau^2 = 1 + h_{ik}h_{kj}u_i u_j. \tag{3.24}
$$

Proof Since $\{\nu_1,\nu_2\}$ is a local orthonormal frame field of $T^\perp M$, we have

$$
\overline{\nabla}^\perp_{e_i^h}\nu_\alpha = G_s(\overline{D}_{e_i^h}\nu_\alpha,\nu_\beta)\nu_\beta, \quad \overline{\nabla}^\perp_{e_i^v}\nu_\alpha = G_s(\overline{D}_{e_i^v}\nu_\alpha,\nu_\beta)\nu_\beta \tag{3.25}
$$

for any $1 \le i \le n$ and $\alpha = 1, 2$. Substituting (3.19)–(3.21) into (3.25), we obtain (3.23). The proof of Proposition 3.1 is completed.

From (3.19)–(3.21), we can immediately obtain the following lemma.

Lemma 3.3 *Denote by $\overline{A}_\alpha$ the shape operator of TM with respect to ν_α for $\alpha = 1, 2$. Then*

$$\begin{cases} \overline{A}_1(e_i^h) = \Big[h_{ik}\overline{g}^{jk} - \dfrac{c}{2}(h_{il}u_l u_k)\overline{g}^{jk} - \dfrac{c}{2}\tau^{-2}u_i(h_{jl}u_l)\Big]e_j^h, \\ \overline{A}_2(e_i^h) = \tau^{-1}\Big\{\Big[h_{ij} - \dfrac{c}{2}(h_{kl}u_k u_l)\delta_{ij} + \dfrac{c}{2}u_i(h_{jl}u_l)\Big]e_j^v + (h_{kli}u_l)\overline{g}^{kj}e_j^h\Big\}, \\ \overline{A}_1(e_i^v) = 0, \quad \overline{A}_2(e_i^v) = \tau^{-1}\Big[h_{ij} - \dfrac{c}{2}(h_{kl}u_k u_l)\delta_{ij} + \dfrac{c}{2}(h_{il}u_l)u_j\Big]\overline{g}^{jk}e_k^h. \end{cases} \tag{3.26}$$

Using Lemma 3.3, we can prove the following proposition.

Proposition 3.2 *The second fundamental form σ of TM is determined by*

$$\begin{cases} \sigma(e_i^h, e_j^h) = \Big[h_{ij} - \dfrac{c}{2}u_i(h_{jk}u_k) - \dfrac{c}{2}u_j(h_{ik}u_k)\Big]\nu_1 + \tau^{-1}(h_{jki}u_k)\nu_2, \\ \sigma(e_i^v, e_j^h) = \tau^{-1}\Big[h_{ij} - \dfrac{c}{2}(h_{kl}u_k u_l)\delta_{ij} + \dfrac{c}{2}u_j(h_{ik}u_k)\Big]\nu_2, \quad \sigma(e_i^v, e_j^v) = 0 \end{cases} \tag{3.27}$$

for any $1 \leqslant i, j \leqslant n$.

Proof It is well-known that $\sigma = \sigma_\alpha \nu_\alpha$, where

$$\sigma_\alpha(\overline{X}, \overline{Y}) = G_s(\overline{A}_\alpha(\overline{X}), \overline{Y}) \tag{3.28}$$

for any $\overline{X}, \overline{Y} \in \Gamma(T_{F(x,u)}TM)$. Substituting (3.26) into (3.28), we obtain

$$\begin{aligned} \sigma_1(e_i^h, e_p^h) &= \Big[h_{ik}\overline{g}^{kj} - \frac{c}{2}(h_{il}u_l)u_k\overline{g}^{kj} - \frac{c}{2}\tau^{-2}u_i(h_{jl}u_l)\Big]\overline{g}_{jp} \\ &= h_{ip} - \frac{c}{2}(h_{il}u_l)u_p - \frac{c}{2}u_i(h_{pk}u_k), \\ \sigma_1(e_i^h, e_p^v) &= \sigma_1(e_p^v, e_i^h) = \sigma_1(e_i^v, e_p^v) = 0, \\ \sigma_2(e_i^h, e_p^h) &= \tau^{-1}(h_{kli}u_l)\overline{g}^{kj}\overline{g}_{jp} = \tau^{-1}(h_{ipl}u_l), \quad \sigma_2(e_i^v, e_p^v) = 0, \\ \sigma_2(e_i^h, e_p^v) &= \tau^{-1}\Big[h_{ip} - \frac{c}{2}(h_{kl}u_k u_l)\delta_{ip} + \frac{c}{2}u_i(h_{pl}u_l)\Big] = \sigma_2(e_p^v, e_i^h). \end{aligned}$$

This completes the proof of Proposition 3.1.

Theorem 3.1 *Let M^n be an immersed hypersurface of a space form $N^{n+1}(c)$. Denote by $\overline{H}$ the mean curvature vector field of TM in $TN^{n+1}(c)$. Suppose that the length of $\overline{H}$ is invariant along every fibre of TM. Then we have that*

(1) *If $c \geq 0$, M is totally geodesic in $N^{n+1}(c)$.*

(2) *If $c < 0$, M is an isoparametric hypersurface with at most three distinct principal curvatures $\{-\sqrt{-c}, 0, \sqrt{-c}\}$ with multiples $\{m_-, m_0, m_+\}$, whose second fundamental form is parallel.*

Proof Let $\overline{H}$ be the mean curvature vector field of TM. Choose $\{e_i\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. Then it follows from Proposition 3.2 that

$$\begin{aligned} 2n\overline{H} &= \overline{g}^{ij}\sigma(e_i^h, e_j^h) \\ &= \overline{g}^{ij}\Big[h_{ij} - \frac{c}{2}u_i(h_{jk}u_k) - \frac{c}{2}u_j(h_{ik}u_k)\Big]\nu_1 + \tau^{-1}\overline{g}^{ij}(h_{ijk}u_k)\nu_2 \\ &= [\delta_{ij} - \tau^{-2}(\lambda_i u_i)(\lambda_j u_j)]\Big[\lambda_i\delta_{ij} - \frac{c}{2}u_i(\lambda_j u_j) - \frac{c}{2}u_j(\lambda_i u_i)\Big]\nu_1 \\ &\quad + \tau^{-1}[\delta_{ij} - \tau^{-2}(\lambda_i u_i)(\lambda_j u_j)](h_{ijk}u_k)\nu_2 \\ &= [nH - \tau^{-2}(\lambda_i^2 + c)(\lambda_i u_i^2)]\nu_1 + \tau^{-1}[n(H_k u_k) - \tau^{-2}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)]\nu_2, \end{aligned} \tag{3.29}$$

where H is the mean curvature of M, and $\{H_k\}$ are the coefficients of the covariant derivative of H. Taking the squared length on both sides of the above equation, we obtain

$$\begin{aligned}4n^2|\overline{H}|^2 &= [nH - \tau^{-2}(\lambda_i^2 + c)(\lambda_i u_i^2)]^2 + \tau^{-2}[n(H_k u_k) - \tau^{-2}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)]^2 \\ &= \tau^{-6}\{\tau^2[(nH)\tau^2 - (\lambda_i^2 + c)(\lambda_i u_i^2)]^2 + [n(H_k u_k)\tau^2 - (\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)]^2\}.\end{aligned}$$

Since the length of $\overline{H}$ is invariant along the fibres, by the above equation, we have

$$4n^2|\overline{H}|^2 = 4n^2|\overline{H}(x,0)|^2 = n^2H^2. \tag{3.30}$$

It follows that

$$\begin{aligned}(n^2H^2)\tau^6 &= \tau^2[(nH)\tau^2 - (\lambda_i^2 + c)(\lambda_i u_i^2)]^2 + [n(H_k u_k)\tau^2 - (\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)]^2 \\ &= (n^2H^2)\tau^6 - 2(nH)\tau^4\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big] + \tau^2\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big]^2 \\ &\quad + n^2\Big(\sum_k H_k u_k\Big)^2\tau^4 - 2n\tau^2\Big(\sum_k H_k u_k\Big)\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k) \\ &\quad + \Big[\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big]^2.\end{aligned}$$

So we have

$$\begin{aligned}0 &= \tau^4\Big\{n^2\Big(\sum_k H_k u_k\Big)^2 - 2(nH)\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big]\Big\} + \tau^2\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big]^2 \\ &\quad - 2n\tau^2\Big(\sum_k H_k u_k\Big)\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k) + \Big[\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big]^2 \\ &= \Big[1 + 2\Big(\sum_i \lambda_i^2 u_i^2\Big) + \Big(\sum_i \lambda_i^2 u_i^2\Big)^2\Big]\Big\{n^2\Big(\sum_k H_k u_k\Big)^2 - 2(nH)\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big]\Big\} \\ &\quad + \Big(1 + \sum_i \lambda_i^2 u_i^2\Big)\Big\{\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big]^2 - 2n\Big(\sum_k H_k u_k\Big)\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big\} \\ &\quad + \Big[\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big]^2 \\ &= \mathrm{F}_2 + \mathrm{F}_4 + \mathrm{F}_6,\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}\mathrm{F}_2 &= n^2\Big(\sum_k H_k u_k\Big)^2 - 2(nH)\Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big], \\ \mathrm{F}_4 &= 2\Big(\sum_i \lambda_i^2 u_i^2\Big)F_2 + \Big[\sum_i(\lambda_i^2 + c)(\lambda_i u_i^2)\Big]^2 - 2n\Big(\sum_k H_k u_k\Big)\Big[\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big], \\ \mathrm{F}_6 &= \Big(\sum_i \lambda_i^2 u_i^2\Big)\Big[F_4 + \Big(\sum_i \lambda_i^2 u_i^2\Big)F_2\Big] + \Big[\sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big]^2.\end{aligned}$$

From (3.31), we have $\mathrm{F}_2 = \mathrm{F}_4 = \mathrm{F}_6 = 0$ for any $(u_1, \cdots, u_n)$, which is equivalent to

$$0 = n\Big(\sum_k H_k u_k\Big) = \sum_i(\lambda_i^2 + c)(\lambda_i u_i^2) = \sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k). \tag{3.32}$$

From the first equality of (3.32), we have $H_k = 0$ for any $1 \leq k \leq n$, which means that M is of the constant mean curvature.

The second equality of (3.32) implies that $\lambda_i(\lambda_i^2 + c) = 0$ for any $1 \leq i \leq n$. It follows that every principal curvature λ_i is constant for all $1 \leq i \leq n$.

When $c \geq 0$, $\lambda_i = 0$, for all $1 \leq i \leq n$. In this case, M^n is totally geodesic in $N^{n+1}(c)$.

When $c < 0$, $\lambda_i = 0$, $-\sqrt{-c}$ or $\sqrt{-c}$. We suppose that $\lambda_i = 0$ for $1 \leq i \leq m_0$, $\lambda_i = -\sqrt{-c}$ for $m_0 + 1 \leq i \leq m_0 + m_-$ and $\lambda_i = -\sqrt{-c}$ for $m_0 + m_- + 1 \leq i \leq m_0 + m_- + m_+ = n$.

The third equality of (3.32) turns into

$$\begin{aligned}
0 &= \sum_{i,j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k) = \sum_{i,k}(\lambda_i^2 u_i^2)(\lambda_{i,k}u_k) + 2\sum_{i<j,k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k) \\
&= \sum_{i,k}(\lambda_i^2 u_i^2)(\lambda_{i,k}u_k) + 2\Big[\sum_{k<i<j}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k) + \sum_{i<j}(\lambda_i u_i)(\lambda_j u_j)(\lambda_{i,j}u_i) \\
&\quad + \sum_{i<k<j}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k) + \sum_{i<j}(\lambda_i u_i)(\lambda_j u_j)(\lambda_{j,i}u_j) + \sum_{i<j<k}(\lambda_i u_i)(\lambda_j u_j)(h_{ijk}u_k)\Big] \\
&= \sum_{i}(\lambda_i^2\lambda_{i,i})(u_i^3) + \sum_{i\neq j}(\lambda_i^2 + 2\lambda_i\lambda_j)\lambda_{i,j}(u_i^2 u_j) + 2\Big[\sum_{i<j<k}(\lambda_i\lambda_k + \lambda_j\lambda_k + \lambda_i\lambda_j)h_{ijk}(u_i u_j u_k)\Big] \\
&= 2\Big[\sum_{i<j<k}(\lambda_i\lambda_k + \lambda_j\lambda_k + \lambda_i\lambda_j)h_{ijk}(u_i u_j u_k)\Big].
\end{aligned}$$

It follows from the above equality and the assumptions that

$$[\lambda_i\lambda_j + \lambda_k\lambda_i + \lambda_j\lambda_k]h_{ijk} = 0 \tag{3.33}$$

for any $1 \leq i < j < k \leq n$. On the other hand, we have

$$h_{ijk} = (\lambda_i - \lambda_j)\theta_{ij}(e_k) = (\lambda_j - \lambda_k)\theta_{jk}(e_i) = (\lambda_k - \lambda_i)\theta_{ki}(e_j). \tag{3.34}$$

Suppose that $1 \leq i \leq j \leq k \leq n$. It is seen from (3.34) that $h_{ijk} = 0$, and now that i, j or j, k lie in the same range of indices. For i, j, k lying in the different ranges of indices, from (3.33), we have $h_{ijk} = 0$. It follows that $h_{ijk} = 0$ for any $1 \leq i, j, k \leq n$, which means that M^n has a parallel second fundamental form in $N^{n+1}(c)$. This completes the proof of Theorem 3.1.

Remark 3.2 Miyaoka [6] studied the geometries of isoparametric hypersurfaces with at most three distinct principal curvatures in a space form $N^{n+1}(c)$ of the constant curvature c with $c \geq 0$. Our result gives a geometrical description of this kind of hypersurfaces with $c < 0$.

By Theorem 3.1, we immediately obtain the following corollary.

Corollary 3.1 *Let M^n be a smooth hypersurface of a space form $N^{n+1}(c)$ with $c \geqslant 0$. Then the following statements are equivalent:*

(1) *The length of the mean curvature field $\overline{H}$ of TM is invariant along the fibres of TM;*
(2) *TM is totally geodesic in $(TN^{n+1}(c), G_s)$;*
(3) *TM is minimal in $(TN^{n+1}(c), G_s)$;*
(4) *M is totally geodesic in $N^{n+1}(c)$.*

Munteanu [8] computed the Riemannian curvature tensor of TN endowed with the general metric $G_{a,b}$. For $(TN^{n+1}(c), G_s)$, we have the following lemma.

Lemma 3.4 (cf. [8]) *Suppose that $(N^{n+1}(c), G)$ is a space form of the constant sectional curvature c. Then the Riemannian curvature tensor $\widehat{R}$ of $(TN^{n+1}(c), G_s)$ is given by*

$$\begin{cases} \widehat{R}_{X^HY^H}Z^H = (R^N_{XY}Z)^H + \frac{1}{4}[R^N_{v(R^N_{XZ}v)}Y - R^N_{v(R^N_{YZ}v)}X + 2R^N_{v(R^N_{XY}v)}Z]^H, \\ \widehat{R}_{X^HY^H}Z^V = (R^N_{XY}Z)^V + \frac{1}{4}[R^N_{Y(R^N_{vZ}X)}v - R^N_{X(R^N_{vZ}Y)}v]^V, \\ \widehat{R}_{X^HY^V}Z^H = \frac{1}{2}(R^N_{XZ}Y)^V - \frac{1}{4}[R^N_{X(R^N_{vY}Z)}v]^V, \\ \widehat{R}_{X^HY^V}Z^V = -\frac{1}{2}(R^N_{YZ}X)^H - \frac{1}{4}[R^N_{vY}(R^N_{vZ}X)]^H, \\ \widehat{R}_{X^VY^V}Z^H = (R^N_{XY}Z)^H + \frac{1}{4}[R^N_{vX}(R^N_{vY}Z) - R^N_{vY}(R^N_{vX}Z)]^H, \\ \widehat{R}_{X^VY^V}Z^V = 0 \end{cases} \tag{3.35}$$

for any $X, Y, Z \in \Gamma(TN)$ at point $(y, v) \in TN$.

By direct computation, we obtain the following lemma.

Lemma 3.5 *Let M^n be a hypersurface of $(N^{n+1}(c), G)$. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field on M, and ν be the unit normal vector field of M. Then*

$$\begin{cases} \widehat{R}_{e_i^He_j^H}e_k^H = \{\delta_{jk}U_{il} - \delta_{ik}U_{jl} - \delta_{jl}U_{ik} + \delta_{il}U_{jk} + c(\delta_{jl}\delta_{ik} - \delta_{il}\delta_{jk})\}e_l^H, \\ \widehat{R}_{e_i^He_j^H}e_k^V = \{\delta_{il}V_{jk} - \delta_{jl}V_{ik} + \delta_{jk}V_{il} - \delta_{ik}V_{jl} + V(\delta_{jl}\delta_{ik} - \delta_{il}\delta_{jk})\}e_l^V, \\ \widehat{R}_{e_i^He_j^V}e_k^H = \Big\{\delta_{il}W_{jk} + \delta_{jk}W_{il} - \delta_{jl}W_{ik} - \frac{c}{2}\delta_{ij}\delta_{kl} + W(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il})\Big\}e_l^V, \end{cases} \tag{3.36}$$

$$\begin{cases} \widehat{R}_{e_i^He_j^V}e_k^V = \Big\{\delta_{jk}W_{il} - \delta_{jl}W_{ik} - \delta_{ik}W_{jl} + \frac{c}{2}\delta_{ij}\delta_{kl} + W(\delta_{jl}\delta_{ik} - \delta_{il}\delta_{jk})\Big\}e_l^H, \\ \widehat{R}_{e_i^Ve_j^V}e_k^H = \{\delta_{kj}V_{il} - \delta_{ik}V_{jl} + \delta_{il}V_{jk} - \delta_{jl}V_{ik} + V(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\}e_l^H, \\ \widehat{R}_{e_i^Ve_j^V}e_k^V = 0, \end{cases} \tag{3.37}$$

$$\begin{cases} \widehat{R}_{e_i^H\nu^H}e_j^H = -U_{ij}\nu^H, \quad \widehat{R}_{e_i^H\nu^H}e_j^V = -V_{ij}\nu^V, \quad \widehat{R}_{e_i^H\nu^V}e_j^H = -\frac{c^2}{4}u_iu_j\nu^V, \\ \widehat{R}_{e_i^H\nu^V}e_j^V = -W_{ij}\nu^H, \quad \widehat{R}_{e_i^V\nu^V}e_j^H = -V_{ij}\nu^H, \quad \widehat{R}_{e_i^V\nu^V}e_j^V = 0, \\ \widehat{R}_{e_i^H\nu^H}\nu^H = U_{ij}e_j^H, \quad \widehat{R}_{e_i^H\nu^H}\nu^V = V_{ij}e_j^V, \quad \widehat{R}_{e_i^H\nu^V}\nu^H = W_{ij}e_j^V, \end{cases} \tag{3.38}$$

$$\begin{cases} \widehat{R}_{e_i^H\nu^V}\nu^V = \frac{c^2}{4}u_iu_je_j^H, \quad \widehat{R}_{e_i^V\nu^V}\nu^H = V_{ij}e_j^H, \quad \widehat{R}_{e_i^V\nu^V}\nu^V = 0, \\ \widehat{R}_{e_i^He_j^H}\nu^H = 0, \quad \widehat{R}_{e_i^He_j^H}\nu^V = 0, \quad \widehat{R}_{e_i^He_j^V}\nu^H = -\frac{c}{2}\delta_{ij}\nu^V, \\ \widehat{R}_{e_i^He_j^V}\nu^V = \frac{c}{2}\delta_{ij}\nu^H, \quad \widehat{R}_{e_i^Ve_j^V}\nu^H = 0, \quad \widehat{R}_{e_i^Ve_j^V}\nu^V = 0, \end{cases} \tag{3.39}$$

where $V = c - \frac{c^2}{4}|u|^2$, $W = \frac{c}{2} - \frac{c^2}{4}|u|^2$ and

$$U_{ij} = c\delta_{ij} - \frac{3c^2}{4}u_iu_j, \quad V_{ij} = V\delta_{ij} + \frac{c^2}{4}u_iu_j, \quad W_{ij} = W\delta_{ij} + \frac{c^2}{4}u_iu_j. \tag{3.40}$$

Lemma 3.6 *Under the assumptions as in Lemma 3.5, suppose in addition that $\{e_i^h; e_j^v\}$*

and $\{\nu_1, \nu_2\}$ *are chosen as in* (3.17). *Then we have*

$$\begin{cases} \widehat{R}_{e_i^h e_j^h}\nu_1 = [(h_{jk}u_k)W_{il} - (h_{ik}u_k)W_{jl}]e_l^V, \quad \widehat{R}_{e_i^v e_j^v}\nu_1 = 0, \\ \widehat{R}_{e_i^h e_j^h}\nu_2 = \tau^{-1}\Big\{\dfrac{c^2}{4}[(h_{jk}u_k)u_i - (h_{ik}u_k)u_j][u^H + (h_{kl}u_k u_l)\nu^V] - (h_{kl}u_l)\widehat{R}_{e_i^H e_j^H} e_k^H\Big\}, \\ \widehat{R}_{e_i^h e_j^v}\nu_2 = \tau^{-1}\Big\{\Big[\dfrac{c}{2}\delta_{ij} - (h_{kp}u_p)(h_{iq}u_q)V_{jk}\Big]\nu^H - (h_{kp}u_p)\widehat{R}_{e_i^H e_j^V} e_k^H\Big\}, \\ \widehat{R}_{e_i^h e_j^v}\nu_1 = -\dfrac{c}{2}\delta_{ij}\nu^V - (h_{il}u_l)V_{jk}e_k^H, \quad \widehat{R}_{e_i^v e_j^v}\nu_2 = -\tau^{-1}(h_{kl}u_l)\widehat{R}_{e_i^V e_j^V} e_k^H. \end{cases} \tag{3.41}$$

Denote by $\overline{R}$ the Riemannian curvature tensor and by $\overline{R}^\perp$ the normal curvature tensor of TM in $TN^{n+1}(c)$. Then we have the following Gauss-Codazzi equations:

$$\overline{R}_{XY}Z = [\widehat{R}_{XY}Z]^\top + A_{\sigma(Y,Z)}(X) - A_{\sigma(X,Z)}(Y), \tag{3.42}$$

$$\overline{R}^\perp_{XY}\xi = [\widehat{R}_{XY}\xi]^\perp + \sigma(X, A_\xi(Y)) - \sigma(Y, A_\xi(X)) \tag{3.43}$$

for any $X, Y, Z \in \Gamma(T(TM))$ and $\xi \in \Gamma(T^\perp(TM))$.

By Proposition 3.1, we have the following theorem.

Theorem 3.2 *Let M^n be a smooth hypersurface of a space form $N^{n+1}(c)$. If the normal bundle of TM in TN is flat, then M^n is flat and totally geodesic in $N^{n+1}(c)$, and vise versa.*

Proof Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field on M such that $h_{ij} = \lambda_i\delta_{ij}$. Then (3.16) and (3.26)–(3.27) turn into

$$\begin{cases} \overline{g}_{ij} = \delta_{ij} + (\lambda_i u_i)(\lambda_j u_j), \quad \overline{g}^{ij} = \delta_{ij} - \tau^{-2}(\lambda_i u_i)(\lambda_j u_j), \quad \overline{A}_1(e_i^v) = 0, \\ \overline{A}_1(e_i^h) = \Phi_{ik}\overline{g}^{kj}e_j^h, \quad \overline{A}_2(e_i^v) = \Psi_{ij}\overline{g}^{jk}e_k^h, \quad \overline{A}_2(e_i^h) = \Psi_{ji}e_j^v + \tau^{-1}(h_{ikl}u_l)\overline{g}^{kj}e_j^h, \\ \sigma(e_i^h, e_j^h) = \Phi_{ij}\nu_1 + \tau^{-1}(h_{ijk}u_k)\nu_2, \quad \sigma(e_i^v, e_j^h) = \Psi_{ij}\nu_2, \quad \sigma(e_i^v, e_j^v) = 0, \end{cases} \tag{3.44}$$

where

$$\Phi_{ij} = \lambda_i\delta_{ij} - \frac{c}{2}[(\lambda_i u_i)u_j + (\lambda_j u_j)u_i], \quad \Psi_{ij} = \tau^{-1}\Big[\lambda_i\delta_{ij} - \frac{c}{2}(\lambda_k u_k^2)\delta_{ij} + \frac{c}{2}(\lambda_i u_i)u_j\Big]. \tag{3.45}$$

Since the normal curvatures of TM are determined by

$$G_s(\overline{R}^\perp(e_i^h, e_j^h)\nu_1, \nu_2), \quad G_s(\overline{R}^\perp(e_i^v, e_j^h)\nu_1, \nu_2), \quad G_s(\overline{R}^\perp(e_i^v, e_j^v)\nu_1, \nu_2),$$

it follows that the normal bundle of TM is flat in TN if and only if

$$G_s(\overline{R}^\perp(e_i^h, e_j^h)\nu_1, \nu_2) = G_s(\overline{R}^\perp(e_i^v, e_j^h)\nu_1, \nu_2) = G_s(\overline{R}^\perp(e_i^v, e_j^v)\nu_1, \nu_2) = 0 \tag{3.46}$$

for any $(x, u) \in TM$ and $1 \le i, j \le n$. From the Wiengarten formula (3.43), we have

$$G_s(\overline{R}^\perp(e_i^h, e_j^h)\nu_1, \nu_2) = G_s((\widehat{R}_{e_i^h e_j^h}\nu_1), \nu_2) + G_s(\sigma(e_i^h, \overline{A}_{\nu_1}(e_j^h)), \nu_2) - G_s(\sigma(e_j^h, \overline{A}_{\nu_1}(e_i^h)), \nu_2),$$

$$G_s(\overline{R}^\perp(e_i^v, e_j^h)\nu_1, \nu_2) = G_s((\widehat{R}_{e_i^v e_j^h}\nu_1), \nu_2) + G_s(\sigma(e_i^v, \overline{A}_{\nu_1}(e_j^h)), \nu_2) - G_s(\sigma(e_j^h, \overline{A}_{\nu_1}(e_i^v)), \nu_2),$$

$$G_s(\overline{R}^\perp(e_i^v, e_j^v)\nu_1, \nu_2) = G_s((\widehat{R}_{e_i^v e_j^v}\nu_1), \nu_2) + G_s(\sigma(e_i^v, \overline{A}_{\nu_1}(e_j^v)), \nu_2) - G_s(\sigma(e_j^v, \overline{A}_{\nu_1}(e_i^v)), \nu_2).$$

Using (3.41) and (3.44), we have $(\widehat{R}_{e_i^h e_j^h}\nu_1)^\perp = 0$, $\widehat{R}_{e_i^v e_j^v}\nu_1 = 0$ and $\overline{A}_1(e_i^v) = 0$, from which we get

$$G_s(\widehat{R}^\perp(e_i^h, e_j^h)\nu_1, \nu_2) = 0, \quad G_s(\overline{R}^\perp(e_i^v, e_j^v)\nu_1, \nu_2) = 0, \quad G_s(\sigma(e_j^h, \overline{A}_{\nu_1}(e_i^v)), \nu_2) = 0.$$

It follows that

$$G_s(\overline{R}^{\perp}(e_i^v, e_j^h)\nu_1, \nu_2) = G_s((\widehat{R}_{e_i^v e_j^h}\nu_1)^{\perp}, \nu_2) + G_s(\sigma(e_i^v, \overline{A}_{\nu_1}(e_j^h)), \nu_2), \tag{3.47}$$

$$G_s(\overline{R}^{\perp}(e_i^h, e_j^h)\nu_1, \nu_2) = G_s(\sigma(e_i^h, \overline{A}_{\nu_1}(e_j^h)), \nu_2) - G_s(\sigma(e_j^h, \overline{A}_{\nu_1}(e_i^h)), \nu_2). \tag{3.48}$$

By a direct but not difficulty computation, we can see that

$$G_s((\widehat{R}_{e_i^v e_j^h}\nu_1)^{\perp}, \nu_2) = \tau^{-1}\Big\{\frac{c}{2}\delta_{ij} + \Big[\Big(\frac{c^2}{4}|u|^2 - c\Big)(\lambda_i\lambda_j) - \frac{c^2}{4}(\lambda_k u_k^2)\lambda_j\Big]u_i u_j\Big\}, \tag{3.49}$$

$$\begin{aligned} G_s(\sigma(e_i^v, \overline{A}_{\nu_1}(e_j^h)), \nu_2) &= \tau^{-1}\Big\{\Big[\lambda_i - \frac{c}{2}(\lambda_p u_p^2)\Big]\lambda_j\delta_{ij} + \Big[\frac{c^2}{4}(\lambda_p u_p^2)\lambda_j - \frac{c}{2}\lambda_i\lambda_j - \frac{c^2}{4}\Big]u_i u_j\Big\} \\ &\quad - \tau^{-3}\Big[\lambda_i^2 - \frac{c}{2}(\lambda_p u_p^2)\lambda_i - \frac{c}{2}\Big]\Big[\lambda_j^2 - \frac{c}{2}(\lambda_k u_k^2)\lambda_j + \frac{c}{2}\Big]u_i u_j, \end{aligned} \tag{3.50}$$

$$\begin{aligned} G_s(\sigma(e_i^h, \overline{A}_{\nu_1}(e_j^h)), \nu_2) &= \tau^{-1}\Big[(\lambda_j)(h_{ijp}u_p) - \frac{c}{2}(h_{kpi}u_k u_p)(\lambda_j u_j)\Big] \\ &\quad - \tau^{-3}\Big[(\lambda_j^2 u_j) + \frac{c}{2}u_j - \frac{c}{2}(\lambda_j u_j)(\lambda_k u_k^2)\Big]h_{lpi}(\lambda_l u_l)u_p. \end{aligned} \tag{3.51}$$

Substituting (3.49)–(3.50) into (3.47) and sorting it, we get

$$\begin{aligned} G_s(\overline{R}^{\perp}(e_i^v, e_j^h)\nu_1, \nu_2) &= \tau^{-3}\Big\{\tau^2\Big\{\frac{c}{2}\delta_{ij} + \Big[\lambda_i - \frac{c}{2}(\lambda_p u_p^2)\Big]\lambda_j\delta_{ij} \\ &\quad + \Big[\Big(\frac{c^2}{4}|u|^2 - c\Big)\lambda_i\lambda_j - \frac{c}{2}\lambda_i\lambda_j - \frac{c^2}{4}\Big]u_i u_j\Big\} \\ &\quad - \Big[\lambda_i^2 - \frac{c}{2}(\lambda_p u_p^2)\lambda_i - \frac{c}{2}\Big]\Big[\lambda_j^2 - \frac{c}{2}(\lambda_p u_p^2)\lambda_j + \frac{c}{2}\Big]u_i u_j\Big\} \end{aligned} \tag{3.52}$$

for all $1 \le i, j \le n$. From (3.48) and (3.51), we obtain

$$\begin{aligned} G_s(\overline{R}^{\perp}(e_i^h, e_j^h)\nu_1, \nu_2) &= \tau^{-1}\Big[(\lambda_j - \lambda_i)(h_{ijp}u_p) + \frac{c}{2}(\delta_{jq}\delta_{il} - \delta_{iq}\delta_{jl})(h_{kpq}u_k u_p)(\lambda_l u_l)\Big] \\ &\quad + \tau^{-3}\Big\{(\delta_{jq}\delta_{ir} - \delta_{iq}\delta_{jr})\Big[\lambda_r^2 + \frac{c}{2} - \frac{c}{2}\lambda_r(\lambda_k u_k^2)\Big]h_{lpq}(\lambda_l u_l)u_p u_r\Big\} \end{aligned} \tag{3.53}$$

for all $1 \le i, j \le n$. Suppose in (3.52) that $u = u_k e_k = 0$, and from $G_s(\overline{R}^{\perp}(e_i^v, e_j^h)\nu_1, \nu_2) = 0$ at any point $x \in M$ for all $1 \le i, j \le n$, we have $2\lambda_i^2 + c = 0$ at any $x \in M$ for all $1 \le i \le n$, which implies that $N^{n+1}(c)$ is of the non-positive curvature. Suppose in (3.52) that $\delta_{ij} = 1$ and $c < 0$, i.e., $\lambda_i \ne 0$ at any $x \in M$ for all $1 \le i \le n$, and we have that

$$\tau^2\lambda_i^3[(\lambda_p u_p^2) + \lambda_i^3|u|^2 u_i^2] + (\lambda_p u_p^2)\lambda_i^5 u_i^2[2 - (\lambda_p u_p^2)\lambda_i] = 0,$$

which implies that $\lambda_i = 0$ at any $x \in M$ for all $1 \le i \le n$. It is contradictory to our assumption, and we have that $G_s\big(\overline{R}^{\perp}(e_i^v, e_j^h)\nu_1, \nu_2\big) = 0$ at any point $x \in M$ for all $1 \le i, j \le n$ if and only if M is flat and totally geodesic in $N^{n+1}(c)$. It is seen from (3.53) that, in this case, $G_s\big(\overline{R}^{\perp}(e_i^h, e_j^h)\nu_1, \nu_2\big) = 0$ for all $1 \le i, j \le n$.

The reverse is trivial. This completes the proof of Theorem 3.2.

4 The Almost Complex Structure on TM

In this section, we study the almost complex structure J on TM, which is compatible with $\overline{g}$ and the Kählerian form ω on TM induced by J.

4.1 The induced almost complex structure on TM

Let $\{e_1, e_2, \cdots, e_n\}$ be a local orthonormal frame field and $\{\theta_j^i\}$ be the associated connection forms on M. We describe the almost complex structure J on TM as follows:

$$Je_i^h = \alpha e_i^v + \beta(h_{ij}u_j)(h_{kl}u_l)e_k^v, \quad Je_i^v = \gamma e_i^h + \rho(h_{ij}u_j)(h_{kl}u_l)e_k^h, \tag{4.1}$$

where α, β, γ and ρ are the smooth functions on TM to be determined. Since J is compatible with $\overline{g}$, we have

$$J^2 = -I, \quad \overline{g}(Je_i^h, Je_j^h) = \delta_{ij} + (h_{ik}u_k)(h_{jl}u_l). \tag{4.2}$$

Substituting (4.1) into (4.2), we get

$$\alpha = 1, \quad \beta = \frac{1}{1+\tau}, \quad \gamma = -1, \quad \rho = \frac{1}{\tau(1+\tau)}. \tag{4.3}$$

Thus, J is determined by

$$Je_i^h = e_i^v + \frac{1}{1+\tau}(h_{ir}u_r)(h_{kl}u_l)e_k^v, \quad Je_i^v = -e_i^h + \frac{1}{\tau(1+\tau)}(h_{ir}u_r)(h_{kl}u_l)e_k^h \tag{4.4}$$

for any $1 \leqslant i \leqslant n$.

The Nijenhuis tensor of J is defined to be

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

for any $X, Y \in \Gamma(T_{(x,u)}TM)$. It is easy to see that

$$N_J(Y,X) = -N_J(X,Y), \quad N_J(X,Y) = -N_J(JX,JY), \quad N_J(X,Y) = JN_J(JX,Y).$$

Therefore, we have

$$N_J(X^v, Y^v) = -N_J(JX^v, JY^v), \quad N_J(X^v, Y^h) = JN_J(JX^v, Y^h)$$

for any point $(x,u) \in TM$ and $X, Y \in T_xM$. Since J is an isomorphism from $\mathcal{H}_{(x,u)}$ to $\mathcal{V}_{(x,u)}$, it follows that J is integrable if and only if $N_J(X^h, Y^h) = 0$ for any $X, Y \in \Gamma(TM)$.

Let us compute $N_J(e_i^h, e_j^h)$ for any $1 \leqslant i \leqslant n$. It is known that

$$[e_i^h, e_j^h] = [e_i, e_j]^h - (R^M_{e_ie_j}u)^v, \quad [e_k^v, e_j^h] = -(\nabla_{e_j}e_k)^v = \theta_p^k(e_j)e_p^v, \quad [e_i^v, e_j^v] = 0, \tag{4.5}$$

from which we have

$$\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)[e_k^v, e_j^h] = \frac{1}{1+\tau}(h_{il}u_l)[h_{pm}\theta_k^p(e_j)u_m]e_k^v.$$

At first, we have

$$\begin{aligned}[Je_i^h, e_j^h] &= \Big[e_i^v + \frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)e_k^v, e_j^h\Big] \\ &= [e_i^v, e_j^h] + \frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)[e_k^v, e_j^h] - e_j^h\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)\Big]e_k^v.\end{aligned} \tag{4.6}$$

From the definition of e_j^h, we can see that

$$\begin{aligned}
& -e_j^h\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)\Big] \\
&= -e_j\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)\Big] + \theta_p^q(e_j)u_p\frac{\partial}{\partial u_q}\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)\Big] \\
&= \frac{1}{(1+\tau)^2\tau}[e_j(h_{pq})](h_{pr}u_ru_q)(h_{il}u_l)(h_{km}u_m) - \frac{1}{1+\tau}[e_j(h_{il})](u_lh_{km}u_m) \\
&\quad - \frac{1}{1+\tau}[e_j(h_{km})](u_mh_{il}u_l) - \frac{1}{(1+\tau)^2\tau}[h_{ps}\theta_q^s(e_j)](h_{pr}u_ru_q)(h_{il}u_l)(h_{km}u_m) \\
&\quad + \frac{1}{1+\tau}[h_{iq}\theta_p^q(e_j)u_p](h_{km}u_m) + \frac{1}{1+\tau}(h_{il}u_l)[h_{kq}\theta_p^q(e_j)u_p] \\
&= \frac{1}{(1+\tau)^2\tau}[e_j(h_{pq}) - h_{ps}\theta_q^s(e_j)](h_{pr}u_ru_q)(h_{il}u_l)(h_{km}u_m) \\
&\quad - \frac{1}{1+\tau}[e_j(h_{ip}) - h_{iq}\theta_p^q(e_j)](u_ph_{km}u_m) - \frac{1}{1+\tau}[e_j(h_{kp}) - h_{kq}\theta_p^q(e_j)](u_ph_{il}u_l) \\
&= \frac{1}{(1+\tau)^2\tau}[h_{pqj} + h_{qs}\theta_p^s(e_j)](h_{pr}u_ru_q)(h_{il}u_l)(h_{km}u_m) \\
&\quad - \frac{1}{1+\tau}[h_{ipj} + h_{qp}\theta_i^q(e_j)](u_ph_{km}u_m) - \frac{1}{1+\tau}[h_{kpj} + h_{qp}\theta_k^q(e_j)](u_ph_{il}u_l) \\
&= \frac{1}{(1+\tau)^2\tau}[h_{pqj}(h_{pr}u_ru_q)(h_{il}u_l)(h_{km}u_m)] - \frac{1}{1+\tau}[h_{ipj}(u_ph_{km}u_m) + h_{kpj}(u_ph_{il}u_l)] \\
&\quad - \frac{1}{1+\tau}[h_{qp}\theta_i^q(e_j)(u_ph_{km}u_m) + h_{qp}\theta_k^q(e_j)(u_ph_{il}u_l)]. \qquad (4.7)
\end{aligned}$$

Substituting (4.7) into (4.6) and using (4.5), we obtain

$$\begin{aligned}
[Je_i^h, e_j^h] &= -(\nabla_{e_j}e_i)^v + \Big\{\frac{1}{(1+\tau)^2\tau}[h_{pqj}(h_{pr}u_ru_q)(h_{il}u_l)(h_{km}u_m)] \\
&\quad - \frac{1}{1+\tau}[h_{ipj}(u_ph_{km}u_m) + h_{kpj}(u_ph_{il}u_l)] - \frac{1}{1+\tau}[h_{qp}\theta_i^q(e_j)(u_ph_{km}u_m)]\Big\}e_k^v, \quad (4.8) \\
[Je_j^h, e_i^h] &= -(\nabla_{e_i}e_j)^v + \Big\{\frac{1}{(1+\tau)^2\tau}[h_{pqi}(h_{pr}u_ru_q)(h_{jl}u_l)(h_{km}u_m)] \\
&\quad - \frac{1}{1+\tau}[h_{jpi}(u_ph_{km}u_m) + h_{kpi}(u_ph_{jl}u_l)] - \frac{1}{1+\tau}[h_{qp}\theta_j^q(e_i)(u_ph_{km}u_m)]\Big\}e_k^v. \quad (4.9)
\end{aligned}$$

Using (4.8)–(4.9), we have

$$\begin{aligned}
& [Je_i^h, e_j^h] + [e_i^h, Je_j^h] \\
&= [Je_i^h, e_j^h] - [Je_j^h, e_i^h] \\
&= [e_i, e_j]^v + \Big\{\frac{1}{(1+\tau)^2\tau}(h_{km}u_m)(h_{pr}u_ru_qu_l)(h_{pqj}h_{il} - h_{pqi}h_{jl}) \\
&\quad + \frac{1}{1+\tau}[h_{kpi}(u_ph_{jl}u_l) - h_{kpj}(u_ph_{il}u_l)] + \frac{1}{1+\tau}[\theta_j^q(e_i) - \theta_i^q(e_j)](h_{qp}u_ph_{km}u_m) \\
&\quad + \frac{1}{1+\tau}(h_{pji} - h_{pij})(u_ph_{km}u_m)\Big\}e_k^v,
\end{aligned}$$

where we have used

$$[e_i, e_j]^v = (\nabla_{e_i}e_j)^v - (\nabla_{e_j}e_i)^v.$$

Therefore,

$$
\begin{aligned}
&J([Je_i^h, e_j^h] + [e_i^h, Je_j^h]) \\
&= J([e_i, e_j]^v) + \Big\{ \frac{1}{1+\tau}(h_{pji} - h_{pij})(u_p h_{km} u_m) \\
&\quad + \frac{1}{1+\tau}[h_{kpi}(u_p h_{jl} u_l) - h_{kpj}(u_p h_{il} u_l)] + \frac{1}{1+\tau}[\theta_j^q(e_i) - \theta_i^q(e_j)](h_{qp} u_p h_{km} u_m) \\
&\quad + \frac{1}{(1+\tau)^2\tau}(h_{km} u_m)(h_{pr} u_r u_q u_l)(h_{pqj} h_{il} - h_{pqi} h_{jl}) \Big\} J(e_k^v) \\
&= [\theta_j^q(e_i) - \theta_i^q(e_j)] \Big[- e_q^h + \frac{1}{\tau(1+\tau)}(h_{qr} u_r)(h_{kl} u_l) e_k^h \Big] \\
&\quad + \Big\{ \frac{1}{1+\tau}[h_{kpi}(u_p h_{jl} u_l) - h_{kpj}(u_p h_{il} u_l)] + \frac{1}{1+\tau}(h_{pji} - h_{pij})(u_p h_{km} u_m) \\
&\quad + \frac{1}{(1+\tau)^2\tau}(h_{km} u_m)(h_{pr} u_r u_q u_l)(h_{pqj} h_{il} - h_{pqi} h_{jl}) \\
&\quad + \frac{1}{1+\tau}[\theta_j^q(e_i) - \theta_i^q(e_j)](h_{qp} u_p h_{km} u_m) \Big\} \Big[- e_k^h + \frac{1}{\tau(1+\tau)}(h_{kr} u_r)(h_{ls} u_l) e_s^h \Big] \\
&= -[\theta_j^q(e_i) - \theta_i^q(e_j)] e_q^h + \frac{1}{\tau(1+\tau)}[\theta_j^q(e_i) - \theta_i^q(e_j)](h_{qr} u_r)(h_{kl} u_l) e_k^h \\
&\quad + \Big\{ \frac{1}{(1+\tau)^2\tau}(h_{km} u_m)(h_{pr} u_r u_q u_l)(h_{pqj} h_{il} - h_{pqi} h_{jl}) + \frac{1}{1+\tau}[h_{kpi}(u_p h_{jl} u_l) - h_{kpj}(u_p h_{il} u_l)] \\
&\quad + \frac{1}{1+\tau}(h_{pji} - h_{pij})(u_p h_{km} u_m) \Big\} \Big[- e_k^h + \frac{1}{\tau(1+\tau)}(h_{kr} u_r)(h_{ls} u_l) e_s^h \Big] \\
&\quad - \Big\{ \frac{1}{1+\tau}[\theta_j^q(e_i) - \theta_i^q(e_j)](h_{qp} u_p h_{km} u_m) \Big\} e_k^h + \frac{\tau - 1}{\tau(1+\tau)} \{ [\theta_j^q(e_i) - \theta_i^q(e_j)](h_{qp} u_p)(h_{sl} u_l) \} e_s^h .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&J([Je_i^h, e_j^h] + [e_i^h, Je_j^h]) \\
&= -[e_i, e_j]^h - \frac{1}{\tau(1+\tau)^2}(h_{km} u_m)(h_{pr} u_r)(u_q u_l)(h_{pqj} h_{il} - h_{pqi} h_{jl}) e_k^h \\
&\quad + \frac{1}{\tau(1+\tau)^2}[(h_{pqi} h_{jl} - h_{pqj} h_{il})(u_q u_l)](h_{pr} u_r)(h_{km} u_m) e_k^h \\
&\quad + \frac{1}{(1+\tau)^3\tau^2}(h_{km} u_m)(h_{pr} u_r u_q u_l)(h_{pqj} h_{il} - h_{pqi} h_{jl})(h_{kr} u_r)(h_{ts} u_t) e_s^h \\
&\quad - \frac{1}{1+\tau}[(h_{kpi} h_{jl} - h_{kpj} h_{il})(u_p u_l)] e_k^h - \frac{1}{\tau(1+\tau)}(h_{pji} - h_{pij})(u_p h_{km} u_m) e_k^h \\
&= -[e_i, e_j]^h - \frac{1}{1+\tau}[(h_{kpi} h_{jl} - h_{kpj} h_{il})(u_p u_l)] e_k^h - \frac{1}{\tau(1+\tau)}(h_{pji} - h_{pij})(u_p h_{km} u_m) e_k^h \\
&\quad + \frac{2}{\tau(1+\tau)^2}(h_{pqi} h_{jl} - h_{pqj} h_{il})(u_q u_l)(h_{pr} u_r)(h_{km} u_m) e_k^h \\
&\quad + \frac{1}{(1+\tau)^3\tau^2}(\tau^2 - 1)(h_{pr} u_r u_q u_l)(h_{pqj} h_{il} - h_{pqi} h_{jl})(h_{kt} u_t) e_k^h \\
&= -[e_i, e_j]^h - \frac{1}{1+\tau}[(h_{kpi} h_{jl} - h_{kpj} h_{il})(u_p u_l)] e_k^h - \frac{1}{\tau(1+\tau)}(h_{pji} - h_{pij})(u_p h_{km} u_m) e_k^h \\
&\quad + \frac{2}{\tau(1+\tau)^2}(h_{pqi} h_{jl} - h_{pqj} h_{il})(u_q u_l)(h_{pr} u_r)(h_{km} u_m) e_k^h
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\tau-1)}{(1+\tau)^2\tau^2}(h_{pr}u_r)(u_qu_l)(h_{pqj}h_{il}-h_{pqi}h_{jl})(h_{kt}u_t)e_k^h\\
&=-[e_i,e_j]^h-\frac{1}{1+\tau}[(h_{kpi}h_{jl}-h_{kpj}h_{il})(u_pu_l)]e_k^h-\frac{1}{\tau(1+\tau)}(h_{pji}-h_{pij})(u_ph_{km}u_m)e_k^h\\
&+\frac{2}{\tau(1+\tau)^2}(h_{pqi}h_{jl}-h_{pqj}h_{il})(u_qu_l)(h_{pr}u_r)(h_{km}u_m)e_k^h\\
&+\frac{1}{(1+\tau)^2\tau}(h_{pr}u_r)(u_qu_l)(h_{pqj}h_{il}-h_{pqi}h_{jl})(h_{kl}u_l)e_k^h\\
&-\frac{1}{(1+\tau)^2\tau^2}(h_{pr}u_r)(u_qu_l)(h_{pqj}h_{il}-h_{pqi}h_{jl})(h_{kt}u_t)e_k^h\\
&=-[e_i,e_j]^h-\frac{1}{1+\tau}[(h_{pqi}h_{jl}-h_{pqj}h_{il})(u_qu_l)\delta_{pk}]e_k^h-\frac{1}{\tau(1+\tau)}(h_{pji}-h_{pij})(u_ph_{km}u_m)e_k^h\\
&+\frac{1}{(1+\tau)\tau^2}(h_{pqi}h_{jl}-h_{pqj}h_{il})(u_qu_l)(h_{pr}u_r)(h_{km}u_m)e_k^h\\
&=-[e_i,e_j]^h-\frac{1}{1+\tau}(h_{pqi}h_{jl}-h_{pqj}h_{il})(u_qu_l)[\delta_{pk}-\tau^{-2}(h_{pr}u_r)(h_{km}u_m)]e_k^h\\
&-\frac{1}{\tau(1+\tau)}(h_{pji}-h_{pij})(u_ph_{km}u_m)e_k^h\\
&=-[e_i,e_j]^h-\frac{1}{1+\tau}(h_{pqi}h_{jl}-h_{pqj}h_{il})(u_qu_l)\overline{g}^{pk}e_k^h\\
&-\frac{1}{\tau(1+\tau)}(h_{pji}-h_{pij})(u_ph_{km}u_m)e_k^h. \qquad (4.10)
\end{aligned}
$$

Since $[e_i^v,e_j^v]=0$, we have

$$
\begin{aligned}
[Je_i^h,Je_j^h]&=\Big[e_i^v,\frac{1}{1+\tau}(h_{jl}u_l)(h_{km}u_m)e_k^v\Big]+\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)e_k^v,e_j^v\Big]\\
&+\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)e_k^v,\frac{1}{1+\tau}(h_{jl}u_l)(h_{km}u_m)e_k^v\Big]. \qquad (4.11)
\end{aligned}
$$

Let $e_i=\xi_i^j\partial_{x^j}$ for all $1\le i\le n$ and $u=u_ie_i=u_i\xi_i^j\partial_{x^j}:=v^j\partial_{x^j}$. Then

$$\Big(\frac{\partial u_k}{\partial v^j}\Big)=\Big(\frac{\partial v^j}{\partial u_k}\Big)^{-1}=(\xi_k^j)^{-1}.$$

It follows that for any differentiable function φ on TM,

$$e_i^v(\varphi)=\xi_i^j\partial_{v^j}\varphi=\xi_i^j\Big(\frac{\partial u_k}{\partial v^j}\Big)\partial_{u_k}\varphi=\partial_{u_i}\varphi. \qquad (4.12)$$

By using (4.12), we have

$$
\begin{aligned}
\Big[e_i^v,\frac{1}{1+\tau}(h_{jl}u_l)(h_{km}u_m)e_k^v\Big]&=e_i^v\Big[\frac{1}{1+\tau}(h_{jl}u_l)(h_{km}u_m)\Big]e_k^v\\
&=\Big\{\frac{1}{1+\tau}[h_{ij}(h_{km}u_m)+(h_{jl}u_l)h_{ik}]\\
&\quad-\frac{1}{(1+\tau)^2\tau}(h_{ip}h_{pq}u_q)(h_{jl}u_l)(h_{km}u_m)\Big\}e_k^v, \qquad (4.13)\\
\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)e_k^v,e_j^v\Big]&=-e_j^v\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)\Big]e_k^v\\
&=\Big\{\frac{1}{(1+\tau)^2\tau}(h_{jp}h_{pq}u_q)(h_{il}u_l)(h_{km}u_m)
\end{aligned}
$$

$$- \frac{1}{1+\tau}[h_{ij}(h_{km}u_m) + (h_{il}u_l)h_{jk}]\Big\}e_k^v \tag{4.14}$$

and

$$\begin{aligned}
&\Big[\frac{1}{1+\tau}(h_{il}u_l)(h_{km}u_m)e_k^v, \frac{1}{1+\tau}(h_{jl}u_l)(h_{km}u_m)e_k^v\Big] \\
&= \frac{1}{1+\tau}(h_{ir}u_r)(h_{qm}u_m)e_q^v\Big[\frac{1}{1+\tau}(h_{js}u_s)(h_{pm}u_m)\Big]e_p^v \\
&\quad - \frac{1}{1+\tau}(h_{js}u_s)(h_{qm}u_m)e_q^v\Big[\frac{1}{1+\tau}(h_{ir}u_r)(h_{pm}u_m)\Big]e_p^v \\
&= \frac{1}{1+\tau}(h_{qm}u_m)\Big\{(h_{ir}u_r)e_q^v\Big[\frac{1}{1+\tau}(h_{js}u_s)(h_{pm}u_m)\Big] \\
&\quad - (h_{js}u_s)e_q^v\Big[\frac{1}{1+\tau}(h_{ir}u_r)(h_{pm}u_m)\Big]\Big\}e_p^v \\
&= \frac{1}{(1+\tau)^2}(h_{pl}u_l)(h_{qm}u_m)[(h_{ir}u_r)e_q^v\big(h_{js}u_s\big) - (h_{js}u_s)e_q^v\big(h_{ir}u_r\big)]e_p^v \\
&= \frac{1}{(1+\tau)^2}(h_{pl}u_l)(h_{qm}u_m)(h_{ir}u_r h_{jq} - h_{js}u_s h_{iq})e_p^v \\
&= \frac{1}{(1+\tau)^2}u_p(h_{ip}h_{jq} - h_{iq}h_{jp})(h_{qm}u_m)(h_{kl}u_l)e_k^v.
\end{aligned} \tag{4.15}$$

Substituting (4.13)–(4.15) into (4.11), we obtain

$$\begin{aligned}
[Je_i^h, Je_j^h] &= \Big\{ - \frac{1}{(1+\tau)^2\tau}(h_{ip}h_{pq}u_q)(h_{jl}u_l)(h_{km}u_m) + \frac{1}{1+\tau}h_{ij}(h_{km}u_m) \\
&\quad + \frac{1}{1+\tau}(h_{jl}u_l)h_{ik}\Big\}e_k^v - \Big\{ - \frac{1}{(1+\tau)^2\tau}(h_{jp}h_{pq}u_q)(h_{il}u_l)(h_{km}u_m) \\
&\quad + \frac{1}{1+\tau}h_{ij}(h_{km}u_m) + \frac{1}{1+\tau}(h_{il}u_l)h_{jk}\Big\}e_k^v \\
&\quad + \frac{1}{(1+\tau)^2}(h_{ip}h_{jq} - h_{iq}h_{jp})u_p(h_{qm}u_m)(h_{kl}u_l)e_k^v \\
&= \frac{1}{(1+\tau)^2\tau}\{(h_{jp}h_{pq}u_q)(h_{ir}u_r) - (h_{ip}h_{pq}u_q)(h_{jr}u_r)\}(h_{kl}u_l)e_k^v \\
&\quad + \frac{1}{1+\tau}\{(h_{jl}u_l)h_{ik} - (h_{il}u_l)h_{jk}\}e_k^v \\
&\quad + \frac{1}{(1+\tau)^2}(h_{ip}h_{jq} - h_{iq}h_{jp})u_p(h_{qm}u_m)(h_{kl}u_l)e_k^v \\
&= \frac{1}{(1+\tau)\tau}(h_{jp}h_{ir} - h_{ip}h_{jr})(h_{pq}u_q u_r)(h_{kl}u_l)e_k^v \\
&\quad + \frac{1}{1+\tau}\{(h_{ik}h_{jl} - h_{il}h_{jk})u_l\}e_k^v.
\end{aligned} \tag{4.16}$$

It follows from (4.5), (4.10) and (4.16) that

$$\begin{aligned}
N_J(e_i^h, e_j^h) &= [e_i^h, e_j^h] + J([Je_i^h, e_j^h] + [e_i^h, Je_j^h]) - [Je_i^h, Je_j^h] \\
&= \frac{1}{(1+\tau)\tau}(h_{ip}h_{jr} - h_{jp}h_{ir})(h_{pq}u_q u_r)(h_{kl}u_l)e_k^v - \frac{1}{1+\tau}\{(h_{ik}h_{jl} - h_{il}h_{jk})u_l\}e_k^v \\
&\quad + (u_l R^M_{ijkl})e_k^v - \frac{1}{1+\tau}(h_{pqi}h_{jl} - h_{pqj}h_{il})(u_q u_l)\overline{g}^{pk}e_k^h
\end{aligned}$$

$$- \frac{1}{\tau(1+\tau)}(h_{pji} - h_{pij})(u_p h_{km} u_m) e_k^h$$

$$= \frac{1}{(1+\tau)\tau}(h_{ip}h_{jr} - h_{jp}h_{ir})(h_{pq}u_q u_r)(h_{kl}u_l)e_k^v$$

$$- \frac{1}{1+\tau}(h_{pqi}h_{jl} - h_{pqj}h_{il})(u_q u_l)\overline{g}^{pk} e_k^h$$

$$+ \Big[R_{ijkl}^M u_l - \frac{1}{1+\tau}(h_{ik}h_{jl} - h_{il}h_{jk})u_l\Big] e_k^v - \frac{1}{\tau(1+\tau)}(h_{pji} - h_{pij})(u_p h_{km} u_m) e_k^h$$

$$= L_{ij}^k e_k^h + P_{ij}^k e_k^v, \tag{4.17}$$

where

$$L_{ij}^k = \frac{1}{1+\tau}(h_{il}h_{pqj} - h_{jl}h_{pqi})(u_q u_l)\overline{g}^{pk} - \frac{1}{\tau(1+\tau)}(h_{pji} - h_{pij})(h_{km}u_m u_p), \tag{4.18}$$

$$P_{ij}^k = \Big[R_{ijkl}^M - \frac{1}{1+\tau}(h_{ik}h_{jl} - h_{il}h_{jk})\Big]u_l + \frac{h_{kl}u_l}{(1+\tau)\tau}(h_{ip}h_{jr} - h_{jp}h_{ir})(h_{pq}u_q u_r). \tag{4.19}$$

Theorem 4.1 *Let M^n be a smooth hypersurface of N^{n+1}, and J be the almost complex structure on $(TM, \overline{g})$. Then J is integrable if and only if M is flat and is locally a product of a part of the principal curvature line and a piece of the $(n-1)$-dimensional totally geodesic submanifold of N^{n+1}.*

Proof From (4.17), one can see that J is integrable if and only if $L_{ij}^k = P_{ij}^k = 0$ at any point $(x, u) \in TM$ for any $1 \leqslant i, j, k \leqslant n$.

Let $\{e_i\}$ be a local orthonormal frame field on M such that $h_{ij} = \lambda_i \delta_{ij}$. Then $\tau^2 = 1 + \lambda_i^2 u_i^2$. It follows from (4.12) that

$$e_l^v(\tau) = (2\tau)^{-1} e_l^v(\tau^2) = (2\tau)^{-1}\partial_{u_l}(1 + \lambda_i^2 u_i^2) = \tau^{-1}\lambda_l^2 u_l, \tag{4.20}$$

$$e_p^v[e_l^v(\tau)] = e_p^v[\tau^{-1}\lambda_l^2 u_l] = \tau^{-1}\lambda_l^2\delta_{pl} - \tau^{-2}e_p^v(\tau)\lambda_l^2 u_l = \tau^{-1}\lambda_l^2\delta_{pl} - \tau^{-3}(\lambda_p\lambda_l)^2(u_p u_l) \tag{4.21}$$

for any $1 \le l, p \le n$. Therefore, we have for any $1 \le l \le n$, that

$$e_l^v\Big[\frac{1}{\tau(1+\tau)}\Big] = \Big[\frac{1}{(1+\tau)^2} - \frac{1}{\tau^2}\Big] e_l^v(\tau) = -\frac{1+2\tau}{(1+\tau)^2\tau^3}\lambda_l^2 u_l. \tag{4.22}$$

Suppose that $P_{ij}^k = 0$ for any $1 \leqslant i, j, k \leqslant n$ at any point $(x, u) \in TM$. Then $e_l^v(P_{ij}^k) = 0$ for any $1 \leqslant i, j, k, l \leqslant n$ at any point $(x, u) \in TM$.

Computing $e_l^v(P_{ij}^k)$ and putting $u = 0$, we get

$$R_{ijkl}^M - \frac{1}{2}(h_{ik}h_{jl} - h_{il}h_{jk}) = 0 \tag{4.23}$$

for any $1 \leqslant i, j, k, l \leqslant n$ at any point $x \in M$. Substituting (4.23) into (4.19), we obtain

$$0 = \frac{1}{2(1+\tau)\tau}\{\tau(\tau-1)(h_{ik}h_{jl} - h_{il}h_{jk})u_l + 2(h_{ip}h_{jr} - h_{jp}h_{ir})(h_{pq}u_q u_r)(h_{kl}u_l)\}. \tag{4.24}$$

At the point $x \in M$ where $\sum\limits_{k=1}^n \lambda_k^2 = 0$, (4.24) is trivial.

Suppose that $\sum\limits_{k=1}^n \lambda_k^2 \neq 0$ at the given point $x \in M$. For $u \in T_x M$ with $\tau \neq 1$, we have

$$0 = \tau(\tau-1)(h_{ik}h_{jl} - h_{il}h_{jk})u_l + 2(h_{ip}h_{jr} - h_{jp}h_{ir})(h_{pq}u_q u_r)(h_{kl}u_l)$$

$$= \tau(\tau-1)\lambda_i\lambda_j(\delta_{ik}u_j - \delta_{jk}u_i) + 2(\lambda_i - \lambda_j)(\lambda_i u_i)(\lambda_j u_j)(\lambda_k u_k). \tag{4.25}$$

Multiplying $\lambda_k u_k$ on both sides of (4.25) and taking sum with respect to k, we obtain

$$\begin{aligned} 0 &= \tau(\tau-1)\lambda_i\lambda_j(\delta_{ik}u_j - \delta_{jk}u_i)(\lambda_k u_k) + 2(\lambda_i - \lambda_j)(\lambda_i u_i)(\lambda_j u_j)(\lambda_k u_k)^2 \\ &= \tau(\tau-1)(\lambda_i - \lambda_j)(\lambda_i u_i)(\lambda_j u_j) + 2(\tau^2 - 1)(\lambda_i - \lambda_j)(\lambda_i u_i)(\lambda_j u_j) \\ &= (\tau-1)(2+3\tau)(\lambda_i - \lambda_j)(\lambda_i u_i)(\lambda_j u_j). \end{aligned}$$

It follows that

$$0 = (\lambda_i - \lambda_j)(\lambda_i u_i)(\lambda_j u_j). \tag{4.26}$$

Substituting (4.26) into (4.25), we get

$$0 = \lambda_i\lambda_j u_j \tag{4.27}$$

for any $i \neq j$. Suppose that $\lambda_1 \neq 0$. Then from (4.27), we have

$$0 = \lambda_j u_j \tag{4.28}$$

for any $j > 1$. The condition that $\tau \neq 1$ implies that $\sum_{k=1}^{n} \lambda_k^2 u_k^2 \neq 0$. It follows from (4.28) that $\lambda_1 u_1 \neq 0$. Taking $j = 1$ in (4.27), we obtain that $\lambda_i = 0$ for any $i > 1$.

So there is at most one nonzero principal curvature of the shape operator A at any point of M. It follows that

$$h_{ik}h_{jl} - h_{il}h_{jk} = \lambda_i\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0$$

for any $i \neq j$ and $k \neq l$, which together with (4.23) implies that $R^M_{ijkl} = 0$ for any $i \neq j$ and $k \neq l$, which implies that M is flat.

Suppose that $L^k_{ij} = 0$ for any $1 \leqslant i, j, k \leqslant n$ at any point $(x, u) \in TM$. Then (4.18) turns into

$$\begin{aligned} 0 &= \tau(h_{il}h_{rpj} - h_{jl}h_{rpi})(u_p u_l) - (h_{pji} - h_{pij})(u_p)(\lambda_r u_r) - (h_{pji} - h_{pij})(u_p)(\tau^2 - 1)(\lambda_r u_r) \\ &= \tau(h_{il}h_{rpj} - h_{jl}h_{rpi})(u_p u_l) - \tau^2(h_{pji} - h_{pij})(u_p)(\lambda_r u_r) \end{aligned}$$

or equivalently

$$(h_{il}h_{rpj} - h_{jl}h_{rpi})(u_p u_l) = \tau(h_{pji} - h_{pij})(h_{rl})(u_p u_l) \tag{4.29}$$

for any fixed $1 \leq i, j, r \leq n$. Denote $\Lambda_{ij} = h_{pij}u_p$. Then (4.29) turns into

$$(\lambda_i u_i)\Lambda_{rj} - (\lambda_j u_j)\Lambda_{ri} = \tau(\Lambda_{ji} - \Lambda_{ij})(\lambda_r u_r) \tag{4.30}$$

for any fixed $1 \leq i, j, r \leq n$ with $i \neq j$.

At the point $x \in M$ where $\sum_{k=1}^{n} \lambda_k^2 = 0$, (4.30) is trivial. At the point $x \in M$ where $\lambda_k \neq 0$ for some $1 \leq k \leq n$, we can suppose that $\lambda_1 \neq 0$ and $\lambda_k = 0$ for $2 \leq k \leq n$.

Suppose that $i = 1$ in (4.30). Then

$$(\lambda_1 u_1)\Lambda_{rj} = 0, \quad (\lambda_1 u_1)[\Lambda_{1j} + \tau(\Lambda_{1j} - \Lambda_{j1})] = 0 \tag{4.31}$$

for any fixed $1 < j, r \leq n$. For any $u \in T_xM$ with $u_1 \neq 0$, we have from (4.31) that

$$\Lambda_{rj} = 0, \quad \Lambda_{1j} + \tau(\Lambda_{1j} - \Lambda_{j1}) = 0 \tag{4.32}$$

for any fixed $1 < j, r \leq n$. Note that

$$h_{ijk} = \lambda_{j,k}\delta_{ij} + (\lambda_i - \lambda_j)\theta_{ij}(e_k) \tag{4.33}$$

for all $1 \leq i, j, k \leq n$ where $\lambda_{j,k} = e_k(\lambda_j)$. It follows from (4.33) that

$$h_{ijk} = 0 \tag{4.34}$$

for any $1 < i, j \leq n$ and $1 \leq k \leq n$. From the first equality in (4.32), we can see that

$$h_{1jk} = \lambda_1\theta_{1j}(e_k) = 0, \tag{4.35}$$

which implies that

$$\theta_{1j}(e_k) = 0 \tag{4.36}$$

for all $1 < j, k \leq n$. Note that

$$\Lambda_{jk} = h_{ijk}u_i = \lambda_{j,k}u_j + (\lambda_1 u_1)\theta_{1j}(e_k) - \lambda_j\theta_{ij}(e_k)u_i \tag{4.37}$$

for all $1 \leq j, k \leq n$. It follows from (4.36)–(4.37) that

$$\Lambda_{j1} = (\lambda_1 u_1)\theta_{1j}(e_1), \quad \Lambda_{1j} = \lambda_{1,j}u_1 \tag{4.38}$$

for any $1 < j \leq n$. Substituting (4.38) into the second equality of (4.32) and noting that $u_1 \neq 0$, we obtain

$$\lambda_{1,j} + \tau[\lambda_{1,j} - \lambda_1\theta_{1j}(e_1)] = 0. \tag{4.39}$$

Taking the partial derivative on both sides of (4.39) with respect to u_1 and using (4.20), we have

$$\lambda_{1,j} - \lambda_1\theta_{1j}(e_1) = 0. \tag{4.40}$$

It follows from (4.36) and (4.39)–(4.40) that

$$\lambda_{1,j} = 0, \quad \theta_{1j}(e_1) = 0 \tag{4.41}$$

for all $1 < j \leq n$. From (4.33) and (4.41), we can see that

$$h_{11k} = \lambda_{1,k} = 0, \quad h_{1j1} = \lambda_1\theta_{1j}(e_1) = 0 \tag{4.42}$$

for all $1 < j, k \leq n$. It follows from (4.34)–(4.35) and (4.42) that all of h_{ijk}'s are zero except for h_{111}. Note that

$$[e_j, e_k] = \nabla_{e_j}(e_k) - \nabla_{e_k}(e_j) = [\theta_{kl}(e_j) - \theta_{jl}(e_k)]e_l = \sum_{l=2}^{n}[\theta_{kl}(e_j) - \theta_{jl}(e_k)]e_l \tag{4.43}$$

for all $1 < j, k \leq n$. It follows that the distribution $L^{n-1} = \mathrm{span}\{e_2, \cdots, e_n\}$ is involutive on M^n.

Let γ be a part of the curvature line with respect to the principal curvature λ_1 and U^{n-1} be the maximal integral submanifold of L^{n-1} through every point of γ. Then it follows from (4.36) that U^{n-1} is totally geodesic in N^{n+1}. The second equality of (4.41) implies that M^n is locally a Cartesian product of γ and U^{n-1}.

The proof of sufficiency is trivial. This completes the proof of Theorem 4.1.

4.2 The induced Kählerian form on TM

It is known that the Kählerian 2-form ω of TM is defined to be

$$\omega(\overline{X},\overline{Y})=\overline{g}(\overline{X},J(\overline{Y})) \tag{4.44}$$

for all vector fields $\overline{X},\overline{Y}\in\mathfrak{X}(TM)$, where J is determined by (4.4). Let $\{e_1,e_2,\cdots,e_n\}$ be an orthonormal frame on M such that $h_{ij}=\lambda_i\delta_{ij}$. By direct computation, we have

$$\omega(e_i^h,e_j^h)=\overline{g}(e_i^h,Je_j^h)=\overline{g}\Big(e_i^h,e_j^v+\frac{1}{1+\tau}(h_{jp}u_p)(h_{kl}u_l)e_k^v\Big)=0, \tag{4.45}$$

$$\omega(e_i^v,e_j^v)=\overline{g}(e_i^v,Je_j^v)=\overline{g}\Big(e_i^v,-e_j^h+\frac{1}{\tau(1+\tau)}(h_{jp}u_p)(h_{kl}u_l)e_k^h\Big)=0, \tag{4.46}$$

$$\begin{aligned}\omega(e_i^v,e_j^h)&=\overline{g}(e_i^v,Je_j^h)=\overline{g}\Big(e_i^v,e_j^v+\frac{1}{1+\tau}(h_{jp}u_p)(h_{kl}u_l)e_k^v\Big)\\&=\delta_{ij}+\frac{1}{1+\tau}(h_{il}u_l)(h_{jr}u_r)=\frac{\tau}{1+\tau}\delta_{ij}+\frac{1}{1+\tau}\overline{g}_{ij},\end{aligned} \tag{4.47}$$

$$\begin{aligned}\omega(e_i^h,e_j^v)&=\overline{g}(e_i^h,Je_j^v)=\overline{g}\Big(e_i^h,-e_j^h+\frac{1}{\tau(1+\tau)}(h_{jp}u_p)(h_{kl}u_l)e_k^h\Big)\\&=-\delta_{ij}-\frac{1}{1+\tau}(h_{il}u_l)(h_{jr}u_r)=-\frac{\tau}{1+\tau}\delta_{ij}-\frac{1}{1+\tau}\overline{g}_{ij}.\end{aligned} \tag{4.48}$$

It is known that the exterior differential of ω is defined by

$$\begin{aligned}\mathrm{d}\omega(\overline{X},\overline{Y},\overline{Z})&=\overline{X}(\omega(\overline{Y},\overline{Z}))+\overline{Y}(\omega(\overline{Z},\overline{X}))+\overline{Z}(\omega(\overline{X},\overline{Y}))\\&\quad-\omega([\overline{X},\overline{Y}],\overline{Z})-\omega([\overline{Y},\overline{Z}],\overline{X})-\omega([\overline{Z},\overline{X}],\overline{Y})\end{aligned} \tag{4.49}$$

for all vector fields $\overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(TM)$.

By using (4.5), we have

$$\begin{aligned}&\mathrm{d}\omega(e_i^h,e_j^h,e_k^h)\\
=&-\omega([e_i^h,e_j^h],e_k^h)-\omega([e_j^h,e_k^h],e_i^h)-\omega([e_k^h,e_i^h],e_j^h)\\
=&-\omega(-R^M_{ijpq}u_pe_q^v,e_k^h)-\omega(-R^M_{jkpq}u_pe_q^v,e_i^h)-\omega(-R^M_{kipq}u_pe_q^v,e_j^h)\\
=&\,R^M_{ijpq}u_p\omega(e_q^v,e_k^h)+R^M_{jkpq}u_p\omega(e_q^v,e_i^h)+R^M_{kipq}u_p\omega(e_q^v,e_j^h)\\
=&\,\frac{1}{1+\tau}[R^M_{ijpq}(h_{kr}u_r)+R^M_{jkpq}(h_{ir}u_r)+R^M_{kipq}(h_{jr}u_r)]u_p(h_{ql}u_l),\\
&\mathrm{d}\omega(e_i^h,e_j^h,e_k^v)\\
=&\,e_j^h(\omega(e_k^v,e_i^h))-e_i^h(\omega(e_k^v,e_j^h))-\omega([e_i^h,e_j^h],e_k^v)+\omega([e_k^v,e_j^h],e_i^h)-\omega([e_k^v,e_i^h],e_j^h)\\
=&\,e_j^h\Big[\delta_{ij}+\frac{1}{1+\tau}(h_{kl}u_l)(h_{ir}u_r)\Big]-e_i^h\Big[\delta_{ij}+\frac{1}{1+\tau}(h_{kl}u_l)(h_{jr}u_r)\Big]\\
&+[\theta_j^l(e_i)-\theta_i^l(e_j)]\Big(\delta_{kl}+\frac{1}{1+\tau}(h_{kq}u_q)(h_{lr}u_r)\Big)\\
&+\theta_p^k(e_j)\Big(\delta_{pi}+\frac{1}{1+\tau}(h_{pq}u_q)(h_{ir}u_r)\Big)-\theta_p^k(e_i)\Big(\delta_{pj}+\frac{1}{1+\tau}(h_{pq}u_q)(h_{jr}u_r)\Big)\\
=&\,e_j^h\Big[\frac{1}{1+\tau}(h_{kl}u_l)(h_{ir}u_r)\Big]-e_i^h\Big[\frac{1}{1+\tau}(h_{kl}u_l)(h_{jr}u_r)\Big]\\
&+[\theta_j^k(e_i)-\theta_i^k(e_j)]+\frac{1}{1+\tau}[\theta_j^l(e_i)-\theta_i^l(e_j)](h_{kq}u_q)(h_{lr}u_r)\end{aligned}$$

$$
\begin{aligned}
&+\theta_i^k(e_j)-\theta_j^k(e_i)+\frac{1}{1+\tau}\theta_p^k(e_j)(h_{pq}u_q)(h_{ir}u_r)-\frac{1}{1+\tau}\theta_p^k(e_i)(h_{pq}u_q)(h_{jr}u_r)\\
&=e_j^h\Big[\frac{1}{1+\tau}(h_{kl}u_l)(h_{ir}u_r)\Big]-e_i^h\Big[\frac{1}{1+\tau}(h_{kl}u_l)(h_{jr}u_r)\Big]\\
&\quad+\frac{1}{1+\tau}[\theta_j^l(e_i)(h_{lr}u_r)(h_{kq}u_q)+\theta_k^p(e_i)(h_{pq}u_q)(h_{jr}u_r)]\\
&\quad-\frac{1}{1+\tau}[\theta_i^l(e_j)(h_{lr}u_r)(h_{kq}u_q)+\theta_k^p(e_j)(h_{pq}u_q)(h_{ir}u_r)]\\
&=\frac{1}{1+\tau}\Big\{[(h_{ipj})u_p(h_{km}u_m)+(h_{kpj}u_p)(h_{il}u_l)]-\frac{1}{(1+\tau)\tau}[(h_{pqj}u_q)(h_{pr}u_r)(h_{il}u_l)(h_{km}u_m)]\\
&\quad-[(h_{jpi}u_p)(h_{km}u_m)+(h_{kpi}u_p)(h_{jl}u_l)]+\frac{1}{(1+\tau)\tau}[(h_{pqi}u_q)(h_{pr}u_r)(h_{jl}u_l)(h_{km}u_m)]\Big\},\\
&\quad \mathrm{d}\omega(e_i^h,e_j^v,e_k^v)\\
&=e_i^h(\omega(e_j^v,e_k^v))+e_j^v(\omega(e_k^v,e_i^h))+e_k^v(\omega(e_i^h,e_j^v))-\omega([e_i^h,e_j^v],e_k^v)\\
&\quad-\omega([e_j^v,e_k^v],e_i^h)-\omega([e_k^v,e_i^h],e_j^v)\\
&=e_j^v(\omega(e_k^v,e_i^h))+e_k^v(\omega(e_i^h,e_j^v))\\
&=e_j^v\Big(\delta_{ki}+\frac{1}{1+\tau}(h_{il}u_l)(h_{kr}u_r)\Big)-e_k^v\Big(\delta_{ij}+\frac{1}{1+\tau}(h_{il}u_l)(h_{jr}u_r)\Big)\\
&=e_j^v\Big(\frac{1}{1+\tau}\Big)(h_{il}u_l)(h_{kr}u_r)+\frac{1}{1+\tau}h_{ij}(h_{kr}u_r)+\frac{1}{1+\tau}(h_{il}u_l)h_{kj}\\
&\quad-e_k^v\Big(\frac{1}{1+\tau}\Big)(h_{il}u_l)(h_{jr}u_r)-\frac{1}{1+\tau}h_{ik}(h_{jr}u_r)-\frac{1}{1+\tau}(h_{il}u_l)h_{jk}\\
&=\frac{1}{\tau(1+\tau)^2}[(h_{kp}h_{jr}-h_{jp}h_{kr})u_r](h_{pq}u_q)(h_{il}u_l)+\frac{1}{1+\tau}[h_{ij}h_{kr}-h_{ik}h_{jr}]u_r\\
&=\frac{1}{\tau(1+\tau)^2}[(\lambda_k^2u_k)(\lambda_ju_j)-(\lambda_j^2u_j)(\lambda_ku_k)](\lambda_iu_i)+\frac{1}{1+\tau}[h_{ij}(\lambda_ku_k)-h_{ik}(\lambda_ju_j)].
\end{aligned}
$$

It follows from the above results that

$$
\mathrm{d}\omega(e_i^h,e_j^h,e_k^h)=\frac{1}{1+\tau}(\lambda_ku_kR_{ijlm}^M+\lambda_ju_jR_{kilm}^M+\lambda_iu_iR_{jklm}^M)\lambda_mu_mu_l, \tag{4.50}
$$

$$
\begin{aligned}
\mathrm{d}\omega(e_i^h,e_j^h,e_k^v)&=\frac{1}{1+\tau}[(h_{ipj}-h_{jpi})u_p(\lambda_ku_k)+(h_{kpj}u_p)(\lambda_iu_i)-(h_{kpi}u_p)(\lambda_ju_j)]\\
&\quad+\frac{1}{(1+\tau)^2\tau}[(h_{pqi}u_q)(\lambda_ju_j)-(h_{pqj}u_q)(\lambda_iu_i)](\lambda_pu_p)(\lambda_ku_k),
\end{aligned} \tag{4.51}
$$

$$
\mathrm{d}\omega(e_i^h,e_j^v,e_k^v)=\frac{(\lambda_k-\lambda_j)}{\tau(1+\tau)^2}(\lambda_iu_i)(\lambda_ju_j)(\lambda_ku_k)+\frac{1}{1+\tau}[h_{ij}(\lambda_ku_k)-h_{ik}(\lambda_ju_j)], \tag{4.52}
$$

$$
\mathrm{d}\omega(e_i^v,e_j^v,e_k^v)=0. \tag{4.53}
$$

Theorem 4.2 *Suppose that M^n is a smooth hypersurface of N^{n+1} and ω is the Kählerian 2-form on $(TM,\overline{g},J)$. If $(TM,\overline{g},J,\omega)$ is almost Kählerian, then M is locally a product of a part of the principal curvature line and a piece of the $(n-1)$-dimensional totally geodesic submanifold of N^{n+1}.*

Proof By definition, $(TM,\overline{g},J,\omega)$ is almost Kählerian if $\mathrm{d}\omega=0$ on TM, which implies that the right-hand sides of (4.50)–(4.53) are zero.

We suppose that $\mathrm{d}\omega=0$ on TM. From (4.52), we have

$$
(1+\tau)\mathrm{d}\omega(e_j^h,e_j^v,e_k^v)=\frac{1}{(1+\tau)\tau}(\lambda_k-\lambda_j)(\lambda_ju_j)^2(\lambda_ku_k)+\lambda_j(\lambda_ku_k)=0 \tag{4.54}
$$

for any fixed $j \neq k$ and all $(x, u) \in TM$.

Choose $u_k = 1$ and $u_j = 0$ in (4.54) for any other $j \neq k$. Then it follows that

$$\lambda_j \lambda_k = 0,$$

which implies that there is at most one nonzero principal curvature at any $x \in M$, whose multiple is 1. When $\lambda_1 = \cdots = \lambda_n = 0$ at $x \in M$, the right-hand sides of (4.50)–(4.53) are identically zero. Without loss of generality, we suppose that $\lambda_1 \neq 0$ and $\lambda_k = 0$ for all $2 \leq k \leq n$ in an open subset W of M. From (4.51), we have

$$0 = \mathrm{d}\omega(e_1^h, e_k^h, e_j^v) = \frac{1}{1+\tau}[(h_{ijk} u_i)(\lambda_1 u_1)], \tag{4.55}$$

$$0 = \mathrm{d}\omega(e_1^h, e_j^h, e_1^v) = \frac{1}{1+\tau}\Big[(2h_{1pj} - h_{jp1}) u_p (\lambda_1 u_1) - \frac{1}{\tau(1+\tau)}(h_{1qj} u_q)(\lambda_1 u_1)^3\Big] \tag{4.56}$$

for all $2 \leq j, k \leq n$ at any $(x, u) \in TM$ where $x \in W$. It follows immediately from (4.55) that

$$h_{ijk} = 0 \tag{4.57}$$

for all $1 \leq i \leq n$ and $2 \leq j, k \leq n$. Substituting (4.56) with $i = 1$ into (4.56), we obtain

$$0 = \tau(1+\tau)(2h_{1pj} - h_{jp1}) u_p - (h_{1pj} u_p)(\lambda_1 u_1)^2, \tag{4.58}$$

from which we have

$$0 = \tau(1+\tau)(2h_{1pj} - h_{jp1}) - h_{1pj}(\lambda_1 u_1)^2, \tag{4.59}$$

$$0 = \tau(1+\tau)(2h_{11j} - h_{j11}) - h_{11j}(\lambda_1 u_1)^2 \tag{4.60}$$

for all $2 \leq j, p \leq n$ at any $(x, u) \in TM$ where $x \in M$ and $u \in T_x M$ with $u_1 \neq 0$. Let u_1 tend to 0 in (4.59) and (4.60). It follows that

$$2h_{1pj} - h_{jp1} = 0, \quad 2h_{11j} - h_{j11} = 0. \tag{4.61}$$

Substituting (4.61) into (4.59) and (4.60), and using (4.61) once more, we obtain

$$h_{ij1} = h_{i1j} = 0 \tag{4.62}$$

for all $1 \leq i \leq n$ and $2 \leq j \leq n$ at any $x \in W$. It follows from (4.57) and (4.62) that all of h_{ijk}'s are zero except for h_{111}. Note that for $h_{ij} = \lambda_i \delta_{ij}$,

$$h_{ijk} = \lambda_{j,k} \delta_{ij} + (\lambda_i - \lambda_j)\theta_{ij}(e_k) \tag{4.63}$$

for all $1 \leq i, j, k \leq n$ where $\lambda_{j,k} = e_k(\lambda_j)$. It follows that

$$h_{111} = \lambda_{1,1}, \quad h_{11i} = \lambda_{1,i} = 0, \quad h_{1jk} = \lambda_1 \theta_{1j}(e_k) = 0, \quad h_{ijk} = 0 \tag{4.64}$$

for any $2 \leq i, j \leq n$ and $1 \leq k \leq n$. From (4.64), we can see that

$$\theta_{1j}(e_k) = 0 \quad \text{for all } 2 \leq j \leq n \text{ and } 1 \leq k \leq n. \tag{4.65}$$

By applying the similar discussion as in the proof of Theorem 4.1, we can see that M is locally a product of a part of the principal curvature line and a piece of the $(n-1)$-dimensional totally geodesic submanifold of N^{n+1}. This completes the proof of Theorem 4.2.

Acknowledgement The authors wish to make a grateful acknowledgement to the referees for their advice on the original manuscript.

References

[1] Deshmukh, S., Al-Odan, H. and Shaman, T. A., Tangent bundle of the hypersurfaces in a Euclidean space, *Acta. Math. Acad. Paedagog. Nyházi* (*N.S.*), **23**(1), 2007, 71–87.

[2] Klingenberg, W. and Sasaki, S., On the tangent sphere bundle of a 2-sphere, *Tohôku Math. J.*, **27**, 1975, 49–56.

[3] Konno, T., Killing vector fields on tangent sphere bundles, *Kodai Math. J.*, **21**, 1998, 61–72.

[4] Konno, T. and Tanno S., Geodesics and Killing vector fields on the tangent sphere bundle, *Nagoya Math. J.,* **151**, 1998, 91–97.

[5] Kowalski, O., Curvature of the induced Riemannian metric of the tangent bundle of Riemannian manifold, *J. Reine Angew. Math.*, **250**, 1971, 124–129.

[6] Miyaoka, R., Complete hypersurfaces in the space form with three principal curvatures, *Math. Z.*, **179**, 1982, 345–354.

[7] Munteanu, M. I., Cheeger Gromoll type metrics on the tangent bundle, *Sci. Ann. Univ. Agric. Sci. Vet. Med.*, **49**(2), 2006, 257–268.

[8] Munteanu, M. I., Some aspects on the geometry of the tangent bundles and tangent shpere bundles of a Riemannian manifold, *Mediterr. J. Math.*, **5**, 2008, 43–59.

[9] Musso, E. and Tricerri, F., Riemannian metrics on tangent bundles, *Ann. Math. Pura Appl.*, **150**(4), 1988, 1–20.

[10] Nagano, T., Isometries on complex-product spaces, *Tensor, New Series*, **9**, 1959, 47–61.

[11] Nagy, P. T., On the tangent sphere bundle of a Riemannian 2-manifold, *Tohôku Math. J.*, **29**, 1977, 203–208.

[12] Nagy, P. T., Geodesics on the tangent sphere bundle of a Riemannian manifold, *Geom. Dedicata.*, **7**(2), 1978, 233–243.

[13] Sasaki, S., On the differential geometry of tangent bundles of Riemannian manifolds I, II, *Tohôku Math. J.*, **10**, 1958, 338–354; **14**, 1962, 146–155.

[14] Sasaki, S., Geodesics on the tangent sphere bundles over space forms, *J. Reine Angew. Math.*, **288**, 1976, 106–120.

[15] Tachibana, S. and Okumura, M., On the almost-complex structure of tangent bundles of Riemannian spaces, *Tohôku Math. J.*, **14**(2), 1962, 156–161.

[16] Tashiro, Y., On contact structure of hypersurfaces in complex manifolds I, II, *Tohôku Math. J.*, **15**, 1963, 62–78; 167–175.

[17] Tashiro, Y., On contact structures of tangent sphere bundles, *Tohôku Math. J.*, **21**, 1969, 117–143.