

Some Smoothness Results for Classical Problems in Optimal Design and Applications*

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(In Honor of the Scientific Contributions of Professor Luc Tartar)

Abstract The author considers two classical problems in optimal design consisting in maximizing or minimizing the energy corresponding to the mixture of two isotropic materials or two-composite material. These results refer to the smoothness of the optimal solutions. They also apply to the minimization of the first eigenvalue.

Keywords Optimal design, Two-phase material, Non-existence, Relaxation

2000 MR Subject Classification 49J20

1 Introduction

This paper deals with a classical problem in optimal design. It consists in mixing two isotropic materials (such as electric, thermic, ...) given by their respective diffusion constants in order to minimize a certain functional. In other words, given a bounded open set $\Omega \subset \mathbb{R}^N$, two constants $0 < \alpha < \beta$ and a distribution $f \in H^{-1}(\Omega)$, we are interested in the problem

$$\begin{cases} \min J(u, \omega), \\ -\operatorname{div}((\alpha\chi_\omega + \beta\chi_{\Omega \setminus \omega})\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \omega \subset \Omega & \text{measurable.} \end{cases} \quad (1.1)$$

Here the control variable is the set ω , where we place the material α . In some interesting situations, one of the materials α or β is better than the other one but also more expensive. Then, it is usual to add to the above problem a bound on the amount of the best material. This type of problems has been considered since the pioneering works of Murat and Tartar. In particular, it is known that there is not a solution in general (see [13–14]). So, it is better to consider a relaxed formulation. This can be done by using the homogenization theory whose aim is to describe what materials can be approximated by using microscopic (or better mesoscopic) mixtures of other ones. Then, to relax (1.1), the idea is to consider the closure of the materials

Manuscript received December 23, 2013.

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*This work was supported by the project of the “Ministerio de Economía y Competitividad” of Spain (No. MTM2011-24457).

of the form $\alpha\chi_\omega + \beta(1 - \chi_\Omega)$ in the sense of the homogenization. We refer, for example, to [1, 15, 18, 20, 23] for the theoretical study of the homogenization theory and to [1, 3, 16, 21–22] for its application to optimal design problems.

An interesting question from both the theoretical and numerical points of view is to study the smoothness properties of the solutions of (1.1) in its relaxed form. This is the problem that we consider in this paper for two very particular cases of functional J corresponding to

$$J(u, \omega) = - \int_{\Omega} (\alpha\chi_\omega + \beta\chi_{\Omega \setminus \omega}) |\nabla u|^2 dx, \quad (1.2)$$

$$J(u, \omega) = \int_{\Omega} (\alpha\chi_\omega + \beta\chi_{\Omega \setminus \omega}) |\nabla u|^2 dx. \quad (1.3)$$

The case where J is given by (1.2), compliance problem, is considered in Section 2. It was specially studied for $f = 1$ and $N = 2$ (see [9–10, 16]), where the problem applies to maximizing the torsional rigidity of a rod and to maximizing the flow rate corresponding to two fluids of different viscosities in a pipe. Most of the results contained in these papers easily generalize to an arbitrary f . It is simple to check that the solution of (1.1) corresponds to $\omega = \Omega$, i.e., material β is better than the material α . Thus, we will add the restriction $|\omega| \leq \kappa$ with $0 < \kappa < |\Omega|$. Then, the problem has not a solution in general, and it is necessary to consider a relaxed formulation. It consists in replacing the materials $(\alpha\chi_\omega + \beta\chi_{\Omega \setminus \omega})$ by those of the form $(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta})^{-1}$, where $\theta \in L^\infty(\Omega; [0, 1])$ represents the proportion of material α which we are using in the homogenized mixture. As it is proved in [16], although an optimal solution $(\hat{\theta}, \hat{u})$ is not necessarily unique, the flux $\hat{\sigma} = (\frac{\theta}{\alpha} + \frac{1-\theta}{\beta})^{-1} \nabla \hat{u}$ is unique. Moreover, there exists $\hat{\mu} \geq 0$ such that $\hat{\theta} = 0$ if $|\hat{\sigma}| < \hat{\mu}$, $\hat{\theta} = 1$ if $|\hat{\sigma}| > \hat{\mu}$. In this paper, we show that if Ω and f are smooth enough then $\hat{\sigma}$ belongs to $H^1(\Omega)^N \cap L^\infty(\Omega)^N$ and $\hat{\theta}$ is derivable in the orthogonal directions of $\hat{\sigma}$, that is, $\partial_i \hat{\theta} \hat{\sigma}_i - \partial_j \hat{\theta} \hat{\sigma}_j$ is in $L^2(\Omega)$ for $1 \leq i \leq j \leq N$. This allows us to improve some results in [16], where they were established by assuming the existence of solutions to be smooth enough.

In Section 3, we consider the case where the functional J is given by (1.3). From the application point of view, it consists in minimizing the potential energy for an electric material or a membrane. In this case, the best material is β , and then we will add the restriction $|\omega| \geq \kappa$ with $0 < \kappa < |\Omega|$. The relaxed formulation consists in replacing the materials $(\alpha\chi_\omega + \beta\chi_{\Omega \setminus \omega})$ by those of the form $\theta\alpha + (1 - \theta)\beta$, with $\theta \in L^\infty(\Omega; [0, 1])$ being the proportion of material α in the homogenized mixture. For this problem, we show that if $(\hat{\theta}, \hat{u})$ is an optimal solution, then \hat{u} is unique and there exists $\hat{\mu} \geq 0$ such that $\hat{\theta} = 1$ if $|\nabla \hat{u}| < \hat{\mu}$, and $\hat{\theta} = 0$ if $|\nabla \hat{u}| > \hat{\mu}$. Moreover, if Ω and f are smooth enough then $\hat{u} \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and $\hat{\theta}$ is derivable in the directions of $\nabla \hat{u}$, that is, $\nabla \hat{\theta} \cdot \nabla \hat{u}$ is in $L^2(\Omega)$.

In Section 4, we show that the results obtained in Section 2 can be applied to another classical problem in optimal design consisting in minimizing the first eigenvalue of the operator

$$u \in H_0^1(\Omega) \mapsto -\operatorname{div}((\alpha\chi_\omega + \beta\chi_{\Omega \setminus \omega}) \nabla u) \in H^{-1}(\Omega).$$

In particular, our smoothness results allow us to obtain an example where the unrelaxed problem has not a solution.

The results in this paper are a summary of the ones in [4–5], where we can find the corresponding proofs.

2 The Compliance Problem

For a bounded open set Ω , three constants α, β, κ with $0 < \alpha < \beta$, $0 < \kappa < |\Omega|$, and a distribution $\tilde{f} \in H^{-1}(\Omega)$, we consider the compliance problem

$$\begin{cases} \max \left\{ \int_{\Omega} (\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}) |\nabla u_{\omega}|^2 dx \right\}, \\ \omega \subset \Omega \text{ measurable, } |\omega| \leq \kappa, \\ -\operatorname{div}((\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}) \nabla u_{\omega}) = \tilde{f} \text{ in } \Omega, \quad u_{\omega} \in H_0^1(\Omega). \end{cases} \quad (2.1)$$

This problem has not a solution in general (see [13–14]), and therefore it is more convenient to work with a relaxed formulation which can be obtained (see [16, 21–23]) by replacing in (2.1) the materials of the form $\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}$ with $\omega \subset \Omega$ measurable by more general materials of the form

$$\left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1}.$$

Here $\theta \in L^{\infty}(\Omega; [0, 1])$ represents the proportion of material α in the homogenized mixture. Then, denoting

$$c = \frac{\beta - \alpha}{\alpha}, \quad f = \frac{1}{\beta} \tilde{f}, \quad (2.2)$$

we get the following relaxed formulation for (2.1):

$$\begin{cases} \max \left\{ \int_{\Omega} \frac{|\nabla u_{\theta}|^2}{1 + c\theta} dx \right\}, \\ \theta \in L^{\infty}(\Omega; [0, 1]), \quad \int_{\Omega} \theta dx \leq \kappa, \\ -\operatorname{div} \frac{\nabla u_{\theta}}{1 + c\theta} = f \text{ in } \Omega, \quad u_{\theta} = 0 \in H_0^1(\Omega), \end{cases} \quad (2.3)$$

which by using the classical characterization of the state equation as a minimum problem can also be written as

$$\begin{cases} \min \left\{ \int_{\Omega} \frac{|\nabla u|^2}{1 + c\theta} dx - 2\langle f, u \rangle \right\}, \\ \theta \in L^{\infty}(\Omega; [0, 1]), \quad \int_{\Omega} \theta dx \leq \kappa, \quad u \in H_0^1(\Omega). \end{cases} \quad (2.4)$$

The solution θ of (2.4) is not unique in general, but one can show the following result (see [16]).

Theorem 2.1 *There exists a unique function $\hat{\sigma} \in L^2(\Omega)^N$ such that for every solution $\hat{\theta}$ of (2.3), we have*

$$\hat{\sigma} = \frac{\nabla u_{\hat{\theta}}}{1 + c\hat{\theta}}. \quad (2.5)$$

This function $\hat{\sigma}$ is characterized as the unique solution of

$$\min_{\substack{\sigma \in L^2(\Omega)^N \\ -\operatorname{div} \sigma = f}} \max_{\substack{\theta \in L^\infty(\Omega; [0,1]) \\ \int_\Omega \theta dx \leq \kappa}} \int_\Omega (1 + c\theta) |\sigma|^2 dx. \quad (2.6)$$

Moreover, $\hat{\theta}$ is a solution of (2.3) if and only if it is a solution of

$$\max_{\substack{\theta \in L^\infty(\Omega; [0,1]) \\ \int_\Omega \theta dx \leq \kappa}} \min_{\substack{\sigma \in L^2(\Omega)^N \\ -\operatorname{div} \sigma = f}} \int_\Omega (1 + c\theta) |\sigma|^2 dx, \quad (2.7)$$

and the minimum in σ for $\hat{\theta}$ is attained in $\hat{\sigma}$.

Using the Kuhn-Tucker theorem to solve the minimum problem in θ in (2.6), we get the following theorem.

Theorem 2.2 Define $\hat{\sigma} \in L^2(\Omega)^N$ by Theorem 2.1, and $\hat{\mu}$ by

$$\hat{\mu} = \min\{\mu \geq 0 : |\{x \in \Omega : |\hat{\sigma}(x)| > \mu\}| \leq \kappa\}. \quad (2.8)$$

Then, every solution $\hat{\theta}$ of (2.3) satisfies

$$\hat{\theta} = 0 \quad \text{a.e. in } \{|\hat{\sigma}| < \hat{\mu}\}, \quad \hat{\theta} = 1 \quad \text{a.e. in } \{|\hat{\sigma}| > \hat{\mu}\}. \quad (2.9)$$

From (2.9), (2.5) and $-\operatorname{div} \sigma = f$ in Ω , we can also prove the following result which is the basis of our smoothness results for (2.3). We refer to [9] for a very related result.

Theorem 2.3 For $\hat{\mu}$ given by (2.8), we define the positive convex function $F \in W^{2,\infty}(0, +\infty)$ by

$$F(s) = \begin{cases} s^2, & \text{if } 0 \leq s < \hat{\mu}, \\ 2\hat{\mu}s - \hat{\mu}^2, & \text{if } \hat{\mu} \leq s \leq (1+c)\hat{\mu}, \\ \frac{s^2}{(1+c)} + c\hat{\mu}^2, & \text{if } s > (1+c)\hat{\mu}. \end{cases} \quad (2.10)$$

Then, if $\hat{\theta} \in L^\infty(\Omega; [0,1])$ is a solution of (2.3), the corresponding function $u_{\hat{\theta}}$ is a solution of

$$\min_{u \in H_0^1(\Omega)} \left\{ \int_\Omega F(|\nabla u|) dx - 2\langle f, u \rangle \right\}. \quad (2.11)$$

Moreover, $\hat{\theta}$ can be obtained from $u_{\hat{\theta}}$ by

$$\hat{\theta}(x) = \begin{cases} 0, & \text{if } 0 \leq |\nabla u_{\hat{\theta}}| < \hat{\mu}, \\ \frac{1}{c} \left(\frac{|\nabla u_{\hat{\theta}}|}{\hat{\mu}} - 1 \right), & \text{if } \hat{\mu} \leq |\nabla u_{\hat{\theta}}| \leq (1+c)\hat{\mu}, \\ 1, & \text{if } |\nabla u_{\hat{\theta}}| > (1+c)\hat{\mu}. \end{cases} \quad (2.12)$$

Using Theorem 2.3, we can now prove the following theorem.

Theorem 2.4 Assume $\Omega \in C^{1,1}$, and define $\hat{\sigma}$ by Theorem 2.2. We have the following assertions:

- (1) If $f \in W^{-1,p}(\Omega)$, $2 \leq p < \infty$, then $\hat{\sigma}$ belongs to $L^p(\Omega)^N$.
- (2) If $f \in L^p(\Omega)$, $p > N$, then $\hat{\sigma}$ belongs to $L^\infty(\Omega)^N$.
- (3) If $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$, then $\hat{\sigma}$ belongs to $H^1(\Omega)^N$ and every solution $\hat{\theta}$ of (2.3) is such that $\partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i$ belongs to $L^2(\Omega)$, $1 \leq i, j \leq N$.

Sketch of the Proof It is based on Theorem 2.3. In other words, if F was C^2 with a strictly positive second derivative, then it is known that f being smooth implies that the solutions of (2.11) are twice derivable (see [8]). Using this result, the idea is then to approximate F for a sequence of smooth functions F_ε strictly convex. The solution u_ε of (2.11) with F replaced by F_ε is twice derivable, and for a subsequence, converges to a solution u of (2.11) in $H_0^1(\Omega)$. Now, it is easy to check that $\nabla \partial_i u_\varepsilon$ satisfies an elliptic problem of the type

$$-\operatorname{div}(M_\varepsilon(x) \nabla \partial_i u_\varepsilon) = \partial_i f \quad \text{in } \Omega, \quad 1 \leq i \leq N. \quad (2.13)$$

Although the matrix functions M_ε are uniformly bounded, they lose the strict positivity when ε tends to zero, due to F not strictly convex. For this reason, we can not deduce the estimate that $\|D^2 u_\varepsilon\|_{L^2(\Omega)^{N \times N}}$ is bounded. However, we can prove that $M_\varepsilon \nabla \partial_i u_\varepsilon \cdot \nabla \partial_i u_\varepsilon$ is bounded in $L^1(\Omega)$. Using the explicit expression of the matrix functions M_ε , this allows us to show that $\hat{\sigma}$ is in $H^1(\Omega)^N$. The estimates of $\hat{\sigma}$ in $L^q(\Omega)^N$ mainly follow from Stampacchia estimates (see [19]).

Remark 2.1 The estimates in Theorem 2.4 are local. That is, if U is an open set in \mathbb{R}^N such that $\partial\Omega \cap U$ is in $C^{1,1}$, then the results above hold with Ω replaced by $\Omega \cap O$ and $O \subset \subset U$ open.

Remark 2.2 The fact that $\hat{\theta}$ is a solution of (2.3) and $f \in L^p(\Omega)$ ($p > N$) implies that the corresponding state function $u_{\hat{\theta}}$ is in $W^{1,\infty}(K)$ for every compact set $K \subset \Omega$ is given in [10] as a simple consequence of the results in [6]. Therefore the main contribution of Theorem 2.4 refers to the estimates on the boundary and specially to $\hat{\sigma} \in H^1(\Omega)^N$.

Theorem 2.4 has some interesting consequences for the unrelaxed problem (2.1). They apply mainly to proving the non-existence of a solution.

Theorem 2.5 Assume $\Omega \in C^{1,1}$ and $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$. We suppose that there exists a solution $\hat{\omega}$ of (2.1) and define $\hat{\sigma}$ by Theorem 2.1. Then, we have the following assertions:

- (1) The curl of $\hat{\sigma}$ is zero.
- (2) If Ω is simply connected, then $\hat{\sigma} = \nabla w$, with w being the solution of

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \Omega. \end{cases} \quad (2.14)$$

Remark 2.3 The second assertion in the previous lemma was obtained in [16] assuming that the solution $\hat{\omega}$ has a smooth boundary. Here the result is a consequence of Theorem 2.4.

Since $\operatorname{curl} \hat{\sigma} = 0$, it is locally the gradient of a function w which, taking into account that $-\operatorname{div} \hat{\sigma} = f$, satisfies the equation $-\Delta w = f$ (the second assertion of Theorem 2.5 is a

consequence of this result). This allows us to prove that if f is in a certain space $W^{k,p}(\Omega)$, then $\hat{\sigma}$ is in $W^{k+1,p}(\Omega)^N$.

3 The Energy Problem

In this section, instead of maximizing the functional in (2.1), let us minimize it. Now, the best material is not α but β . For this reason, we do not ask the measure of ω to be small but large. In other words, given a bounded open set Ω , two constants α, β , with $0 < \alpha < \beta$, a distribution $\tilde{f} \in H^{-1}(\Omega)$, and a constant κ with $0 < \kappa < |\Omega|$, let us consider the problem

$$\begin{cases} \min \left\{ \int_{\Omega} (\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}) |\nabla u_{\omega}|^2 dx \right\}, \\ \omega \subset \Omega \text{ measurable, } |\omega| \geq \kappa, \\ -\operatorname{div} \left((\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}) \nabla u_{\omega} \right) = \tilde{f} \text{ in } \Omega, \quad u_{\omega} \in H_0^1(\Omega). \end{cases} \quad (3.1)$$

As for the compliance problem, it is better to work with a relaxed formulation, because problem (3.1) has not a solution in general. This relaxation can be obtained by replacing the materials of the form $\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}$ by those of the form $\alpha \theta + \beta(1 - \theta)$ (see [1, 16, 21–23]). Taking

$$d = \frac{\beta - \alpha}{\beta}, \quad f = \frac{1}{\beta} \tilde{f}, \quad (3.2)$$

we then get the relaxed formulation of (3.1)

$$\begin{cases} \min \left\{ \int_{\Omega} (1 - d\theta) |\nabla u_{\theta}|^2 dx \right\}, \\ \theta \in L^{\infty}(\Omega; [0, 1]), \quad \int_{\Omega} \theta dx \geq \kappa, \\ -\operatorname{div}((1 - d\theta) \nabla u_{\theta}) = f \text{ in } \Omega, \quad u_{\theta} \in H_0^1(\Omega), \end{cases} \quad (3.3)$$

or equivalently

$$\max_{\substack{\theta \in L^{\infty}(\Omega; [0, 1]) \\ \int_{\Omega} \theta dx \geq \kappa}} \min_{u \in H_0^1(\Omega)} \left(\int_{\Omega} (1 - d\theta) |\nabla u|^2 dx - 2 \langle f, u \rangle \right). \quad (3.4)$$

Instead of using the variables u and θ , it is also possible to write the problem in the variables σ and θ with $\sigma = (1 - d) \nabla u$, that is, (3.3) is also equivalent to

$$\min_{\substack{\theta \in L^{\infty}(\Omega; [0, 1]) \\ \int_{\Omega} \theta dx \geq \kappa}} \min_{\substack{\sigma \in L^2(\Omega)^N \\ -\operatorname{div} \sigma = f}} \int_{\Omega} \frac{|\sigma|^2}{1 - d\theta} dx. \quad (3.5)$$

Remark that the structure of these problems is similar to those of (2.4) and (2.6), but the roles of σ and ∇u are exchanged. Thus we expect to prove properties similar to those obtained for the optimal $\hat{\sigma}$ in the compliance problem for the gradient of an optimal state function \hat{u} corresponding to the energy problem. This is what the results we state below essentially show.

We start with the following uniqueness result for (3.3).

Theorem 3.1 *There exists a unique function $\hat{u} \in H_0^1(\Omega)$ such that if $\hat{\theta}$ is a solution of (3.3), then the corresponding state equation $u_{\hat{\theta}}$ agrees with \hat{u} . Moreover, defining*

$$\hat{\mu} = \min\{\mu \geq 0 : |\{x \in \Omega : |\nabla \hat{u}(x)| \leq \mu\}| \geq \kappa\}, \quad (3.6)$$

we have

$$\hat{\theta}(x) = \begin{cases} 1, & \text{if } |\nabla \hat{u}(x)| < \hat{\mu}, \\ 0, & \text{if } |\nabla \hat{u}(x)| > \hat{\mu} \end{cases} \quad (3.7)$$

for every solution $\hat{\theta}$ of (3.3).

Analogous to Theorem 2.3 for (2.3), the following result is the key ingredient to obtaining some smoothness properties for the solutions of (3.3).

Theorem 3.2 *Let \hat{u} and $\hat{\mu}$ be given by Theorem 3.1. We define $G \in W^{2,\infty}(0, +\infty)$ by*

$$G(s) = \begin{cases} \frac{s^2}{1-d}, & \text{if } 0 \leq s < (1-d)\hat{\mu}, \\ 2\hat{\mu}s - (1-d)|\hat{\mu}|^2, & \text{if } (1-d)\hat{\mu} \leq s \leq \hat{\mu}, \\ s^2 + d|\hat{\mu}|^2, & \text{if } s > \hat{\mu}. \end{cases} \quad (3.8)$$

Then, a function $\hat{\theta} \in L^\infty(\Omega)$ is a solution of (3.3) if and only if

$$\hat{\sigma} = \frac{\nabla \hat{u}}{1-d\hat{\theta}} \quad (3.9)$$

is a solution of

$$\min_{\substack{\sigma \in L^2(\Omega)^N \\ -\operatorname{div} \sigma = f}} \int_{\Omega} G(|\sigma|) dx. \quad (3.10)$$

Moreover, the function $\hat{\theta}$ can be obtained from $\hat{\sigma}$ by

$$\hat{\theta}(x) = \begin{cases} 1, & \text{if } 0 \leq |\hat{\sigma}| < (1-d)\hat{\mu}, \\ \frac{1}{d} \left(1 - \frac{|\hat{\sigma}|}{\hat{\mu}}\right), & \text{if } (1-d)\hat{\mu} \leq |\hat{\sigma}| \leq \hat{\mu}, \\ 0, & \text{if } \hat{\mu} < |\hat{\sigma}|. \end{cases} \quad (3.11)$$

Our main result related to the smoothness properties of the solutions of (3.3) is given by the following theorem.

Theorem 3.3 *Assume $\Omega \in C^{1,1}$, and define \hat{u} by Theorem 3.1. Then, we have the following assertions:*

- (1) *If $f \in W^{-1,p}(\Omega)$, $p \geq 2$, then \hat{u} belongs to $W_0^{1,p}(\Omega)^N$.*
- (2) *If $f \in L^p(\Omega)$, $p > N$, then \hat{u} belongs to $W^{1,\infty}(\Omega)^N$. Besides, if $N = 2$, then $\hat{u} \in C^1(\overline{\Omega})$.*
- (3) *If $f \in L^2(\Omega)$, then \hat{u} belongs to $H^2(\Omega)$, and every solution $\hat{\theta}$ of (3.3) is such that $\nabla \hat{\theta} \cdot \nabla \hat{u} \in L^2(\Omega)$.*

Sketch of the Proof As in the proof of Theorem 2.5, the idea is to approximate problem (3.10) by another one obtained by regularizing the function G defined by (3.8), which allows to construct a sequence of smooth functions u_ε converging to \hat{u} at least in $H^1(\Omega)$. The second derivatives of u_ε satisfy an elliptic equation of the form

$$-\operatorname{div} A_\varepsilon \nabla (\partial_i u_\varepsilon) = \partial_i f \quad \text{in } \Omega,$$

where, contrary to (2.13), the matrix functions A_ε are uniformly elliptic and independent of ε , but they have not any upper bound. The proof of $\hat{u} \in H^2(\Omega)$ follows from this ellipticity condition, while the estimate in $W^{1,\infty}(\Omega)$ follows from Stampacchia estimates. In the special case $N = 2$, we can show that \hat{u} is in $C^1(\Omega)$ by using the results in [2] (see also [7, 12]).

Remark 3.1 Because in the proof of Theorem 2.5, we deal with a diffusion coefficient matrix bounded but not strictly elliptic and in Theorem 3.3 with a diffusion coefficient matrix strictly elliptic but unbounded, there are important differences in the proofs. Moreover, there are also some important differences in the result. In this sense, contrary to Theorem 3.3, we do not know whether the estimates in Theorem 3.3 are local, i.e., whether Ω is smooth just in a portion of the boundary, we do not know whether \hat{u} is smooth in a neighborhood of this portion.

Remark 3.2 In Theorem 3.3, we proved that the optimal solutions $\hat{\theta}$ of (2.3) are derivable in the orthogonal directions to $\hat{\sigma}$ and then to $\nabla \hat{u}$. For 3.3, we have that $\hat{\theta}$ is derivable in the direction of $\nabla \hat{u}$.

We finish this section by giving some consequences of Theorem 3.3 when it applies to the unrelaxed problem (3.1).

Theorem 3.4 Assume $\Omega \in C^{1,1}$ and $f \in L^2(\Omega)$. We suppose that there exists a solution $\hat{\omega} \subset \Omega$ of (3.1) and define \hat{u} by Theorem 3.1. Then, we have the following assertions:

(1)

$$-\Delta \hat{u} = \left(\frac{1}{1-d} \chi_{\hat{\omega}} + \chi_{\Omega \setminus \hat{\omega}} \right) f \quad \text{in } \Omega. \quad (3.12)$$

(2)

$$-(1-d) \operatorname{div}(\nabla \hat{u} \chi_{\hat{\omega}}) = f \chi_{\hat{\omega}} \quad \text{in } \Omega. \quad (3.13)$$

(3)

$$-\Delta |\nabla \hat{u}|^2 + 2|D^2 \hat{u}|^2 = \left(\frac{1}{1-d} \chi_{\hat{\omega}} + \chi_{\Omega \setminus \hat{\omega}} \right) \nabla f \cdot \nabla \hat{u} \quad \text{in } \Omega. \quad (3.14)$$

(4) If f does not change its sign and $\hat{\mu}$ defined by (3.6) does not vanish, then $\hat{\theta} = \chi_{\hat{\omega}}$ is the unique solution of (3.3).

(5) Assume f to be strictly positive or strictly negative, and consider an open set $O \subset \Omega$ and a neighborhood \mathcal{N} of ∂O . Then we have

$$\mathcal{N} \subset \hat{\omega} \rightarrow O \subset \hat{\omega}, \quad \mathcal{N} \subset O \setminus \hat{\omega} \rightarrow O \subset \Omega \setminus \hat{\omega}.$$

Remark 3.3 (3.12) implies that if $f \in L^p(\Omega)$, then $\hat{u} \in W^{2,p}(\Omega)$. Combined with (3.14) and the Sobolev imbedding theorem, this provides the existence of two derivatives for $|\nabla \hat{u}|^2$. This is related to the smoothness properties of the interfaces corresponding to the two materials α and β , which we recall, are contained in the sets $|\nabla \hat{u}| = \hat{\mu}$.

Equation (3.13) shows that the Neumann condition $\frac{\partial \hat{u}}{\partial \nu} = 0$ on $\partial\omega$ is satisfied in a weak sense.

4 Applications to the Minimization of the First Eigenvalue

Another interesting problem in the optimal design of two-composites is the minimization of the first eigenvalue for the corresponding diffusion operator. Mathematically, it can be formulated as

$$\min_{\substack{\omega \subset \Omega \\ |\omega| \leq \kappa}} \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} (\alpha \chi_{\omega} + \beta \chi_{\Omega \setminus \omega}) |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}, \quad (4.1)$$

where, similar to the previous sections, we assume $0 < \alpha < \beta$, $0 < \kappa < |\Omega|$.

Define c by (2.2), and the corresponding relaxed formulation is given by

$$\min_{\substack{\theta \in L^\infty(\Omega; [0,1]) \\ \int_{\Omega} \theta dx \leq \kappa}} \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} \frac{|\nabla u|^2}{1+c\theta} dx}{\int_{\Omega} |u|^2 dx}. \quad (4.2)$$

A different formulation of (4.1) can be obtained by using the following result.

Lemma 4.1 *Assume $A \in L^\infty(\Omega)^{N \times N}$ to be uniformly elliptic. Then the first eigenvalue of the operator $-\operatorname{div}(A \nabla u)$ with Dirichlet conditions*

$$\lambda_1(A) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} A \nabla u \cdot \nabla u dx}{\int_{\Omega} |u|^2 dx}$$

can be characterized by

$$\frac{1}{\lambda_1(A)} = - \min_{\substack{u \in H_0^1(\Omega) \\ \|f\|_{L^2(\Omega)} \leq 1}} \left(\int_{\Omega} A \nabla u \cdot \nabla u dx - 2 \int_{\Omega} f u dx \right).$$

This lemma proves that (4.1) is equivalent to

$$\min_{\|f\|_{L^2(\Omega)} \leq 1} \min_{\substack{\theta \in L^\infty(\Omega; [0,1]) \\ \int_{\Omega} \theta dx \leq \kappa}} \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \left(\int_{\Omega} \frac{|\nabla u|^2}{1+c\theta} dx - 2 \int_{\Omega} f u dx \right). \quad (4.3)$$

Therefore, comparing with (2.4), we see that the eigenvalue problem consists in solving the compliance problem (2.4) for every $f \in L^2(\Omega)$ with a unitary norm, and then minimizing in f . Moreover, it is not difficult to check that if f is a function for which the maximum is attained, then f is an eigenfunction for the corresponding optimal mixture. From Theorem 2.4, we then have the following smoothness result.

Theorem 4.1 Assume $\Omega \in C^{1,1}$. Then, if $(\hat{\theta}, \hat{u})$ is an optimal solution of (4.1), we have

$$\begin{aligned} \hat{u} &\in W^{1,\infty}(\Omega), \quad \hat{\sigma} = \frac{\nabla \hat{u}}{1 + c\hat{\theta}} \in H^1(\Omega)^N, \\ \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i &\in L^2(\Omega), \quad 1 \leq i, j \leq N. \end{aligned}$$

The results obtained in Theorem 2.5 can be used to obtain a counterexample to the existence of solution for the unrelaxed problem (4.1), that is, we have the following theorem.

Theorem 4.2 Assume $\Omega = (-\frac{\pi}{4}, \frac{\pi}{4}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Then for $\varepsilon > 0$ small enough, the problem

$$\min_{\substack{\omega \subset \Omega \\ |\omega| \leq |\Omega| - \varepsilon}} \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} (\chi_{\omega} + 2\chi_{\Omega \setminus \omega}) |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \quad (4.4)$$

has not a solution.

Sketch of the Proof We assume that there exists a solution $\omega_{\varepsilon}, u_{\varepsilon}$ of (4.4), with $u_{\varepsilon} > 0$, $\|u\|_{L^2(\Omega)} = 1$. Taking into account that $|\omega_{\varepsilon}| = |\Omega| - \varepsilon$, we get that u_{ε} is very close to the eigenfunction u corresponding to the first eigenvector of the Laplacian in Ω , which is given by

$$u(x) = \frac{2}{\sqrt{3\pi}} \cos(2x_1) \cos(x_2), \quad \forall (x_1, x_2) \in \Omega.$$

Now, the optimality conditions for (4.4) show that $\partial \omega_{\varepsilon}$ is composed by level lines of u_{ε} . Then we deduce that there exists a component O_{ε} of $\Omega \setminus \omega_{\varepsilon}$ which is close to the ellipse

$$\frac{x_1^2}{8} + \frac{x_2^2}{2} = 1 - c_{\varepsilon},$$

where c_{ε} tends to zero with ε . However, using our smoothness results for the solution of (4.4) (Ω is not smooth, but by Remark 2.1, we have local regularity) and Serrin's theorem (see [17]), we must have that O_{ε} is a ball. This contradiction shows the result.

Figures 1–3 below correspond to the numerical solution of problem

$$\min_{\substack{\theta \in L^{\infty}(\Omega; [0,1]) \\ \int_{\Omega} \theta dx \leq |\Omega| - \varepsilon}} \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} \frac{|\nabla u|^2}{1+c\theta} dx}{\int_{\Omega} |u|^2 dx}, \quad (4.5)$$

with Ω given by Theorem 4.2. This is the relaxed formulation of (4.4). The figures correspond to $\varepsilon = 0.5 \sim 0.1 |\Omega|$, $\varepsilon = 2 \sim 0.4 |\Omega|$ and $\varepsilon = 3 \sim 0.6 |\Omega|$. In these pictures, the white zones correspond to the material α , the black zones to the material β and the grey figures to the homogenized materials. We observe that although in Theorem 4.2, we supposed that ε is small, grey zones appear in all the cases, and in fact, they are larger when ε is larger.

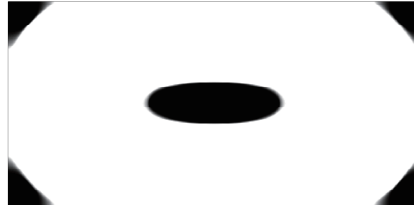
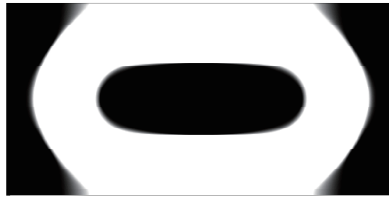


Figure 1 Optimal solution of (4.5) with $\varepsilon = 0.5$.

Figure 2 Optimal solution of (4.5) with $\varepsilon = 2$.Figure 3 Optimal solution of (4.5) with $\varepsilon = 3$.

References

- [1] Allaire, G., Shape Optimization by the Homogenization Method, Springer-Verlag, New York, 2002.
- [2] Briane, M. and Casado-Díaz, J., Uniform convergence of sequences of solutions of two-dimensional linear elliptic equations with unbounded coefficients, *J. Differential Equations*, **245**, 2008, 2038–2054.
- [3] Casado-Díaz, J., Couce-Calvo, J. and Martín-Gómez, J. D., Relaxation of a control problem in the coefficients with a functional of quadratic growth in the gradient, *SIAM J. Control Optim.*, **43**, 2008, 1428–1457.
- [4] Casado-Díaz, J., Smoothness properties for the optimal mixture of two isotropic materials, the compliance and eigenvalue problems, *SIAM J. Control Optim.*, to appear.
- [5] Casado-Díaz, J., Some smoothness results for the optimal design of a two-composite material which minimizes the energy, *Calc. Var. PDE*, **53**, 2015, 649–673.
- [6] Chipot, M. and Evans, L., Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations, *Proc. Roy. Soc. Edinburgh A*, **102**, 1986, 291–303.
- [7] Franchi, B., Serapioni, R. and Serra Cassano, F., Irregular solutions of linear degenerate elliptic equations, *Potential Anal.*, **9**, 1998, 201–216.
- [8] Gilbarg, D. and Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 2001.
- [9] Goodman, J., Kohn, R. V. and Reyna, L., Numerical study of a relaxed variational problem for optimal design, *Comput. Methods Appl. Mech. Eng.*, **57**, 1986, 107–127.
- [10] Kawohl, B., Stara, J. and Wittum, G., Analysis and numerical studies of a problem of shape design, *Arc. Rational Mech. Anal.*, **114**, 1991, 343–363.
- [11] Lions, J. L., Some Methods in the Mathematical Analysis of Systems and Their Control, Science Press, Beijing, 1981.
- [12] Manfredi, J. J., Weakly monotone functions, *J. Geom. Anal.*, **4**, 1994, 393–402.
- [13] Murat, F., Un contre-exemple pour le problème du contrôle dans les coefficients, *C. R. A. S. Sci. Paris A*, **273**, 1971, 708–711.
- [14] Murat, F., Théorèmes de non existence pour des problèmes de contrôle dans les coefficients, *C. R. A. S. Sci. Paris A*, **274**, 1972, 395–398.
- [15] Murat, F., *H-Convergence*, Séminaire d'Analyse Fonctionnelle et Numérique, Université d'Alger, 1977–1978; English translation: *H-Convergence*, Topics in the Mathematical Modelling of Composite Materials, F. Murat and L. Tartar (eds.), Birkhäuser, Boston, 1998, 21–43.

- [16] Murat, F. and Tartar L., Calcul des variations et homogénéisation, Les Méthodes de L'homogénéisation: Theorie et Applications en Physique, Eirolles, Paris, 1985, 319–369; English translation: Calculus of variations and homogenization, Topics in the Mathematical Modelling of Composite Materials, F. Murat and L. Tartar (eds.), Birkhäuser, Boston, 1998, 139–174.
- [17] Serrin J., A symmetry problem in potential theory, *Arch. Rat. Mech. Anal.*, **43**, 1971, 304–318.
- [18] Spagnolo, S., Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 1968, 571–597.
- [19] Stampacchia, G., Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier Grenoble*, **15**, 1965, 189–258.
- [20] Tartar, L., Estimations fines de coefficients Homogénéisés, Ennio de Giorgi Colloquium, Pitman, London, 1985, 168–187.
- [21] Tartar, L., Remarks on optimal design problems, Calculus of Variations, Homogenization and Continuum Mechanics, World Scientific, Singapore, 1994, 279–296.
- [22] Tartar, L., An introduction to the homogenization method in optimal design, Optimal Shape Design, Springer-Verlag, Berlin, 2000, 47–156.
- [23] Tartar, L., The general theory of the homogenization, A Personalized Introduction, Springer-Verlag, Berlin, 2009.