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Hochschild Cohomology Rings of Temperley-Lieb Algebras*

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Abstract The authors first construct an explicit minimal projective bimodule resolution $(\mathbb{P}, \delta)$ of the Temperley-Lieb algebra A, and then apply it to calculate the Hochschild cohomology groups and the cup product of the Hochschild cohomology ring of A based on a comultiplicative map $\Delta : \mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P}$. As a consequence, the authors determine the multiplicative structure of Hochschild cohomology rings of both Temperley-Lieb algebras and representation-finite q-Schur algebras under the cup product by giving an explicit presentation by generators and relations.

Keywords Hochschild cohomology, Cup product, Temperley-Lieb algebra, q-Schur algebra

2000 MR Subject Classification 16E40, 16G10

1 Introduction

The Temperley-Lieb algebras were first introduced in 1971 in [24] to study the single bond transfer matrices for the ice model and the Potts model. Later they were independently found by Jones when he characterized the algebras arising from the tower construction of semisimple algebras in the study of subfactors (see [18]). These algebras have played a central role in the discovery by Jones of his new polynomial invariant of knots and links (see [19]), and in the subsequent developments over the past four decades relating to knot theory, topological quantum field theory, and statistical physics (see [20]). Their relationship with knot theory comes from their role in the definition of the Jones polynomial. The theory of quantum invariants of links nowadays involves many research fields. Thus, many important kinds of algebras related to the invariants of braids or links, such as Birman Wenzl algebras (see [5]), Hecke algebras and Brauer algebras, have been of great interest in mathematics and physics. They are all deformations of certain group algebras or other well-known algebras.

Let K be a field and m a positive integer. Recall that the Temperley-Lieb algebra $A_m(\delta)$ for $\delta \in K$ is defined to be a K-algebra with identity generated by $t_1, t_2, \cdots, t_{m-1}$ subject to

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the relations:

$$\begin{aligned} &t_it_jt_i = t_i, && \text{if } |j-i| = 1,\\ &t_it_j = t_jt_i, && \text{if } |j-i| > 1,\\ &t_i^2 = \delta t_i && \text{for } 1 \leqslant i \leqslant m-1. \end{aligned}$$

It was shown in [25] that a block of a non-semisimple Temperley-Lieb algebra is Morita equivalent to the quotient algebra $A = A_m = KQ/I$ given by the quiver

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left, "\beta_1"] & \underset{2}{\bullet} \arrow[l, shift left, "\alpha_1"] \arrow[r, shift left, "\beta_2"] & \underset{3}{\bullet} \arrow[l, shift left, "\alpha_2"] & \cdots & \underset{m-1}{\bullet} \arrow[r, shift left, "\beta_{m-1}"] & \underset{m}{\bullet} \arrow[l, shift left, "\alpha_{m-1}"] \end{tikzcd}$$

and the relations $I = \langle \alpha_{i+1}\alpha_i, \beta_i\beta_{i+1}, \beta_{i+1}\alpha_{i+1} - \alpha_i\beta_i, \alpha_{m-1}\beta_{m-1} \mid i = 1, 2, \cdots, m-2\rangle$. As was shown in [26], the non-trivial block of the representation-finite q-Schur algebras $S_q(n, r)$ with $n \geqslant r$ is also Morita equivalent to an algebra of the form A_m.

Hochschild cohomology $\mathrm{HH}^*(A, M)$ of A with coefficients in M was introduced in [16] in order to classify, up to equivalence, all extensions of A with Kernel M, which is one-to-one correspondence with $\mathrm{HH}^2(A, M)$. Many other applications of this cohomology have been discovered since then (see [15]). For example, separable algebras are characterized by the fact that their Hochschild (cohomology) dimension is zero, that is, $\mathrm{HH}^1(A, M) = 0$ for every bimodule M (see [16]); the deformation theory of an algebra is controlled by its Hochschild cohomology as a graded Lie algebra under the Gerstenhaber bracket (see [11]); Hochschild cohomology is also closely related to simple connectedness, formal smoothness (or quasi-freeness in literature) (see [1, 22]) and so on. It is well known that $\mathrm{HH}^*(A) = \bigoplus_{i=0}^{\infty} \mathrm{HH}^i(A)$ is endowed with the so-called Gerstenhaber algebra structure under the cup product

$$\smile : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m}(A)$$

and the Gerstenhaber Lie bracket

$$[-, -] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m-1}(A).$$

However, for most finite dimensional algebras, little is known about the Hochschild cohomology groups and even less about the Hochschild cohomology rings (see [2, 4, 7–10, 13, 27]).

Since Hochschild cohomology is invariant under Morita equivalence (see [15]), to describe the Hochschild cohomology rings of both the Temperley-Lieb algebras and the representation-finite q-Schur algebras $S_q(n, r)$ for $n \geqslant r$, it is sufficient to deal with the basic algebra A defined as above. The K-dimensions of Hochschild cohomology groups of A were obtained in [17] by a long exact sequence of cohomology groups relating to a homological epimorphism of K-algebras, but there K-bases were not given. We begin the paper by giving a minimal projective resolution of A as an A^e-module, and then apply it to obtain K-bases of the cohomology groups in terms of parallel paths. In Section 4 we give an explicit description of the "comultiplicative" map $\Delta : \mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P}$ to determine the cup product of $\mathrm{HH}^*(A)$ using the composition

$$\mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P} \to A \otimes_A A \to A.$$

As a consequence, we will give an explicit presentation of the multiplicative structure of $\mathrm{HH}^*(A)$ under the cup product by generators and relations.

2 The Minimal Projective Bimodule Resolution

Throughout the paper we always assume that A is the algebra defined as in the introduction. We denote by e_i the trivial path at the vertex i. Given a path p in Q, we denote by $o(p)$ and $t(p)$ the origin and the terminus of p respectively.

We will employ the strategy due to Green et al in [12, 14] to construct a minimal projective A^e-module resolution of A. Set $g^0_{0,i} = e_i,\ i = 1, 2, \cdots, m$. For $1 \leqslant n \leqslant 2m-2$, one defines the following elements recursively:

$$g^n_{r,i} = g^{n-1}_{r,i}\beta_{i+n-2r-1} + (-1)^n g^{n-1}_{r-1,i}\alpha_{i+n-2r}. \tag{2.1}$$

Noticing that $\mathrm{gl.dim}\, A = 2m-2$, one takes $g^n_{r,i} = 0$ if $n > 2m-2$. Note that $g^n_{r,i}$ is just an algebraic sum of paths of length n with the original i and containing exactly r arrows of type α. Denote by g^n the set of elements of the form $g^n_{r,i}$. Then,

$$\begin{aligned} &g^0 = \{e_1, e_2, \cdots, e_m\},\\ &g^1 = \{-\alpha_1, -\alpha_2, \cdots, -\alpha_{m-1}, \beta_1, \beta_2, \cdots, \beta_{m-1}\},\\ &g^2 = \{-\alpha_{i+1}\alpha_i, \beta_i\beta_{i+1}, \beta_{i+1}\alpha_{i+1} - \alpha_i\beta_i \mid 1 \le i \le m-2\} \cup \{-\alpha_{m-1}\beta_{m-1}\}. \end{aligned}$$

For $3 \leqslant n \leqslant 2m-2$, when $n = 2k$,

$$g^n = \{g^n_{r,i} \mid 0 \leqslant r \leqslant k-1, r+1 \leqslant i \leqslant m-(n-2r)\} \cup \{g^n_{r,i} \mid k \leqslant r \leqslant n, r+1 \leqslant i \leqslant m\};$$

when $n = 2k+1$,

$$g^n = \{g^n_{r,i} \mid 0 \leqslant r \leqslant k, r+1 \leqslant i \leqslant m-(n-2r)\} \cup \{g^n_{r,i} \mid k+1 \leqslant r \leqslant n, r+1 \leqslant i \leqslant m\}.$$

In particular, we have

$$|g^n| = \begin{cases} (2k+1)m - 3k^2 - 2k, & \text{if } n = 2k,\\ 2(k+1)m - 3k^2 - 5k - 2, & \text{if } n = 2k+1. \end{cases}$$

In order to define the differential δ, we need the following lemma so that we have two different ways of expressing the elements of the set g^n in terms of the elements of the set g^{n-1}. The proof of Lemma 2.1 is straightforward and therefore omitted.

Lemma 2.1 *For $n \geqslant 1$, we have*

$$g^n_{r,i} = g^{n-1}_{r,i}\beta_{i+n-2r-1} + (-1)^n g^{n-1}_{r-1,i}\alpha_{i+n-2r} = (-1)^r\beta_i g^{n-1}_{r,i+1} + (-1)^r \alpha_{i-1} g^{n-1}_{r-1,i-1}.$$

Denote $\otimes := \otimes_K$. Define

$$P_n = \bigoplus_{g^n_{r,i} \in g^n} Ao(g^n_{r,i}) \otimes t(g^n_{r,i})A,$$

and for $1 \leqslant n \leqslant 2m-2,\ \delta_n : P_n \to P_{n-1}$ is given by

$$o(g^n_{r,i}) \otimes t(g^n_{r,i}) \mapsto ((-1)^n e_i \otimes \beta_{i+n-2r-1} + e_i \otimes \alpha_{i+n-2r}) + ((-1)^r \beta_i \otimes e_{i+n-2r} + (-1)^r \alpha_{i-1} \otimes e_{i+n-2r}).$$

The following theorem follows immediately from Lemma 2.1 and [12, Theorem 2.1].

Theorem 2.1 *With the above notation, the complex*

$$(\mathbb{P},\delta):\quad 0\to P_{2m-2}\overset{\delta_{2m-2}}{\to}\cdots\overset{\delta_{n+2}}{\to}P_{n+1}\overset{\delta_{n+1}}{\to}P_n\overset{\delta_n}{\to}\cdots\overset{\delta_2}{\to}P_1\overset{\delta_1}{\to}P_0\overset{\delta_0}{\to}A\to 0$$

is a minimal projective A^e-resolution of A, where $\delta_0: P_0\to A$ is the multiplication map.

Proof Let $X=g^1$ and $R=g^2$ be the set of generators of I as above. Since A is a Koszul algebra, by [3, Sect. 9], it suffices to show that g^n is a K-basis of the K-vector space $K_n:=\bigcap\limits_{p+q=n-2}X^pRX^q$ for $n\geq 2$.

We first show that all the $g^n_{r,i}$ belong to K_n inductively. It is trivial for $n=2$. Assume that the assertion holds for $n-1$ and we prove it for n. By the induction hypothesis and the formula (2.1), $g^n_{r,i}\in RX^{n-2}\cap K_{n-1}X$. The induction hypothesis and Lemma 2.1 show that $g^n_{r,i}\in X^{n-2}R\cap XK_{n-1}$. The assertion follows from the fact that $K_n=RX^{n-2}\cap X^{n-2}R\cap XK_{n-1}\cap K_{n-1}X$.

Next, g^n is K-linearly independent since they have distinct supports. Also, the quadratic duality $A^!=kQ/I^\perp$ of A is isomorphic to the Yoneda algebra $E(A)$ of A, where $I^\perp$ is the ideal of KQ generated by $R^\perp=\{\beta_1\alpha_1,\beta_{i+1}\alpha_{i+1}+\alpha_i\beta_i\mid i=1,2,\cdots,m-2\}$. So the Betti number of the minimal projective resolution of A over A^e is

$$\dim K_n=\begin{cases}(2k+1)m-3k^2-2k, & \text{if } n=2k,\\ 2(k+1)m-3k^2-5k-2, & \text{if } n=2k+1.\end{cases}$$

Hence g^n is a K-basis of K_n. Then the result follows.

3 Hochschild Cohomology Groups

This section is devoted to finding K-bases of the Hochschild cohomology groups of A based on the minimal projective A^e-resolution constructed in the previous section.

Applying $\mathrm{Hom}_{A^e}(-,A)$ to the minimal resolution $(\mathbb{P},\delta)$, we have the complex

$$0\to \mathrm{Hom}_{A^e}(P_0,A)\overset{\delta_1^*}{\to}\mathrm{Hom}_{A^e}(P_1,A)\overset{\delta_2^*}{\to}\cdots\overset{\delta_{2m-2}^*}{\to}\mathrm{Hom}_{A^e}(P_{2m-2},A)\to 0.$$

Let $B=\{e_1,e_2,\cdots,e_m,\beta_1,\beta_2,\cdots,\beta_{m-1},\alpha_1,\alpha_2,\cdots,\alpha_{m-1},\beta_1\alpha_1,\beta_2\alpha_2,\cdots,\beta_{m-1}\alpha_{m-1}\}$ be a K-basis of algebra A, and $K(B//g^n)$ denote the vector space with a K-basis $B//g^n=\{(b,g^n_{r,i})\mid o(b)=o(g^n_{r,i}),t(b)=t(g^n_{r,i})\}$. We say that two paths α and β are parallel if $o(\alpha)=o(\beta)$ and $t(\alpha)=t(\beta)$.

The following lemma is immediate, see [6, 21] for details.

Lemma 3.1 $\mathrm{Hom}_{A^e}(P_n,A)\cong K(B//g^n)$ *as vector spaces.*

Proof It is easy to see that

$$\begin{aligned}\mathrm{Hom}_{A^e}(P_n,A)&\cong\mathrm{Hom}_{A^e}\Big(\bigoplus_{\gamma\in g^n}(o(\gamma)\otimes t(\gamma))A^e,A\Big)\cong\bigoplus_{\gamma\in g^n}Ao(\gamma)\otimes t(\gamma)\\ &=\bigoplus_{\gamma\in g^n}o(\gamma)At(\gamma)\cong K(B//g^n)\end{aligned}$$

as vector spaces.

We fix an isomorphism $\phi : K(B//g^n) \to \mathrm{Hom}_{A^e}(P_n, A)$ sending $(b,\gamma) \in (B//g^n)$ to the A^e-homomorphism $f_{(b,\gamma)} \in \mathrm{Hom}_{A^e}(P_n, A)$, which assigns $o(\gamma) \otimes t(\gamma)$ to b, and zero otherwise. The cochain complex above changes into

$$0 \to K(B//g^0) \overset{\delta_1^*}{\to} K(B//g^1) \overset{\delta_2^*}{\to} \cdots \overset{\delta_{2m-3}^*}{\to} K(B//g^{2m-3}) \overset{\delta_{2m-2}^*}{\to} K(B//g^{2m-2}) \to 0, \qquad (3.1)$$

where we still denote by δ_i^* the induced linear maps.

Lemma 3.2 $\mathrm{Ker}\,\delta_1^*$ *has a K-basis* $\Big\{\sum\limits_{i=1}^{m}(e_i,e_i), (\beta_1\alpha_1, e_1), (\beta_2\alpha_2, e_2), \cdots, (\beta_{m-1}\alpha_{m-1}, e_{m-1})\Big\}$ *and* $\dim_K \mathrm{Im}\,\delta_1^* = m-1$.

Proof Under the K-bases,

$$B//g^0 = \{(e_1,e_1),(e_2,e_2),\cdots,(e_m,e_m),(\beta_1\alpha_1,e_1),(\beta_2\alpha_2,e_2),\cdots,(\beta_{m-1}\alpha_{m-1},e_{m-1})\}$$

and

$$B//g^1 = \{(\beta_1,\beta_1),(\beta_2,\beta_2),\cdots,(\beta_{m-1},\beta_{m-1})(\alpha_1,-\alpha_1),(\alpha_2,-\alpha_2),\cdots,(\alpha_{m-1},-\alpha_{m-1}\}.$$

It is not difficult to calculate the matrix of the linear map δ_1^* which is

$$A_1 = \begin{pmatrix}
-1 & 1 & & & & & & & & \\
 & -1 & 1 & & & & & & & \\
 & & -1 & \ddots & & & & & & \\
 & & & \ddots & 1 & & & & & \\
 & & & & -1 & 1 & & & & \\
 & & & & & -1 & 1 & 0 & \cdots & 0 \\
-1 & 1 & & & & & & & & \\
 & -1 & 1 & & & & & & & \\
 & & -1 & \ddots & & & & & & \\
 & & & \ddots & 1 & & & & & \\
 & & & & -1 & 1 & & & & \\
 & & & & & -1 & 1 & 0 & \cdots & 0
\end{pmatrix}_{(2m-2)\times(2m-1)}$$

with the right $m-1$ columns zero. It is clear that $\mathrm{rank}\,A_1 = m-1$, and hence $\dim_K \mathrm{Im}\,\delta_1^* = \mathrm{rank}\,A_1 = m-1$, and $\dim_K \mathrm{Ker}\,\delta_1^* = |B//g^0| - \mathrm{rank}\,A_1 = (m+m-1)-(m-1) = m$. One can easily check that

$$\delta_1^*((\beta_1\alpha_1,e_1)) = 0, \quad \delta_1^*((\beta_2\alpha_2,e_2)) = 0, \ \cdots,$$
$$\delta_1^*((\beta_{m-1}\alpha_{m-1},e_{m-1})) = 0, \quad \delta_1^*\Big(\sum_{i=1}^{m}(e_i,e_i)\Big) = 0.$$

Since $\Big\{\sum\limits_{i=1}^{m}(e_i,e_i), (\beta_1\alpha_1,e_1), (\beta_2\alpha_2,e_2), \cdots, (\beta_{m-1}\alpha_{m-1},e_{m-1})\Big\}$ is K-linear independent and has m elements, it is a K-basis of $\mathrm{Ker}\,\delta_1^*$.

Noticing that $\mathrm{HH}^n(A) = \mathrm{Ker}\,\delta_{n+1}^*/\mathrm{Im}\,\delta_n^*$, we next find out a K-basis of the kernel space $\mathrm{Ker}\,\delta_{n+1}^*$ and the image space $\mathrm{Im}\,\delta_n^*$ for $n > 0$, respectively. They will be discussed in four cases.

Case I: $n=4t$, $t\neq 0$. Set

$$\begin{aligned}
U&=\{(\beta_{2t+j}\alpha_{2t+j},g^{n}_{2t,2t+j})\mid j=1,2,\cdots,m-2t-1\},\\
V&=\{(\alpha_{2t+j},g^{n-1}_{2t,2t+j+1})-(\beta_{2t+j},g^{n-1}_{2t-1,2t+j})\mid j=0,1,\cdots,m-2t-1\}\\
&\qquad\cup\Big\{\sum_{i=2t+1}^{m}(-1)^{i}(\alpha_{i-1},g^{n-1}_{2t,i})\Big\}.
\end{aligned}$$

Case II: $n=4t+1$, $t\neq 0$. Set

$$\begin{aligned}
U&=\{(\alpha_{2t+j},g^{n}_{2t+1,2t+j+1})+(\beta_{2t+j},g^{n}_{2t,2t+j})\mid j=1,2,\cdots,m-2t-1\},\\
V&=\{(\beta_{2t+j}\alpha_{2t+j},g^{n-1}_{2t,2t+j})\mid j=1,2,\cdots,m-2t-1\}\cup\Big\{\sum_{i=2t+1}^{m}(e_{i},g^{n-1}_{2t,i})\Big\}.
\end{aligned}$$

Case III: $n=4t+2$. Set

$$\begin{aligned}
U&=\{(\beta_{2t+j}\alpha_{2t+j},g^{n}_{2t+1,2t+j})\mid j=2,3,\cdots,m-2t-1\},\\
V&=\{(\alpha_{2t+j},g^{n-1}_{2t+1,2t+j+1})+(\beta_{2t+j},g^{n-1}_{2t,2t+j})\mid j=1,2,\cdots,m-2t-1\}\\
&\qquad\cup\Big\{\sum_{i=2t+2}^{m}(\alpha_{i-1},g^{n-1}_{2t+1,i})\Big\}.
\end{aligned}$$

Case IV: $n=4t+3$. Set

$$\begin{aligned}
U&=\{(\alpha_{2t+j},g^{n}_{2t+2,2t+j+1})-(\beta_{2t+j},g^{n}_{2t+1,2t+j})\mid j=2,3,\cdots,m-2t-1\},\\
V&=\{(\beta_{2t+j}\alpha_{2t+j},g^{n-1}_{2t+1,2t+j})\mid j=2,3,\cdots,m-2t-1\}\\
&\qquad\cup\Big\{\sum_{i=2t+2}^{m}(-1)^{i}(e_{i},g^{n-1}_{2t+1,i})\Big\}.
\end{aligned}$$

Lemma 3.3 *U forms a K-basis of* $\operatorname{Im}\delta_n^{*}$ *and V forms a K-basis of* $\operatorname{Ker}\delta_n^{*}$.

Proof We only prove the case I, and the other cases are similar and their proofs are omitted here. It is not difficult to calculate the matrix of the linear map δ_n^{*} under the K-bases $B//g^{n-1}=\{(\alpha_{2t},g^{n-1}_{2t,2t+1}),(\alpha_{2t+1},g^{n-1}_{2t,2t+2}),\cdots,(\alpha_{m-1},g^{n-1}_{2t,m}),(\beta_{2t},g^{n-1}_{2t-1,2t}),(\beta_{2t+1},g^{n-1}_{2t-1,2t+1}),\cdots,(\beta_{m-1},g^{n-1}_{2t-1,m-1})\}$ and $B//g^{n-1}=\{(e_{2t+1},g^{n}_{2t,2t+1}),(e_{2t+2},g^{n}_{2t,2t+2}),\cdots,(e_{m},g^{n}_{2t,m}),(\beta_{2t+1}\alpha_{2t+1},g^{n}_{2t,2t+1}),(\beta_{2t+2}\alpha_{2t+2},g^{n}_{2t,2t+2}),\cdots,(\beta_{m-1}\alpha_{m-1},g^{n}_{2t,m-1})\}$. The matrix A_n is

$$A_n=\left(\begin{array}{cccccc|cccccc}
0 & & & & & & 0 & & & & & \\
 & & \cdots & & & & & & \cdots & & & \\
0 & & & & & & 0 & & & & & \\
1 & 1 & & & & & 1 & 1 & & & & \\
 & 1 & 1 & & & & & 1 & 1 & & & \\
 & & \ddots & \ddots & & & & & \ddots & \ddots & & \\
 & & & 1 & 1 & & & & & 1 & 1 & \\
 & & & & 1 & 1 & & & & & 1 & 1
\end{array}\right)_{(2m-4t-1)\times(2m-4t)}$$

whose first $m-2t$ rows are zero. The rank of A_n is $m-2t-1$ and hence $\dim_K\operatorname{Im}\delta_n^{*}=\operatorname{rank}A_n=m-2t-1$ and $\dim_K\operatorname{Ker}\delta_n^{*}=|B//g^{n-1}|-\dim_K\operatorname{Im}\delta_n^{*}=2(m-2t)-(m-2t-1)=m-2t+1$.

It is easy to see that

$$\begin{aligned}
(\beta_{2t+1}\alpha_{2t+1}, g^n_{2t,2t+1}) &= \delta^*_n((\alpha_{2t}, g^{n-1}_{2t,2t+1})),\\
(\beta_{2t+2}\alpha_{2t+2}, g^n_{2t,2t+2}) &= \delta^*_n((\alpha_{2t+1}, g^{n-1}_{2t,2t+2}) - (\alpha_{2t}, g^{n-1}_{2t,2t+1})),\\
(\beta_{2t+3}\alpha_{2t+3}, g^n_{2t,2t+3}) &= \delta^*_n((\alpha_{2t+1}, g^{n-1}_{2t,2t+3}) - (\alpha_{2t+1}, g^{n-1}_{2t,2t+2}) + (\alpha_{2t}, g^{n-1}_{2t,2t+1})),\\
&\cdots\\
(\beta_{m-1}\alpha_{m-1}, g^n_{2t,m-1}) &= \delta^*_n((\alpha_{m-2}, g^{n-1}_{2t,m-1}) - (\alpha_{m-3}, g^{n-1}_{2t,m-2}) + \cdots + (-1)^{m-2t}(\alpha_{2t}, g^{n-1}_{2t,2t+1})).
\end{aligned}$$

Since the set $\{(\beta_{2t+1}\alpha_{2t+1}, g^n_{2t,2t+1}), (\beta_{2t+2}\alpha_{2t+2}, g^n_{2t,2t+2}), \cdots, (\beta_{m-1}\alpha_{m-1}, g^n_{2t,m-1})\} \subset (B//g^n)$ is K-linear independent and has $m-2t-1$ elements, it is a K-basis of $\operatorname{Im}\delta^*_n$.

Clearly,

$$\begin{aligned}
&\delta^*_n((\alpha_{2t}, g^{n-1}_{2t,2t+1}) - (\beta_{2t}, g^{n-1}_{2t-1,2t})) = 0,\\
&\delta^*_n((\alpha_{2t+1}, g^{n-1}_{2t,2t+2}) - (\beta_{2t+1}, g^{n-1}_{2t-1,2t+1})) = 0,\\
&\qquad\vdots\\
&\delta^*_n((\alpha_{m-1}, g^{n-1}_{2t,m}) - (\beta_{m-1}, g^{n-1}_{2t-1,m-1})) = 0,\\
&\delta^*_n\Big(\sum_{i=2t+1}^{m}(-1)^i(\alpha_{i-1}, g^{n-1}_{2t,i})\Big) = 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\{(\alpha_{2t+j}, g^{n-1}_{2t,2t+j+1}) - (\beta_{2t+j}, g^{n-1}_{2t-1,2t+j}) \mid j = 0, 1, \cdots, m-2t-1\}\\
&\cup\Big\{\sum_{i=2t+1}^{m}(-1)^i(\alpha_{i-1}, g^{n-1}_{2t,i})\Big\} \subseteq \operatorname{Ker}\delta^*_n
\end{aligned}$$

which is obviously K-linear independent and has $m-2t+1$ elements, so it is a K-basis of $\operatorname{Ker}\delta^*_n$. The proof is finished.

Now it is a position to give a K-basis of the Hochschild cohomological space $\mathrm{HH}^n(A)$.

Theorem 3.1 *Let $A = KQ/I$ be the K-algebra defined as in the introduction. Then we have*

(1) $\dim_K \mathrm{HH}^i(A) = \begin{cases} m, & i = 0,\\ 1, & 1 \leqslant i \leqslant 2m-2,\\ 0, & i > 2m-2.\end{cases}$

(2) $\mathrm{HH}^0(A)$ *has a basis* $\sum_{i=1}^{m}(e_i, e_i), (\beta_1\alpha_1, e_1), (\beta_2\alpha_2, e_2), \cdots, (\beta_{m-1}\alpha_{m-1}, e_{m-1})$; *and*

$$\begin{aligned}
&\mathrm{HH}^{4t}(A) \textit{ has a basis } \sum_{i=2t+1}^{m}(e_i, g^{4t}_{2t,i}) \textit{ for } t \neq 0,\\
&\mathrm{HH}^{4t+1}(A) \textit{ has a basis } \sum_{i=2t+2}^{m}(\alpha_{i-1}, g^{4t+1}_{2t+1,i}),\\
&\mathrm{HH}^{4t+2}(A) \textit{ has a basis } \sum_{i=2t+2}^{m}(-1)^i(e_i, g^{4t+2}_{2t+1,i}),\\
&\mathrm{HH}^{4t+3}(A) \textit{ has a basis } \sum_{i=2t+3}^{m}(-1)^i(\alpha_{i-1}, g^{4t+3}_{2t+2,i}).
\end{aligned}$$

Here these basis elements represent the representatives of the corresponding elements in $\mathrm{HH}^n(A)$.

Proof It follows from Lemmas 3.2–3.3 and the fact that $\mathrm{HH}^i(A) = \mathrm{Ker}\,\delta_{i+1}^*/\mathrm{Im}\,\delta_i^*$ directly.

Remark 3.1 The dimension of the Hochschild cohomological space $\mathrm{HH}^n(A)$ was obtained by de la Pe$\widetilde{n}$a and Xi in [17] in a different way.

4 The Cup Product

In this section we will describe the multiplicative structure of the Hochschild cohomology ring of A in terms of parallel paths. In [23] it was shown that for any projective A^e-resolution $\mathbb{P}$ of a finite dimensional algebra A, there exists a unique (up to homotopy) chain map $\Delta : \mathbb{P} \to \mathbb{P}\otimes_A\mathbb{P}$ lifting the identity. $\mathbb{P}$ gives rise to a "cup product" of two elements η in $\mathrm{HH}^m(A)$ and θ in $\mathrm{HH}^n(A)$ by using the composition

$$\mathbb{P} \xrightarrow{\Delta} \mathbb{P}\otimes_A \mathbb{P} \xrightarrow{\eta\otimes\theta} A\otimes_A A \xrightarrow{\nu} A$$

coinciding with the ordinary cup product and being independent of the projective resolution $\mathbb{P}$ of A and the chain map Δ.

The following lemma provides an explicit description of the so-called "comultiplicative structure" of the generators of each P_n in $(\mathbb{P}, \delta)$, which is key to defining a chain map Δ.

Lemma 4.1 *For any given $p = 0, 1, \cdots, n$, we have*

$$g_{r,i}^n = \sum_{s=0}^{r}(-1)^{(r-s)(n+1-p+r-s)} g_{s,i}^{n-p} g_{r-s,i+n-p-2s}^p.$$

Proof We use induction on p. There is nothing to prove provided that $p = 0$. If $p = 1$, then $g_{r,i}^n = g_{r,i}^{n-1}\beta_{i+n-2r-1} + (-1)^n g_{r-1,i}^{n-1}\alpha_{i+n-2r}$, which is just the defining formula of $g_{r,i}^n$.

Suppose now that the formula holds true for $p = k$. We consider the case $p = k+1$. By the induction hypothesis and the formula (2.1), we have

$$\begin{aligned}
g_{r,i}^n &= \sum_{s=0}^{r}(-1)^{(r-s)(n+1-k+r-s)} g_{s,i}^{n-k} g_{r-s,i+n-k-2s}^k \\
&= \sum_{s=0}^{r}(-1)^{(r-s)(n+1-k+r-s)} \big[g_{s,i}^{n-k-1}\beta_{i+n-k-2s-1} + (-1)^{n-k} g_{s-1,i}^{n-k-1}\alpha_{i+n-k-2s}\big] g_{r-s,i+n-k-2s}^k \\
&= \sum_{s=0}^{r}(-1)^{(r-s)(n+1-k+r-s)} g_{s,i}^{n-k-1}\beta_{i+n-k-2s-1} g_{r-s,i+n-k-2s}^k \\
&\quad + \sum_{s=1}^{r}(-1)^{(r-s)(n+1-k+r-s)}(-1)^{n-k} g_{s-1,i}^{n-k-1}\alpha_{i+n-k-2s} g_{r-s,i+n-k-2s}^k \\
&= \sum_{s=0}^{r}(-1)^{(r-s)(n-k+r-s)} g_{s,i}^{n-k-1} g_{r-s,i+n-k-2s-1}^{k+1}.
\end{aligned}$$

The result follows.

The lemma allows us to give the definition of the map $\Delta : \mathbb{P} \to \mathbb{P}\otimes_A\mathbb{P}$. First we recall the tensor product chain complex $(\mathbb{P}\otimes_A\mathbb{P}, D)$ of $(\mathbb{P}, \delta)$. Its n-th object is $(\mathbb{P}\otimes_A\mathbb{P})_n = \bigoplus\limits_{i+j=n} P_i\otimes_A P_j$ and the differential $D_n : (\mathbb{P}\otimes_A\mathbb{P})_n \to (\mathbb{P}\otimes_A\mathbb{P})_{n-1}$ is given by $D_n = \sum\limits_{i=0}^{n-1}(\delta_{i+1}\otimes 1 + (-1)^i 1\otimes\delta_{n-i})$.

By abuse of notations, we denote by $o(g^n_{r,i})$ (resp. $t(g^n_{r,i})$) the corresponding idempotent $e_{o(g^n_{r,i})}$ (resp. $e_{t(g^n_{r,i})}$), and by $\varepsilon^n_{r,i}$ the generator $o(g^n_{r,i}) \otimes t(g^n_{r,i})$ of P_n.

Definition 4.1 *The A-A-bimodule map $\Delta = (\Delta_n) : \mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P}$ is defined by*

$$\Delta_n(\varepsilon^n_{r,i}) = \sum_{p=0}^{n} \sum_{s=0}^{r} (-1)^{(r-s)(n+1-p+r-s)} \varepsilon^{n-p}_{s,i} \otimes_A \varepsilon^p_{r-s,i+n-p-2s}$$

for $0 \leq n \leq 2m-2$ and the other Δ_n are all zero.

Lemma 4.2 *The map $\Delta : (\mathbb{P}, \delta) \to (\mathbb{P} \otimes_A \mathbb{P}, D)$ defined as above is a chain map.*

Proof To prove the result, it suffices to show that the diagram

$$\begin{array}{ccc} P_n & \xrightarrow{\delta_n} & P_{n-1} \\ \Delta_n \downarrow & & \downarrow \Delta_{n-1} \\ (P \otimes_A P)_n & \xrightarrow{D_n} & (P \otimes_A P)_{n-1} \end{array}$$

is commutative for $n \geqslant 1$.

Let $(\Delta_{n-1} \circ \delta_n(\varepsilon^n_{r,i}))_{(t,n-1-t)}$ denote the element of $P^t \otimes_A P^{n-1-t}$. By the definition of Δ and $\delta_n(\varepsilon^n_{r,i}) = (-1)^n e_i \otimes \beta_{i+n-2r-1} + e_i \otimes \alpha_{i+n-2r} + (-1)^r \beta_i \otimes e_{i+n-2r} + (-1)^r \alpha_{i-1} \otimes e_{i+n-2r}$, we have

$$\begin{aligned}(\Delta_{n-1} \circ \delta_n(\varepsilon^n_{r,i}))_{(t,n-1-t)} &= (-1)^n \sum_{s=0}^{r} (-1)^{(r-s)(t+1+r-s)} \varepsilon^t_{s,i} \otimes_A \varepsilon^{n-1-t}_{r-s,i+t-2s} \beta_{i+n-2r-1} \\ &+ \sum_{s=0}^{r} (-1)^{(r-s-1)(t+r-s)} \varepsilon^t_{s,i} \otimes_A \varepsilon^{n-1-t}_{r-1-s,i+t-2s} \alpha_{i+n-2r} \\ &+ (-1)^r \sum_{s=0}^{r} (-1)^{(r-s)(t+1+r-s)} \beta_i \varepsilon^t_{s,i+1} \otimes_A \varepsilon^{n-1-t}_{r-s,i+1+t-2s} \\ &+ (-1)^r \sum_{s=0}^{r} (-1)^{(r-s-1)(t+r-s)} \alpha_{i-1} \varepsilon^t_{s,i-1} \otimes_A \varepsilon^{n-1-t}_{r-s-1,i-1+t-2s}.\end{aligned}$$

On the other hand, noting that

$$(\Delta_n(\varepsilon^n_{r,i}))_{(t,n-t)} = \sum_{s=0}^{r} (-1)^{(r-s)(t+1+r-s)} \varepsilon^t_{s,i} \otimes_A \varepsilon^{n-t}_{r-s,i+t-2s}$$

and

$$(\Delta_n(\varepsilon^n_{r,i}))_{(t+1,n-1-t)} = \sum_{s=0}^{r} (-1)^{(r-s)(t+2+r-s)} \varepsilon^{t+1}_{s,i} \otimes_A \varepsilon^{n-t-1}_{r-s,i+t-2s+1},$$

we can directly check that

$$[(-1)^t \otimes_A \delta_{n-t}]((\Delta_n(\varepsilon^n_{r,i}))_{(t,n-t)}) + [\delta_{t+1} \otimes_A 1]((\Delta_n(\varepsilon^n_{r,i}))_{(t+1,n-1-t)}) = (\Delta_{n-1} \circ \delta_n(\varepsilon^n_{r,i}))_{(t,n-1-t)}$$

and thus $\Delta_{n-1}\delta_n = D_n \Delta_n$ as desired. The proof is finished.

In order to give an explicit description of the Hochschild cohomology ring of A, we first give the cup product on the level of cochains, which is essentially juxtaposition of parallel paths up to sign.

Lemma 4.3 *Let $A = KQ/I$ be the K-algebra defined as in the introduction. Then*

$$(b_1, g^{n_1}_{r_1,i}) \smile (b_2, g^{n_2}_{r_2,j}) = \begin{cases} (-1)^{r_2(n_1+1+r_2)}(b_1b_2, g^{n_1+n_2}_{r_1+r_2,i}), & \text{if } j = i + n_1 - 2r_1, \\ 0, & \text{otherwise.} \end{cases}$$

Here $(b_1b_2, g^{n_1+n_2}_{r_1+r_2,i})$ is viewed as 0 whenever $b_1b_2 \in I$.

Proof Let $\eta_{n_1} = (b_1, g^{n_1}_{r_1,i})$ and $\eta_{n_2} = (b_2, g^{n_2}_{r_2,j})$. Using the composition

$$\mathbb{P} \xrightarrow{\Delta} \mathbb{P} \otimes_A \mathbb{P} \xrightarrow{\eta\otimes\theta} A \otimes_A A \xrightarrow{\nu} A,$$

we have

$$\begin{aligned} &\eta_{n_1} \smile \eta_{n_2}(\varepsilon^{n_1+n_2}_{r,k}) \\ &= \nu(\eta_{n_1} \otimes \eta_{n_2})\Delta_{n_1+n_2}(\varepsilon^{n_1+n_2}_{r,k}) \\ &= \nu(\eta_{n_1} \otimes \eta_{n_2}) \sum_{p=0}^{n_1+n_2} \sum_{s=0}^{r} (-1)^{(r-s)(n_1+n_2+1-p+r-s)} \varepsilon^{n_1+n_2-p}_{s,k} \otimes_A \varepsilon^{p}_{r-s,k+n_1+n_2-p-2s} \\ &= \sum_{s=0}^{r} (-1)^{(r-s)(n_1+1+r-s)} \eta_{n_1}(\varepsilon^{n_1}_{s,k}) \cdot \eta_{n_2}(\varepsilon^{n_2}_{r-s,k+n_1-2s}). \end{aligned}$$

When $s \neq r_1$ or $i \neq k$, we have $\eta_{n_1}(\varepsilon^{n_1}_{s,k}) = 0$. And when $r - s \neq r_2$ or $j \neq k + n_1 - 2s$, we have $\eta_{n_2}(\varepsilon^{n_2}_{r-s,k+n_1-2s}) = 0$. Thus, only in the case of $s = r_1,\ i = k,\ r-s = r_2$ and $j = i+n_1-2r_1$ we have $\eta_{n_1} \smile \eta_{n_2}(\varepsilon^{n_1+n_2}_{r,k}) = (-1)^{r_2(n_1+1+r_2)})b_1b_2$. By the isomorphism of Lemma 3.1, it is easy to see that in the case of $j = i + n_1 - 2r_1$, we have $\eta_{n_1} \smile \eta_{n_2} = (-1)^{r_2(n_1+1+r_2)}(b_1b_2, g^{n_1+n_2}_{r_1+r_2,i})$ and otherwise is zero.

Theorem 4.1 *Let $A = KQ/I$ be the K-algebra defined as in the introduction.*

(1) $\sum\limits_{i=1}^{m}(e_i, e_i)$ *is the identity of* $\mathrm{HH}^*(A)$*, and for any* $\eta_j = (\beta_j\alpha_j, e_j) \in \mathrm{HH}^0(A)$*,* $\xi \in \mathrm{HH}^*(A)$*,* $\xi \notin K$*, we have* $\eta_j \smile \xi = \xi \smile \eta_j = 0$.

(2) *Let η_{n_1} and η_{n_2} be the unique basis elements of* $\mathrm{HH}^{n_1}(A)$ *and* $\mathrm{HH}^{n_2}(A)$ *with $n_1n_2 > 0$, respectively. We have*

$$\eta_{n_1} \smile \eta_{n_2} = \begin{cases} \eta_{n_1+n_2}, & \text{if } n_1n_2 = 2k \text{ and } n_1 + n_2 \leq 2m - 2, \\ 0, & \text{if } n_1n_2 = 2k + 1 \text{ or } n_1 + n_2 > 2m - 2. \end{cases}$$

Proof It follows from Lemma 4.3 directly.

Now we can give a description of the multiplication structure of the Hochschild cohomology ring of A by giving an explicit presentation by generators and relations. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{m-1}, \mathbf{y}, \mathbf{z}$ be the indeterminates of degree $0, 0, \cdots, 0, 1, 2$ respectively. Let $\Lambda = K[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{m-1}, \mathbf{y}, \mathbf{z}]/J$, where J is the two-sided ideal of the polynomial algebra $K[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{m-1}, \mathbf{y}, \mathbf{z}]$ generated by

$$\mathbf{x}_i\mathbf{x}_j = 0, \quad \mathbf{x}_i\mathbf{y} = 0, \quad \mathbf{x}_i\mathbf{z} = 0, \quad 1 \leq i, j \leq m-1, \quad \mathbf{y}^2 = 0, \quad \mathbf{z}^m = 0, \quad \mathbf{y}\mathbf{z}^{m-1} = 0.$$

Theorem 4.2 *Let $A = KQ/I$ be the K-algebra defined as in the introduction. Then* $\mathrm{HH}^*(A) \cong \Lambda$.

Proof We omit the symbol of the cup product $\smile$ of two elements of $\mathrm{HH}^*(A)$ for simplicity. Clearly $\sum\limits_{i=1}^{m}(e_i, e_i)$ is the identity of $\mathrm{HH}^*(A)$. Denote

$$x_1 = (\beta_1\alpha_1, e_1), \quad x_2 = (\beta_2\alpha_2, e_2), \quad \cdots, \quad x_{m-1} = (\beta_{m-1}\alpha_{m-1}, e_{m-1}),$$
$$y = \sum_{i=2}^{m}(\alpha_{i-1}, g^1_{1,i}), \quad z = \sum_{i=2}^{m}(-1)^i(e_i, g^2_{1,i}).$$

By Theorem 4.1, we have

$$\sum_{i=2t+1}^{m}(e_i, g^{4t}_{2t,i}) = z^{2t}, \quad t \neq 0, \quad \sum_{i=2t+2}^{m}(\alpha_{i-1}, g^{4t+1}_{2t+1,i}) = z^{2t}y,$$
$$\sum_{i=2t+2}^{m}(-1)^i(e_i, g^{4t+2}_{2t+1,i}) = z^{2t+1}, \quad \sum_{i=2t+3}^{m}(-1)^i(\alpha_{i-1}, g^{4t+3}_{2t+2,i}) = z^{2t+1}y.$$

Hence $\mathrm{HH}^*(A)$ can be generated by $x_1, x_2, \cdots, x_{m-1}, y, z$ over K. Also, by Theorem 4.1, it is easy to find that any two elements in $\mathrm{HH}^*(A)$ are commutative and the following relations hold true:

$$x_ix_j = 0, \quad x_iy = 0, \quad x_iz = 0, \quad 1 \le i, j \le m-1, \quad y^2 = 0, \quad z^m = 0, \quad yz^{m-1} = 0.$$

Then we construct an epimorphic algebra homomorphism

$$\varphi: K[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{m-1}, \mathbf{y}, \mathbf{z}] \to \mathrm{HH}^*(A)$$

sending $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{m-1}, \mathbf{y}, \mathbf{z}$ to $x_1, x_2, \cdots, x_{m-1}, y, z$, respectively. Clearly, $J \subseteq \operatorname{Ker}\varphi$ by the relations above. Noticing that $\Lambda = K[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{m-1}, \mathbf{y}, \mathbf{z}]/J = \bigoplus\limits_i \Lambda_i$ as a graded algebra satisfies that $\dim_K\Lambda_0 = m$ and $\dim_K\Lambda_j = 1$ for $j \ge 1$, we can immediately obtain that $\mathrm{HH}^*(A) \cong \Lambda$ by comparing the dimensions of graded algebras $\mathrm{HH}^*(A)$ and Λ.

Remark 4.1 Since the Hochschild cohomology of algebras is Morita-invariant, the above theorem describes the Hochschild cohomology rings of both the Temperley-Lieb algebras and the representation-finite q-Schur algebras $S_q(n, r)$ for $n \geqslant r$.

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