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On the Dedekind Sums and Two-Term Exponential Sums*

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Abstract In this paper, the authors use the analytic methods and the properties of character sums mod p to study the computational problem of one kind of mean value involving the classical Dedekind sums and two-term exponential sums, and give an exact computational formula for it.

Keywords Dedekind sums, Two-term exponential sums, Mean value, Computational formula
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1 Introduction

Let q be a natural number and h an integer prime to q. The classical Dedekind sums

$$S(h,q)=\sum_{a=1}^{q}\Big(\Big(\frac{a}{q}\Big)\Big)\Big(\Big(\frac{ah}{q}\Big)\Big),$$

where

$$((x))=\begin{cases}x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer},\\ 0, & \text{if } x \text{ is an integer},\end{cases}$$

describes the behaviour of the logarithm of the eta-function (see [10–11]) under modular transformations. About the arithmetical properties of $S(h,q)$, one can find them in [1, 6, 9, 13–14].

Recently, the second author and Zhang [12] studied the computational problem of the mean value

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}|C(m,n,k,h;p)|^2\,S(m\overline{n},p),$$

and obtained several interesting computational formulae for it, where the two-term exponential sums $C(m,n,k,h;q)$ are defined as follows:

$$C(m,n,k,h;q)=\sum_{a=1}^{q}e\Big(\frac{ma^k+na^h}{q}\Big),$$

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$e(y) = e^{2\pi i y}$ and $n\overline{n} \equiv 1 \bmod p$. Some results related to $C(m, n, k, h; q)$ can be found in references [2–5].

In this paper as a note of [12], we consider the hybrid mean value

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} C^3(m, n, 3, 1; p) S(m\overline{n}, p), \tag{1.1}$$

and then use the analytic method to give an exact computational formula for (1.1). That is, we shall prove the following theorem.

Theorem 1.1 *For any odd prime $p > 3$, we have the computational formulae:*

(A) *If $p \equiv 1 \bmod 4$, then*

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} C^3(m, n, 3, 1; p) \cdot S(m\overline{n}, p) = 0.$$

(B) *If $p = 12k + 11$, and u_0 is the solution of the congruent equation $u^3 \equiv -4 \bmod p$, then*

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} C^3(m, n, 3, 1; p) \cdot S(m\overline{n}, p) = \left(\left(\frac{1 + u_0}{p} \right) - 1 \right) \cdot p^2 \cdot h_p^2.$$

(C) *If $p = 12k + 7$, and the congruent equation $u^3 \equiv -4 \bmod p$ has no solution, then*

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} C^3(m, n, 3, 1; p) \cdot S(m\overline{n}, p) = -p^2 \cdot h_p^2.$$

(D) *If $p = 12k + 7$, and the congruent equation $u^3 \equiv -4 \bmod p$ has three integer solutions u_1, u_2 and u_3, then*

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} C^3(m, n, 3, 1; p) \cdot S(m\overline{n}, p) = \begin{cases} -4 \cdot p^2 \cdot h_p^2, & \text{if all } \left(\frac{1+u_i}{p}\right) = 1,\ i = 1, 2, 3, \\ 0, & \text{otherwise,} \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol, and h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

It is clear from this theorem that we may immediately deduce the following corollary.

Corollary 1.1 *Let $p = 12k + 7$ be an odd prime such that 4 is not a cubic residue* mod p, *and then we have the identity*

$$h_p = \frac{1}{p} \cdot \left\{ \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} C^3(m, n, 3, 1; p) \cdot S(-m\overline{n}, p) \right\}^{\frac{1}{2}}.$$

For general integers $q \geq 3$ and $k \geq 4$, whether there exists an exact computational formula for the mean value

$$\sum_{m=1}^{q}{}' \sum_{n=1}^{q}{}' \ C^3(m, n, 3, 1; q) \cdot S(m\overline{n}, q)$$

and

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} C^k(m,n,3,1;p)\cdot S(m\overline{n},p)$$

is an open problem, where p is an odd prime.

2 Several Lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorem. First we have the following lemma.

Lemma 2.1 *Let $p>3$ be a prime. Then for any integer n with $(n,p)=1$, we have the identity*

$$\sum_{a=1}^{p}\left(\frac{a^2+n}{p}\right)=-1,$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol.

Proof See Lemma 1 of [12].

Lemma 2.2 *Let $p>3$ be a prime with $p\equiv 3 \bmod 4$, and then we have the identities*:

(U) *If $p=12k+11$, and u_0 satisfies the congruence $u^3\equiv -4 \bmod p$, then*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a+b+1)(a^3+b^3+1)}{p}\right)=\left(1-\left(\frac{1+u_0}{p}\right)\right)\cdot p+6.$$

(V) *If $p=12k+7$, and the congruence $u^3\equiv -4 \bmod p$ has no solution, then*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a+b+1)(a^3+b^3+1)}{p}\right)=p.$$

(W) *If $p=12k+7$ and the congruence $u^3\equiv -4 \bmod p$ has three solutions u_1, u_2 and u_3, then*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a+b+1)(a^3+b^3+1)}{p}\right)=\left(1+\sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\right)\cdot p.$$

Proof First let $a+b=u$, and then from the properties of the Legendre's symbol, we have the identity

$$\begin{aligned}
M &\equiv \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a+b+1)(a^3+b^3+1)}{p}\right)\\
&= \sum_{a=1}^{p}\sum_{b=1}^{p}\left(\frac{(a+b+1)(a^3+b^3+1)}{p}\right)-1-2\cdot\sum_{a=1}^{p-1}\left(\frac{(a+1)(a^3+1)}{p}\right)\\
&= \sum_{u=1}^{p}\sum_{a=1}^{p}\left(\frac{(u+1)(a^3+(u-a)^3+1)}{p}\right)-1-2\cdot\sum_{a=1}^{p-1}\left(\frac{(a+1)^2(a^2-a+1)}{p}\right)
\end{aligned}$$

$$= \sum_{u=1}^{p} \left(\frac{u+1}{p}\right) \sum_{a=1}^{p} \left(\frac{3ua^2 - 3u^2a + u^3 + 1}{p}\right) - 1 - 2 \cdot \sum_{a=1}^{p-2} \left(\frac{a^2 - a + 1}{p}\right)$$
$$= \sum_{u=1}^{p-1} \left(\frac{3 + 3\overline{u}}{p}\right) \sum_{a=1}^{p} \left(\frac{a^2 - ua + \overline{3u}(u^3 + 1)}{p}\right) + p - 1 - 2 \cdot \sum_{a=2}^{p-1} \left(\frac{a^2 + a + 1}{p}\right)$$
$$= \sum_{u=1}^{p-1} \left(\frac{3 + 3\overline{u}}{p}\right) \sum_{a=1}^{p} \left(\frac{(2a - u)^2 + 4 \cdot \overline{3u}(u^3 + 1) - u^2}{p}\right) + p - 1 - 2 \cdot \sum_{a=2}^{p-1} \left(\frac{a^2 + a + 1}{p}\right)$$
$$= \sum_{u=1}^{p-1} \left(\frac{3 + 3\overline{u}}{p}\right) \sum_{a=1}^{p} \left(\frac{a^2 + 4 \cdot \overline{3u}(u^3 + 1) - u^2}{p}\right) + p - 1 - 2 \cdot \sum_{a=2}^{p-1} \left(\frac{a^2 + a + 1}{p}\right). \tag{2.1}$$

Now for prime $p > 3$, from Lemma 2.1 we have the identity

$$\sum_{a=2}^{p-1} \left(\frac{a^2 + a + 1}{p}\right) = \sum_{a=1}^{p} \left(\frac{a^2 + a + 1}{p}\right) - 1 - \left(\frac{3}{p}\right)$$
$$= \sum_{a=1}^{p} \left(\frac{4a^2 + 4a + 4}{p}\right) - 1 - \left(\frac{3}{p}\right)$$
$$= \sum_{a=1}^{p} \left(\frac{(2a + 1)^2 + 3}{p}\right) - 1 - \left(\frac{3}{p}\right)$$
$$= \sum_{a=1}^{p} \left(\frac{a^2 + 3}{p}\right) - 1 - \left(\frac{3}{p}\right) = -2 - \left(\frac{3}{p}\right). \tag{2.2}$$

If $p = 12k+11$, then $\left(\frac{3}{p}\right) = 1$ and $(3, p-1) = 1$. This time, if u_0 passes through a residue system mod p, then u_0^3 also passes through the residue system, so there exists one and only one integer $1 \le u_0 \le p - 1$ such that the congruence $u_0^3 \equiv -4 \bmod p$ or $4 \cdot \overline{3u_0}(u_0^3 + 1) - u_0^2 \equiv 0 \bmod p$. For other u, we have

$$(p, 4 \cdot \overline{3u}(u^3 + 1) - u^2) = 1.$$

By Lemma 2.1, noting that

$$\left(\frac{u_0}{p}\right) = \left(\frac{\overline{u_0}}{p}\right) = -1,$$

we have

$$\sum_{u=1}^{p-1} \left(\frac{3 + 3\overline{u}}{p}\right) \sum_{a=1}^{p} \left(\frac{a^2 + 4 \cdot \overline{3u}(u^3 + 1) - u^2}{p}\right)$$
$$= \sum_{\substack{u=1 \\ u \ne u_0}}^{p-1} \left(\frac{1 + \overline{u}}{p}\right) \sum_{a=1}^{p} \left(\frac{a^2 + 4 \cdot \overline{3u}(u^3 + 1) - u^2}{p}\right)$$
$$+ \left(\frac{1 + \overline{u_0}}{p}\right) \sum_{a=1}^{p} \left(\frac{a^2 + 4 \cdot \overline{3u_0}(u_0^3 + 1) - u_0^2}{p}\right)$$
$$= - \sum_{\substack{u=1 \\ u \ne u_0}}^{p-1} \left(\frac{1 + \overline{u}}{p}\right) + \left(\frac{1 + \overline{u_0}}{p}\right) \cdot (p - 1)$$

$$= -\sum_{u=1}^{p-1}\left(\frac{1+\overline{u}}{p}\right) + \left(\frac{1+\overline{u_0}}{p}\right)\cdot p = 1 - \left(\frac{1+u_0}{p}\right)\cdot p. \tag{2.3}$$

So if $p = 12k+11$, then combining (2.1)–(2.3), we have

$$M = 1 - \left(\frac{1+u_0}{p}\right)\cdot p + p - 1 - 2\cdot(-2-1) = \left(1 - \left(\frac{1+u_0}{p}\right)\right)\cdot p + 6. \tag{2.4}$$

If $p = 12k+7$, then $\left(\frac{3}{p}\right) = -1$ and $(3, p-1) = 3$. So the number of the solutions of the congruence equation $u^3 \equiv -4 \bmod p$ is 0 or 3.

(I) If the equation $u^3 \equiv -4 \bmod p$ has no solutions, then from the method of proving (2.4), we have

$$M = \sum_{u=1}^{p-1}\left(\frac{1+\overline{u}}{p}\right) + p - 1 - 2\cdot(-2+1) = p. \tag{2.5}$$

(II) If the equation $u^3 \equiv -4 \bmod p$ has three solutions: u_1, u_2 and u_3. Then $\left(\frac{u_i}{p}\right) = -1$, $i = 1, 2, 3$. From the method of proving (2.4), we have

$$\begin{aligned} M &= \sum_{\substack{u=1\\ u\neq u_1,u_2,u_3}}^{p-1}\left(\frac{1+\overline{u}}{p}\right) - (p-1)\sum_{i=1}^{3}\left(\frac{1+\overline{u_i}}{p}\right) + p - 1 - 2\cdot(-2+1) \\ &= \sum_{u=1}^{p-1}\left(\frac{1+\overline{u}}{p}\right) + 1 + p - p\sum_{i=1}^{3}\left(\frac{1+\overline{u_i}}{p}\right) = \left(1 + \sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\right)\cdot p. \end{aligned} \tag{2.6}$$

Now Lemma 2.2 follows from (2.4)–(2.6).

Lemma 2.3 *Let $p > 3$ be a prime, and χ be any odd character* mod p. *Then we have the identities:*

(a) *If* $p \equiv 1 \bmod 4$, *then*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m, n, 3, 1; p) = 0.$$

(b) *If $p = 12k+11$, and u_0 satisfies the congruent equation* $u^3 \equiv -4 \bmod p$, *then*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m, n, 3, 1; p) = \left(\left(\frac{1+u_0}{p}\right) - 1\right)\cdot p^2(p-1).$$

(c) *If $p = 12k+7$, and the congruent equation* $u^3 \equiv -4 \bmod p$ *has no solutions, then*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m, n, 3, 1; p) = -p^2(p-1).$$

(d) *If $p = 12k+7$, and the congruent equation* $u^3 \equiv -4 \bmod p$ *has three solutions u_1, u_2 and u_3, then*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m, n, 3, 1; p) = -\left(1 + \sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\right)\cdot p^2(p-1).$$

Proof From the definition and the properties of the Gauss sums $\tau(\chi)$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m,n,3,1;p)\\
&=\sum_{a=1}^{p}\sum_{b=1}^{p}\sum_{c=1}^{p}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\overline{n})e\Big(\frac{m(a^3+b^3+c^3)+n(a+b+c)}{p}\Big)\\
&=\tau(\chi)\cdot\tau(\overline{\chi})\sum_{a=1}^{p}\sum_{b=1}^{p}\sum_{c=1}^{p}\chi(a+b+c)\overline{\chi}(a^3+b^3+c^3)\\
&=-p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a+b+c)\overline{\chi}(a^3+b^3+c^3)\\
&\quad-3p\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b)\overline{\chi}(a^3+b^3)-3p\cdot\sum_{a=1}^{p-1}\chi(a)\overline{\chi}(a^3)\\
&=-p\cdot\Big(\sum_{c=1}^{p-1}\overline{\chi}^2(c)\Big)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+1)\overline{\chi}(a^3+b^3+1)\\
&\quad-3p\cdot\Big(\sum_{b=1}^{p-1}\overline{\chi}^2(b)\Big)\sum_{a=1}^{p-1}\chi(a+1)\overline{\chi}(a^3+1)-3p\cdot\sum_{a=1}^{p-1}\overline{\chi}^2(a),
\end{aligned}\tag{2.7}$$

where we have used the fact that $\tau(\chi)\cdot\tau(\overline{\chi})=-p$, if χ is an odd character mod p.

If $p\equiv 1 \bmod 4$, then for any odd character χ mod p, $\overline{\chi}^2$ is a non-principal character mod p. So we have

$$\sum_{a=1}^{p-1}\overline{\chi}^2(a)=0.$$

Therefore, from (2.7) we can deduce that

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m,n,3,1;p)=0. \tag{2.8}$$

If $p\equiv 3 \bmod 4$, then for any non-real odd character χ mod p, $\overline{\chi}^2$ is a non-principal character mod p. So we also have

$$\sum_{a=1}^{p-1}\overline{\chi}^2(a)=0.$$

If χ is the Legendre's symbol, then χ^2 is the principal character mod p, and

$$\sum_{a=1}^{p-1}\overline{\chi}^2(a)=p-1.$$

Therefore, from (2.2), (2.7) and (U) of Lemma 2.2, we can deduce that if $p=12k+11$, and u_0 satisfies the congruent equation $u^3\equiv -4 \bmod p$, then

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m,n,3,1;p)$$

$$= -p(p-1)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a+b+1)(a^3+b^3+1)}{p}\right)$$

$$-\,3p(p-1)\cdot\sum_{a=2}^{p-1}\left(\frac{a^2+a+1}{p}\right)-3p(p-1)$$

$$= -p(p-1)\cdot\left(\left(1-\left(\frac{1+u_0}{p}\right)\right)\cdot p+6\right)+9p(p-1)-3p(p-1)$$

$$= \left(\left(\frac{1+u_0}{p}\right)-1\right)\cdot p^2(p-1). \tag{2.9}$$

If $p = 12k+7$, and the congruent equation $u^3 \equiv -4 \bmod p$ has no solutions, then from (2.2), (2.7) and (V) of Lemma 2.2, we can deduce that

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m,n,3,1;p)$$

$$= -p^2(p-1)-3p(p-1)(-2+1)-3p(p-1) = -p^2(p-1). \tag{2.10}$$

If $p = 12k+7$, and the congruent equation $u^3 \equiv -4 \bmod p$ has three solutions u_1, u_2 and u_3, then from (2.2), (2.7) and (W) of Lemma 2.2, we can deduce that

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\cdot\overline{n})\cdot C^3(m,n,3,1;p)$$

$$= -p^2(p-1)\Big(1+\sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\Big)-3p(p-1)(-2+1)-3p(p-1)$$

$$= -\Big(1+\sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\Big)\cdot p^2(p-1). \tag{2.11}$$

Now Lemma 2.3 follows from (2.8)–(2.11).

Lemma 2.4 *Let $p = 12k+7$ be an odd prime such that the congruent equation $u^3 \equiv -4 \bmod p$ has three integer solutions u_1, u_2 and u_3. Then we have the identity*

$$\sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right) = \begin{cases} 3, & \text{if all } \left(\frac{1+u_i}{p}\right)=1,\ i=1,2,3, \\ -1, & \text{otherwise,} \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol.

Proof Since u_1, u_2 and u_3 are the three integer solutions of the congruent equation $u^3 \equiv -4 \bmod p$, from the properties of the polynomial congruence, we have

$$(u-u_1)(u-u_2)(u-u_3) \equiv 0 \bmod p$$

or

$$u^3 - u^2(u_1+u_2+u_3)+u(u_1u_2+u_2u_3+u_3u_1)-u_1u_2u_3 \equiv 0 \bmod p.$$

Comparing this formula with $u^3+4 \equiv 0 \bmod p$, we have the congruences

$$u_1+u_2+u_3 \equiv 0 \bmod p, \quad u_1u_2u_3 \equiv -4 \bmod p$$

and

$$u_1u_2 + u_2u_3 + u_3u_1 \equiv 0 \bmod p.$$

Then using these three congruences we have

$$\begin{aligned}A &= \Big(\frac{1+u_1}{p}\Big)\Big(\frac{1+u_2}{p}\Big)\Big(\frac{1+u_3}{p}\Big)\\ &= \Big(\frac{(1+u_1)(1+u_2)(1+u_3)}{p}\Big)\\ &= \Big(\frac{u_1u_2u_3 + u_1u_2 + u_2u_3 + u_3u_1 + u_1 + u_2 + u_3 + 1}{p}\Big)\\ &= \Big(\frac{-4+1}{p}\Big) = \Big(\frac{-3}{p}\Big) = -\Big(\frac{3}{12k+7}\Big) = 1.\end{aligned}$$

From this identity and noting that $\big(\frac{u_i+1}{p}\big) = \pm 1$, we may immediately deduce

$$\sum_{i=1}^{3}\Big(\frac{1+u_i}{p}\Big) = \begin{cases} 3, & \text{if all } \big(\frac{1+u_i}{p}\big) = 1, \ \ i = 1, 2, 3, \\ -1, & \text{otherwise.}\end{cases}$$

This proves Lemma 2.4.

Lemma 2.5 *Let $q > 2$ be an integer, and then for any integer a with $(a, q) = 1$, we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to the character $\chi \bmod d$.

Proof See Lemma 2 of [14].

3 Proof of Theorem 1.1

In this section, we shall complete the proof of our theorem. By Lemma 2.5, we have

$$S(a,p) = \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2. \tag{3.1}$$

If $p \equiv 1 \bmod 4$, then from (3.1), the definition of $C(m,n,3,1;p)$ and (a) of Lemma 2.3, we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}C^3(m,n,3,1;p)\cdot S(m\overline{n},p)\\ &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(m\overline{n})C^3(m,n,3,1;p) = 0. \end{aligned}\tag{3.2}$$

If $p = 12k+11$, and u_0 is the solution of the congruent equation $u^3 \equiv -4 \bmod p$, then noting that $L(1,\chi_2) = \frac{\pi h_p}{\sqrt{p}}$ (see [7, p. 50]), from (3.1), the definition of $C(m,n,3,1;p)$ and (b) of

Lemma 2.3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} C^3(m,n,3,1;p)\cdot S(m\overline{n},p)\\
&= \frac{p\cdot\pi^{-2}}{p-1}\,|L(1,\chi_2)|^2\cdot\left(\left(\frac{1+u_0}{p}\right)-1\right)\cdot p^2(p-1)\\
&= \left(\left(\frac{1+u_0}{p}\right)-1\right)\cdot p^2\cdot h_p^2. \qquad (3.3)
\end{aligned}$$

If $p = 12k+7$, and the congruent equation $u^3 \equiv -4 \bmod p$ has no solutions, then from (3.1), the definition of $C(m,n,3,1;p)$ and (c) of Lemma 2.3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} C^3(m,n,3,1;p)\cdot S(m\overline{n},p)\\
&= -\frac{p\cdot\pi^{-2}}{p-1}\,|L(1,\chi_2)|^2\cdot p^2(p-1) = -p^2\cdot h_p^2. \qquad (3.4)
\end{aligned}$$

If $p = 12k+7$, and the congruent equation $u^3 \equiv -4 \bmod p$ has three solutions u_1, u_2 and u_3, then from (3.1), Lemma 2.4, the definition of $C(m,n,3,1;p)$ and (d) of Lemma 2.3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} C^3(m,n,3,1;p)\cdot S(m\overline{n},p)\\
&= -\frac{p\cdot\pi^{-2}}{p-1}\,|L(1,\chi_2)|^2\cdot\left(1+\sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\right)\cdot p^2(p-1)\\
&= -\left(1+\sum_{i=1}^{3}\left(\frac{1+u_i}{p}\right)\right)\cdot p^2\cdot h_p^2\\
&= \begin{cases} -4\cdot p^2\cdot h_p^2, & \text{if all } \left(\frac{1+u_i}{p}\right)=1,\ \ i=1,2,3,\\ 0, & \text{otherwise.}\end{cases} \qquad (3.5)
\end{aligned}$$

Now our conclusion follows from (3.2)–(3.5). This completes the proof of our theorem.

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