Weak Continuity and Compactness for Nonlinear Partial Differential Equations*

Gui-Qiang G. CHEN[1]

(*In Honor of the Scientific Contributions of Professor Luc Tartar*)

Abstract This paper presents several examples of fundamental problems involving weak continuity and compactness for nonlinear partial differential equations, in which compensated compactness and related ideas have played a significant role. The compactness and convergence of vanishing viscosity solutions for nonlinear hyperbolic conservation laws are first analyzed, including the inviscid limit from the Navier-Stokes equations to the Euler equations for homentropic flow, the vanishing viscosity method to construct the global spherically symmetric solutions to the multidimensional compressible Euler equations, and the sonic-subsonic limit of solutions of the full Euler equations for multi-dimensional steady compressible fluids. Then the weak continuity and rigidity of the Gauss-Codazzi-Ricci system and corresponding isometric embeddings in differential geometry are revealed. Further references are also provided for some recent developments on the weak continuity and compactness for nonlinear partial differential equations.

Keywords Weak continuity, Compensated compactness, Nonlinear partial differential equations, Euler equations, Gauss-Codazzi-Ricci system
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1 Introduction

Nonlinear partial differential equations (PDEs) can be written in the following general form:

$$\mathcal{N}[U] = 0, \tag{1.1}$$

where $\mathcal{N}[\cdot]$ is a nonlinear mapping, and U is an unknown function that is called a solution if U solves (1.1).

Two of the fundamental issues for nonlinear PDEs (1.1) are the following:

(i) Weak Continuity and Rigidity: Let $\{U^\varepsilon\}_{\varepsilon>0}$ be a sequence of exact solutions satisfying

$$\begin{cases} \mathcal{N}[U^\varepsilon] = 0, \\ U^\varepsilon \rightharpoonup U \quad \text{in some topology as } \varepsilon \to 0. \end{cases} \tag{1.2}$$

Issue 1.1 Does the limit function U satisfy

$$\mathcal{N}[U] = 0, \tag{1.3}$$

or

$$\widetilde{\mathcal{N}}[U] = 0 \tag{1.4}$$

for a different nonlinear mapping $\widetilde{\mathcal{N}}[\cdot]$ associated with the original nonlinear mapping $\mathcal{N}[\cdot]$ and the solution sequence $\{U^\varepsilon\}_{\varepsilon>0}$?

Such an issue arises in rigidity problems in geometry, mechanics, among others.

(ii) Compactness and Convergence: Let $\{U^\varepsilon\}_{\varepsilon>0}$ be a sequence of approximate or multiscale solutions satisfying

$$\begin{cases} \mathcal{N}^\varepsilon[U^\varepsilon] = 0, \\ U^\varepsilon \rightharpoonup U \quad \text{in some topology as } \varepsilon \to 0. \end{cases} \tag{1.5}$$

Issue 1.2 Does the limit function U satisfy (1.3), or (1.4) for a different nonlinear mapping $\widetilde{\mathcal{N}}[\cdot]$ associated with the nonlinear mappings $\mathcal{N}^\varepsilon[\cdot]$ and the solution sequence $\{U^\varepsilon\}_{\varepsilon>0}$?

This issue arises in the viscosity methods, relaxation methods, numerical methods, as well as problems for homogenization, hydrodynamic limits, and search for effective equations, among others.

This paper presents several examples of these fundamental problems involving weak continuity and compactness for nonlinear PDEs, in which compensated compactness and related ideas, developed by Luc Tartar [85–89] and François Murat [70–74], have played a significant role (see also [91]). In particular, in Section 2, we first analyze the compactness and convergence of vanishing viscosity solutions to hyperbolic conservation laws. In Section 3, we reveal the weak continuity and rigidity of the Gauss-Codazzi-Ricci system and corresponding isometric embeddings in differential geometry. Further references are also provided for some recent developments on the weak continuity and compactness for nonlinear PDEs. We finally remark that, as we will see in Sections 3–4, many fundamental problems in this direction are still open, which require further new mathematical ideas, techniques, and approaches that deserve our special attention.

2 Compactness and Convergence of Vanishing Viscosity Solutions to Hyperbolic Conservation Laws

Consider the following one-dimensional nonlinear hyperbolic conservation laws of the form:

$$\partial_t U + \partial_x F(U) = 0, \quad U \in \mathbb{R}^N, \tag{2.1}$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear mapping so that all the eigenvalues of $\nabla_U F(U)$ are real.

To solve these nonlinear PDEs, one of the important approaches is the viscosity method for which one honors the physical or designs an artificial $N \times N$ matrix function as follows:

$$D : \mathbb{R}^N \to M^{N \times N}, \quad D(U) \geq 0, \tag{2.2}$$

so that

(i) $\partial_t U + \partial_x F(U) = \varepsilon \partial_x (D(U) \partial_x U)$ admits a global solution $U^\varepsilon(t, x)$ for each fixed $\varepsilon > 0$;

(ii) $U^\varepsilon(t, x) \to U(t, x)$ in some topology as $\varepsilon \to 0$, and $U(t, x)$ is an entropy solution.

This method for the multidimensional case can be analogously formulated.

The idea of the vanishing viscosity method originates in the philosophy of regarding the inviscid gas as the limit of viscous gases, which can date back to the 19th century, including the works by Stokes (1848), Rankine (1870), Hugoniot (1889), Rayleigh (1910), Taylor (1910) and Weyl (1949), among others (see also [28] and the references cited therein). This idea has played an essential role in developing the mathematical theory of hyperbolic conservation laws (such as discontinuous solutions, entropy conditions, existence, uniqueness, and solution behavior), as well as numerical methods and related applications (such as shock capturing, upwind, and kinetic schemes). This method becomes increasingly important, especially for understanding the recently observed non-uniqueness phenomena for the weak solutions satisfying the entropy equality for the multidimensional Euler equations (see [29–30]). On the other hand, the realization of this method is truly challenging in mathematics, since it involves several fundamental difficulties in analysis, including singular limits, nonlinearity, discontinuity, singularity, oscillation, cavitation, and concentration.

2.1 Compactness and convergence via BV-estimates

This compactness framework is based on the compactness theorem in BV, which is a sufficient framework to ensure the strong compactness and convergence of exact/approximate solutions. On the other hand, achieving the BV-estimates of exact/approximate solutions is usually very challenging for the nonlinear systems, even though it is relatively easier for the scalar case.

2.1.1 Scalar conservation laws

Consider the Cauchy problem for scalar conservation laws ($N = 1$) as follows:

$$\partial_t U + \partial_x F(U) = \varepsilon \partial_{xx} U \tag{2.3}$$

with the initial data $U|_{t=0} = U_0 \in BV \cap L^\infty(\mathbb{R})$. It can be shown that there exists C independent of ε such that the viscous solutions $U^\varepsilon = U^\varepsilon(t, x)$ of (2.3) satisfy

(i) Maximum principle: $\|U^\varepsilon\|_{L^\infty} \leq C$;

(ii) BV-estimate: $\|\partial_x U^\varepsilon\|_{L^1} + \|\partial_t U^\varepsilon\|_{L^1} \leq C$.

See [51, 57, 76] for the one-dimensional case, and [55, 92] for the multidimensional case.

One of the approaches to achieve the BV-estimate is due to Vol'pert [92], which yields

$$\partial_t(|\partial_x U^\varepsilon|) + \partial_x(F'(U^\varepsilon)|\partial_x U^\varepsilon|) \leq \varepsilon \partial_{xx}(|\partial_x U^\varepsilon|),$$

$$\partial_t(|\partial_t U^\varepsilon|) + \partial_x(F'(U^\varepsilon)|\partial_t U^\varepsilon|) \leq \varepsilon \partial_{xx}(|\partial_t U^\varepsilon|)$$

in the sense of distributions, leading to the BV-estimate.

Then the compactness theorem in BV implies the strong convergence of $U^\varepsilon(t, x)$.

Similar arguments can yield the L^1-equicontinuity of U^ε directly, which is also a corollary of the L^1-stability and the comparison principle via Kruzhkov's method (see [55]).

The same arguments also work for multidimensional scalar conservation laws (see [55, 92]); see also [11] for scalar conservation laws with memory.

2.1.2   Hyperbolic systems of conservation laws: BV-estimate via Glimm's approach

Glimm [41] first developed a random choice method, the Glimm scheme, and derived the BV-estimate of the corresponding Glimm approximate solutions, based on the Glimm functional and corresponding wave interaction estimates. The techniques developed have been successfully employed to establish the global existence of solutions in BV and analyze the behavior of solutions in BV (structure, uniqueness, stability, and asymptotic behavior of solutions in BV) when the total variation of the initial data is small (see also [28, 33, 42, 61] and the references cited therein).

Theorem 2.1 *For a strictly hyperbolic system* (2.1) *on* U *in a neighborhood of a compact set* $K \subset \mathbb{R}^N$, *there exist constants* $\delta > 0$ *and* C *such that, if*

$$\text{Tot.Var.}\{U_0\} < \delta, \quad \lim_{x \to -\infty} U_0(x) \in K, \tag{2.4}$$

then there exists a global solution $U(t, x)$ *such that*

$$\text{Tot.Var.}\{U(t, \cdot)\} \leq C \, \text{Tot.Var.}\{U_0\}.$$

Glimm's approach was further employed to handle the front-tacking method and developed to analyze the L^1-stability of global solutions obtained by either the Glimm scheme or the front tracking method (see [5, 28, 50, 58, 62] and the references cited therein). The approach was also developed to analyze the well-posedness for two-dimensional steady supersonic Euler flows past a Lipschitz wedge in [16, 25].

2.1.3   Hyperbolic systems of conservation laws: BV–estimate for the artificial viscosity method

Consider the following Cauchy problem for one-dimensional nonlinear hyperbolic systems of conservation laws with vanishing artificial viscosity (i.e., $D(U) = I_{N \times N}$):

$$\partial_t U + \partial_x F(U) = \varepsilon \partial_{xx} U \tag{2.5}$$

and the initial data $U(0, x) = U_0(x) \in BV(\mathbb{R}^N)$.

Theorem 2.2 (Bianchini-Bressan [4]) *For a strictly hyperbolic system* (2.1) *on U in a neighborhood of a compact set $K \subset \mathbb{R}^N$, there exist constants $\delta > 0$ and $C_j, j = 1, 2, 3$, such that, if U_0 satisfies* (2.4), *then, for any fixed $\varepsilon > 0$, there exists a unique solution $U^\varepsilon(t, \cdot) := S_t^\varepsilon U_0(\cdot)$ of the Cauchy problem* (2.5) *such that*

(i) *BV bound*: $\mathrm{Tot.Var.}\{S_t^\varepsilon U_0\} \leq C_1 \, \mathrm{Tot.Var.}\{U_0\}$;

(ii) L^1-*stability*:

$$\|S_t^\varepsilon U_0 - S_t^\varepsilon V_0\|_{L^1} \leq C_2 \, \|U_0 - V_0\|_{L^1},$$
$$\|S_t^\varepsilon U_0 - S_s^\varepsilon U_0\|_{L^1} \leq C_3 \left(|t - s| + |\sqrt{\varepsilon t} - \sqrt{\varepsilon s}|\right).$$

These imply the strong convergence and L^1-stability of the limit solution to (2.1).

The strategies to achieve the BV-estimate include the following steps:

(i) Employ the heat kernel to estimate the solution for $t \in [0, \tau_\varepsilon]$,

$$\|\partial_x U^\varepsilon(t, \cdot)\|_{L^1} \leq \kappa \, \delta,$$

where κ is small, independent of ε and δ.

(ii) Decompose $\partial_x U^\varepsilon$ along a suitable basis of unit vectors $\{\mathbf{r}_1, \cdots, \mathbf{r}_N\}$,

$$\partial_x U^\varepsilon = \sum v_i^\varepsilon \mathbf{r}_i \quad \text{(sum of gradients of viscous travelling waves)}.$$

(iii) Derive a system of N equations for these scalar components,

$$\partial_t v_i^\varepsilon + \partial_x(\widetilde{\lambda}_i v_i^\varepsilon) - \varepsilon \partial_{xx} v_i^\varepsilon = \phi_i^\varepsilon, \quad i = 1, \cdots, N.$$

Then, as the scalar case, we obtain that, for all $t \geq \tau_\varepsilon$,

$$\|v_i^\varepsilon(t, \cdot)\|_{L^1} \leq \|v_i^\varepsilon(\tau_\varepsilon, \cdot)\|_{L^1} + \int_{\tau_\varepsilon}^\infty \int_{-\infty}^\infty |\phi_i^\varepsilon(t, x)| \mathrm{d}x \mathrm{d}t.$$

(iv) Construct the basis $\{\mathbf{r}_1, \cdots, \mathbf{r}_N\}$ in an appropriate way so that, for $t \geq \tau_\varepsilon$,

$$\int_{\tau_\varepsilon}^\infty \int_{-\infty}^\infty |\phi_i^\varepsilon(t, x)| \mathrm{d}x \mathrm{d}t \leq \widehat{C},$$

which implies

$$\mathrm{Tot.Var.}\{U^\varepsilon(t, \cdot)\} = \|U_x^\varepsilon(t, \cdot)\|_{L^1} \leq \sum_i \|v_i^\varepsilon(t, \cdot)\|_{L^1} \leq C,$$

where $\widehat{C}$ and C are independent of $\varepsilon > 0$.

Remark 2.1 The results above still hold even for non-conservative strictly hyperbolic systems. On the other hand, this approach requires both the artificial viscosity (i.e., $D(U) = I_{N \times N}$) and the total variation of the initial data sufficiently small.

Remark 2.2 A longstanding open problem is the BV-estimate and convergence of vanishing viscosity approximation U^ε governed by the general form as follows:

$$\partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_x (D(U^\varepsilon) \partial_x U^\varepsilon) \tag{2.6}$$

for general viscosity matrices $D(U)$, including the Navier-Stokes viscosity matrices. This especially includes the fundamental problem in mathematical fluid dynamics, the inviscid limit of solutions of the Navier-Stokes equations to the Euler equations for homentropic flow via the BV-estimate, which is still open.

2.2 Compactness and convergence via compensated compactness

We now discuss the compactness and convergence of exact/approximate solutions to conservation laws via compensated compactness and related ideas, which only require much weak bounds that may be obtained easily through natural energy/entropy estimates as our examples below indicate.

2.2.1 Scalar conservation laws

Consider the Cauchy problem for scalar conservation laws (2.3) ($N = 1$) with the following initial data:

$$U|_{t=0} = U_0 \in L^\infty(\mathbb{R}).$$

Then it can be easily shown that there exists C, independent of ε, such that the viscous solutions U^ε satisfy the following natural estimates:

(i) Maximum principle: $\|U^\varepsilon\|_{L^\infty} \leq C$ or $\|U^\varepsilon\|_{L^p} \leq C$.

(ii) Dissipation estimate: $\|\sqrt{\varepsilon}U_x^\varepsilon\|_{L^2_{\mathrm{loc}}} \leq C$.

The second estimate is a direct corollary of the natural energy estimate as follows:

$$\varepsilon|\partial_x U^\varepsilon|^2 = -\partial_t\Big(\frac{|U^\varepsilon|^2}{2}\Big) - \partial_x\Big(\int^{U^\varepsilon} wF'(w)\mathrm{d}w\Big) + \varepsilon\partial_{xx}\Big(\frac{|U^\varepsilon|^2}{2}\Big).$$

These estimates imply that, for any $\eta \in C^2$ with entropy flux $q(U) = \int^U \eta'(w)F'(w)\mathrm{d}w$,

$$\partial_t\eta(U^\varepsilon) + \partial_x q(U^\varepsilon) \quad \text{is compact in } H^{-1}_{\mathrm{loc}}.$$

Then the compensated compactness arguments yield the weak continuity of $F(U^\varepsilon)$, or even the strong convergence of $U^\varepsilon(t, x)$ a.e.

For the convex case, Tartar [85] was the first to employ one entropy pair $(\eta_*(U), q_*(U)) = (U^2, 2\int^U wF'(w)\mathrm{d}w)$ to conclude the strong convergence, which initiated the successful applications of compensated compactness to nonlinear hyperbolic conservation laws. For the nonconvex case, the entropy pair $(\eta_*(U), q_*(U)) = (F(U), \int^U (F'(w))^2)\mathrm{d}w)$ also suffices to conclude the weak continuity with respect to the general equation, and the strong convergence when the equation is genuinely nonlinear for almost all U, as observed by Chen-Lu [18] and Luc Tartar independently. See also [36, 80, 84].

The approach also applies to (2.6) ($N = 1$) with more general viscosity terms, as well as scalar conservation laws with memory (see [27]).

For these, the following div-curl lemma plays an essential role.

Lemma 2.1 (see [70, 85]) *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be open bounded. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that, for any $\varepsilon > 0$, two vector fields*

$$u^\varepsilon \in L^p(\Omega; \mathbb{R}^d), \quad v^\varepsilon \in L^q(\Omega; \mathbb{R}^d)$$

satisfy the following:

(i) *$u^\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0$;*

(ii) *$v^\varepsilon \rightharpoonup v$ weakly in $L^q(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0$;*

(iii) $\operatorname{div} u^\varepsilon$ *are confined in a compact subset of* $W_{\mathrm{loc}}^{-1,p}(\Omega; \mathbb{R})$;

(iv) $\operatorname{curl} v^\varepsilon$ *are confined in a compact subset of* $W_{\mathrm{loc}}^{-1,q}(\Omega; \mathbb{R}^{d\times d})$.

Then the scalar products of u^ε *and* v^ε *are weakly continuous:*

$$u^\varepsilon \cdot v^\varepsilon \rightarrow u \cdot v$$

in the sense of distributions.

Various variations of Lemma 2.1 for different applications/purposes were developed (see [6, 91] and the references therein).

2.2.2 Hyperbolic systems of conservation laws: Compensated compactness for the artificial viscosity method

Consider (2.5) with artificial viscosity. Assume that there exists a strictly convex entropy function $\eta_*(U), \nabla^2\eta_*(U) > 0$. In many cases, it can be shown that there exists C independent of ε such that

(i) Invariant regions: $\|U^\varepsilon\|_{L^\infty} \leq C$;

(ii) Dissipation estimate: $\|\sqrt{\varepsilon}\partial_x U^\varepsilon\|_{L^2_{\mathrm{loc}}} \leq C$.

In fact, the dissipation estimate is natural as the scalar case, directly from the energy estimate as follows:

$$\varepsilon(\partial_x U^\varepsilon)^{\mathrm{T}}\nabla^2\eta_*(U^\varepsilon)\partial_x U^\varepsilon = -\partial_t\eta_*(U^\varepsilon) - \partial_x q_*(U^\varepsilon) + \varepsilon\partial_{xx}\eta_*(U^\varepsilon).$$

Then, for any $\eta \in C^2$ with entropy flux q, i.e., $\nabla q(U) = \nabla\eta(U)\nabla F(U)$,

$$\partial_t\eta(U^\varepsilon) + \partial_x q(U^\varepsilon) \text{ is compact in } H_{\mathrm{loc}}^{-1}.$$

The compensated compactness arguments can yield the strong convergence of $U^\varepsilon(t, x)$ when the system has strong nonlinearity.

The similar compensated compactness arguments apply to the systems with more general viscosity matrices (2.6) for $\nabla^2\eta_*(U)D(U) \geq c_0 > 0$. Another advantage of this approach is to allow the initial data of large oscillation without bounded variation.

In order to achieve the strong compactness, as first indicated by Tartar [85], combining the div-curl lemma (see [70, 85]) and the Young measure representation theorem (see [1, 3, 85]), we have the following commutation identity for the associated Young measure $\nu = \nu_{(t,x)}(\boldsymbol{\lambda})$ (probability measure) for the sequence $U^\varepsilon(t, x)$:

$$\begin{aligned}
&\langle\nu(\boldsymbol{\lambda}), \eta_1(\boldsymbol{\lambda})q_2(\boldsymbol{\lambda}) - q_1(\boldsymbol{\lambda})\eta_2(\boldsymbol{\lambda})\rangle \\
&= \langle\nu(\boldsymbol{\lambda}), \eta_1(\boldsymbol{\lambda})\rangle\langle\nu(\boldsymbol{\lambda}), q_2(\boldsymbol{\lambda})\rangle - \langle\nu(\boldsymbol{\lambda}), q_1(\boldsymbol{\lambda})\rangle\langle\nu(\boldsymbol{\lambda}), \eta_2(\boldsymbol{\lambda})\rangle
\end{aligned} \tag{2.7}$$

for any entropy pairs $(\eta_j, q_j),\ j = 1, 2$, and

$$\partial_t\langle\nu, \eta(\boldsymbol{\lambda})\rangle + \partial_x\langle\nu, q(\boldsymbol{\lambda})\rangle \leq 0$$

in the sense of distributions for any convex entropy pair $(\eta, q), \nabla^2\eta \geq 0$. Then the main mathematical issue is whether ν is a Dirac measure. The key point is the imbalance of regularity

of the operator in the commutation identity, that is, the operator on the left is more regular than the one on the right due to cancellation when the system has strong nonlinearity. If so, the compactness of $U^\varepsilon(t, x)$ in L^1 follows.

For strict hyperbolicity with $N = 2$, there are two families of entropy pairs determined by two arbitrary functions, which yield an affirmative answer to the issue (see [17, 28, 35, 68, 77, 81]).

Further challenges include nonstrictly hyperbolic systems, viscosity matrices with

$$\nabla^2 \eta_*(U)D(U) \geq 0$$

but not positive definite, and initial data of large oscillation with only energy bounds (without bounded variation or L^∞-uniform bound). We now start with a fundamental example of nonstrictly hyperbolic systems.

2.2.3   Homentropic Euler equations: Compensated compactness for the artificial viscosity method

The homentropic Euler equations take the following form:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\dfrac{m^2}{\rho} + p(\rho) \right) = 0, \end{cases} \tag{2.8}$$

where ρ is the density, $u = \frac{m}{\rho}$ is the fluid velocity that is well-defined when $\rho > 0$, and $p = p(\rho) = \rho^2 e'(\rho)$ is the pressure with internal energy $e(\rho)$.

For a polytropic perfect gas,

$$p(\rho) = \kappa\rho^\gamma, \quad e(\rho) = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}, \tag{2.9}$$

where $\gamma > 1$ is the adiabatic exponent, and the constant κ in the pressure-density relation may be chosen as $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ without loss of generality.

One of the main difficulties for solving (2.8) is that strict hyperbolicity fails when $\rho \to 0$.

An entropy function $\eta(\rho, m)$ is called a weak entropy if $\eta(\rho, m)|_{\rho=0} = 0$. The weak entropy pairs can be represented as

$$\eta^\psi(\rho, \rho u) = \int_{\mathbb{R}} \chi(s)\psi(s)\mathrm{d}s, \quad q^\psi(\rho, \rho u) = \int_{\mathbb{R}} (\theta s + (1-\theta)u)\chi(s)\psi(s)\mathrm{d}s \tag{2.10}$$

for any C^2-test function $\psi(s)$, where $\chi(s)$ is the weak entropy kernel as follows:

$$\chi(s) := [\rho^{2\theta} - (u-s)^2]_+^\lambda, \quad \theta = \frac{\gamma - 1}{2}, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}. \tag{2.11}$$

The following mechanical energy-energy flux pair (η_*, q_*):

$$\eta_*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \quad q_*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + m\left(e(\rho) + \frac{p}{\rho}\right)$$

is a convex entropy pair for (2.8).

Consider the following homentropic Euler equations with artificial viscosity:

$$\begin{cases} \partial_t \rho + \partial_x m = \varepsilon \partial_x^2 \rho, \\ \partial_t m + \partial_x \left(\dfrac{m^2}{\rho} + p(\rho) \right) = \varepsilon \partial_x^2 m. \end{cases} \tag{2.12}$$

It can be shown for (2.12) that there exists $C > 0$, independent of $\varepsilon > 0$, such that

(i) Invariant regions for the L^∞-estimate:

$$0 \le \rho^\varepsilon(t,x) \le C, \quad |m^\varepsilon(t,x)| \le C\rho^\varepsilon(t,x) \quad \text{a.e.};$$

(ii) Dissipation estimate:

$$\sqrt{\varepsilon} \|\partial_x(\rho^\varepsilon, m^\varepsilon)\|_{L^2([0,T] \times \mathbb{R})} \le C,$$

via the mechanical energy pair (η_*, q_*) that is strictly convex for $1 < \gamma \le 2$, and convex for $\gamma > 2$ for which a corresponding weighted dissipation estimate can be obtained.

These estimates yield that, for any C^2 weak entropy pair (η, q),

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \text{ is compact in } H_{\text{loc}}^{-1}.$$

Then the convergence problem for $(\rho^\varepsilon, m^\varepsilon)$ is reduced to the reduction problem for a measure-valued solution $\nu_{t,x}$ as follows.

Problem 2.1 If $\operatorname{supp} \nu_{t,x}$ is bounded, then

$$\nu_{t,x} = \nu_{(\rho(t,x), m(t,x))}, \tag{2.13}$$

that is, $(\rho^\varepsilon(t,x), m^\varepsilon(t,x)) \to (\rho(t,x), m(t,x))$ a.e. (t,x).

This problem was solved by DiPerna [34] for $\gamma = \frac{N+2}{N}$, $N \ge 5$ odd, Ding-Chen-Luo [31–32] and Chen [8] for $\gamma \in (1, \frac{5}{3}]$, Lions-Perthame-Tadmor [64] for $\gamma \ge 3$, Lions-Perthame-Souganidis [63] for $\gamma \in (\frac{5}{3}, 3)$, and Chen-LeFloch [14] for general pressure laws. The key point is to employ effectively the weak entropy pairs in the commutation identity (2.7) for the associated Young measure $\nu_{t,x}$ with compact support.

The convergence of related numerical methods with corresponding numerical viscosity matrices including the Lax-Friedrichs scheme and Godunov scheme was also established in [31–32] (see also [10]).

The isothermal case $\gamma = 1$ was also handled by Huang-Wang [52] (see also [59]).

Some further important problems include the inviscid limit from the compressible Navier-Stokes equations to the compressible Euler equations (see Subsection 2.3) and the existence of global spherically symmetric solutions to the compressible Euler equations (see Subsection 2.4).

2.3 Navier-Stokes equations: Inviscid limit

Consider the following Cauchy problem:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = \varepsilon u_{xx} \end{cases} \tag{2.14}$$

with the initial conditions

$$(\rho, u)|_{t=0} = (\rho_0^\varepsilon(x), u_0^\varepsilon(x)), \quad \lim_{x \to \pm\infty} (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\rho^\pm, u^\pm), \tag{2.15}$$

where $(\rho^\pm, u^\pm)$ are constant end-states with $\rho^\pm > 0$, and the viscosity coefficient $\varepsilon \in (0, \varepsilon_0]$ for some fixed ε_0.

The existence of C^2-solutions $(\rho^\varepsilon, u^\varepsilon)(t, x)$ for large initial data was obtained by Kanel [54] for the same ending states and by Hoff [48] for different ending states.

Inviscid Limit Problem Does the solution sequence $(\rho^\varepsilon, u^\varepsilon)(t, x)$ of (2.14) strongly converge to a solution to the homentropic Euler equations (2.8) when $\varepsilon \to 0$?

This problem was addressed by Gilbarg [40], Hoff-Liu [49], and Gùes-Métivier-Williams-Zumbrun [45] for some physical cases with special structures for which the limit solution contains only one shock.

For the general case, several new difficulties arise, which include

(i) No invariant regions: Only energy norms;

(ii) Direct derivative estimates only partially: $\|\sqrt{\varepsilon}\partial_x u^\varepsilon\|_{L^2_{\mathrm{loc}}} \le C$;

(iii) No a priori bounded support of the measure-valued solution $\nu_{t,x}$.

Nevertheless, the following theorem has been established.

Theorem 2.3 (Chen-Perepelitsa [19]) *Let the initial functions $(\rho_0^\varepsilon, u_0^\varepsilon)$ satisfy*

$$\int_{-\infty}^{\infty} \Phi_*(\rho_0^\varepsilon(x), m_0^\varepsilon(x)) \mathrm{d}x \le E_0 < \infty,$$
$$\int_{-\infty}^{\infty} \left(\varepsilon^2 \frac{|\rho_{0,x}^\varepsilon(x)|^2}{\rho_0^\varepsilon(x)^3} + 2\varepsilon \frac{|\rho_{0,x}^\varepsilon(x) u_0^\varepsilon(x)|}{\rho_0^\varepsilon(x)} + \rho_0^\varepsilon(x)|u_0^\varepsilon(x)| \right) \mathrm{d}x \le E_1 < \infty,$$

where $\Phi_*(\rho, m) = \eta_*(\rho, m) - \eta_*(\overline{\rho}, \overline{m}) - \nabla\eta_*(\overline{\rho}, \overline{m}) \cdot (\rho - \overline{\rho}, m - \overline{m}) \ge 0$ for $\overline{m} = \overline{\rho u}$, $(\overline{\rho}, \overline{u})$ is a pair of smooth monotone functions satisfying $(\overline{\rho}(x), \overline{u}(x)) = (\rho^\pm, u^\pm)$ when $\pm x \ge L_0$ for some large $L_0 > 0$, and both E_0 and E_1 are independent of ε. Let $(\rho^\varepsilon, m^\varepsilon)$, $m^\varepsilon = \rho^\varepsilon u^\varepsilon$, be the solution of the Cauchy problem for the Navier-Stokes equations (2.14) for each fixed $\varepsilon > 0$. Then, when $\varepsilon \to 0$, there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ that converges strongly almost everywhere to a finite-energy solution (ρ, m) to the Cauchy problem for the homentropic Euler equations (2.8) for any $\gamma > 1$.

The strategies for this include the following steps:

(i) Derive the finite-energy bound and higher integrability bound (replacing L^∞ bound).

(ii) Derive a new derivative estimate for $\varepsilon\partial_x\rho^\varepsilon$.

(iii) Show the H^{-1}-compactness of weak entropy dissipation measures only for weak entropy pairs with compactly supported C^2-test functions.

(iv) Prove that any connected component of support of the measure-valued solution $\nu_{t,x}$ must be bounded, which reduces to the case when the support of $\nu_{t,x}$ is bounded as in Subsection 2.2.3.

To achieve these, the following key estimates of solutions to the Navier-Stokes equations are essential: There exist $C_1 > 0$ and $C_2 = C_2(E_0, E_1, K, \gamma, t)$ independent of ε for any compact set $K \subset \mathbb{R}$ such that, for any $t > 0$,

(i) Energy estimate:

$$\int_{-\infty}^{\infty} \Phi_*(\rho^\varepsilon(t,x), m^\varepsilon(t,x)) \, \mathrm{d}x + \int_0^t \int_{-\infty}^{\infty} \varepsilon |u_x^\varepsilon|^2 \, \mathrm{d}x \mathrm{d}\tau \le E_0;$$

(ii) New derivative estimate for the density:

$$\varepsilon^2 \int \frac{|\rho_x^\varepsilon(t,x)|^2}{\rho^\varepsilon(t,x)^3} \mathrm{d}x + \varepsilon \int_0^t \int_{-\infty}^{\infty} (\rho^\varepsilon)^{\gamma-3} |\rho_x^\varepsilon|^2 \, \mathrm{d}x \mathrm{d}\tau \le C_1 (E_0 + E_1);$$

(iii) Higher integrability bound:

$$\int_0^t \int_K (\rho^\varepsilon |u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta} + (\rho^\varepsilon)^{\gamma+1}) \, \mathrm{d}x \mathrm{d}\tau \le C_2.$$

The higher integrability estimate (iii) is motivated by the related work by Lions-Perthame-Tadmor [64] and LeFloch-Westdickenberg [60]. For some related earlier work on the convergence of approximate solutions in the L^p-framework, see [82] for a 2×2 system of elasticity with severe growth conditions, and [60] for the convergence of approximate solutions with full dissipation in the energy norms for the homentropic Euler equations with $\gamma \in (1, \frac{5}{3}]$.

Let $\nu_{t,x}$ be the Young measure determined by the solutions of the Navier-Stokes equations (2.14). Then $\nu_{t,x}$ is confined by

$$\theta(s_2 - s_1)(\overline{\chi(s_1)\chi(s_2)} - \overline{\chi(s_1)}\ \overline{\chi(s_2)})$$
$$= (1 - \theta)(\overline{u\chi(s_2)}\ \overline{\chi(s_1)} - \overline{u\chi(s_1)}\ \overline{\chi(s_2)}) \quad \text{for a.e. } s_1, s_2 \in \mathbb{R},$$

for the entropy kernel $\chi(s) := [\rho^{2\theta} - (u-s)^2]_+^\lambda$ with $\theta = \frac{\gamma-1}{2}$ and $\lambda = \frac{3-\gamma}{2(\gamma-1)}$, where $\overline{f(s)} := \langle \nu_{t,x}, f(s; \rho, u) \rangle$.

The goal is to establish that the Young measure is a Dirac mass in the phase plane for (ρ, m). The new difficulty is now that supp $\nu_{t,x}$ is unbounded in general.

We divide the proof into three cases.

Case 1 $\gamma = 3$. The same argument for the bounded support of $\nu_{t,x}$ applies as in [64]. In this case, $\theta = 1$ and the commutation relation becomes

$$\overline{\chi(s_1)\chi(s_2)} = \overline{\chi(s_1)}\ \overline{\chi(s_2)},$$

which implies $\overline{\chi(s)}^2 = \overline{\chi(s)^2}$ by taking $s_1 = s_2$, that is,

$$\langle \nu_{t,x}, (\chi(s) - \overline{\chi(s)})^2 \rangle = 0 \quad \text{for any } s \in \mathbb{R}.$$

This implies that ν must be a Dirac mass on the set $\{\rho > 0\}$ or be supported completely in the vacuum $V = \{\rho = 0\}$, that is, the measure-valued solution $\nu_{t,x}$ is a Dirac mass (2.13) in the phase plane for (ρ, m).

Case 2 $\gamma > 3$. Let $A := \cup \{(u - \rho^\theta, \rho^\theta + u) : (\rho, u) \in \text{supp} \, \nu\}$. Let $J = (s_-, s_+)$ be any connected component of A. Note that supp $\chi(s) = \{(\rho, u) : u - \rho^\theta \le s \le u + \rho^\theta\}$.

Claim 2.1 *The connected component J is bounded for $\gamma > 3$.*

On the contrary, let $\inf\{s : s \in J\} = -\infty$. Our strategy is to fix M_0 first such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0 + 1)$, then choose sufficiently small $s_1 \leq -2|M_0|$ to reach the contradiction.

To achieve this, two following estimates are essential:

(i) $\int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \leq C(\lambda)|s_1|^\lambda$ for $\lambda < 0$, which is our key new observation.

(ii) $\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} \geq \overline{\chi(s_2)}$ a.e. $s_1, s_2 \in J$, $s_1 < s_2$, by employing Lions-Perthame-Tadmor's argument in [64].

Combining the two estimates, we have

$$\int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \geq \int_{M_0}^{M_0+1} \overline{\chi(s_2)} ds_2 = C(M_0, \lambda) > 0,$$

which implies that, when $s_1 \to -\infty$,

$$0 < C(M_0, \lambda) = \int_{M_0}^{M_0+1} \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} ds_2 \leq C(\lambda)|s_1|^\lambda \to 0.$$

This arrives at the contradiction.

The case when J is unbounded from above can be treated similarly.

This indicates that any connected component J of the support of the Young measure ν is bounded for $\gamma > 3$, which reduces to the Lions-Perthame-Tadmor's case for $\gamma > 3$ in [64].

Case 3 $\gamma \in (1, 3)$. On the contrary, suppose that a connected component J of the support is unbounded from below.

Let $M_0 = \sup\{s : s \in J\} \in (-\infty, \infty]$. Let $s_1, s_2, s_3 \in (-\infty, M_0)$ with $s_1 < s_2 < s_3$. The commutation relation leads to

$$(s_3 - s_1)\overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} = (s_2 - s_1)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + (s_3 - s_2)\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}}. \tag{2.16}$$

Differentiating this equation in s_2 and dividing by $(s_3 - s_1)$, we obtain

$$\overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} = \frac{s_2 - s_1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi'(s_2)}}{\overline{\chi(s_1)}} + \frac{s_3 - s_2}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi'(s_2)}}{\overline{\chi(s_3)}}$$
$$+ \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} - \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}}. \tag{2.17}$$

Our strategy is to take $s_1 \to -\infty$ first and then show that the left-hand side has a smaller order than the right-hand side to arrive at the contradiction.

To do this, we divide the argument into five steps:

(i) Show the estimate

$$\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} \geq 1 \quad \text{for any } s_1, s_3 \in J$$

by employing Lions-Perthame-Tadmor's argument in [64].

(ii) Show that $\overline{\chi(s)} \geq 0$, but is not identically zero, and $\overline{\chi(s)} \to 0$ as $s \to \inf J, \sup J$. This yields that there exists s_2 such that $\overline{\chi'(s_2)} > 0$ and $\overline{\chi(s_2)} > 0$.

(iii) Let $s_3 > s_2$ be the points such that $\overline{\chi(s_3)} > 0$, and let $s_1 \to -\infty$. From the first identity (2.16),

$$\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} = \overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} + o(1) \quad \text{as } s_1 \to -\infty.$$

(iv) Show that $[\chi'(s)]_+ \leq \frac{2\lambda}{s-s_1}\chi(s)$.

(v) From the second equation (2.17), by throwing away the negative terms, we have

$$\overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} \leq \frac{2\lambda+1}{s_3-s_1}\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + o(1),$$

which implies

$$\left(\overline{\chi'(s_2)} - \frac{2\lambda+1}{s_3-s_1}\overline{\chi(s_2)}\right)\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} \leq o(1).$$

This arrives at the contradiction as $s_1 \to -\infty$.

Another different proof was given by LeFloch-Westdickenberg [60] for $1 < \gamma \leq \frac{5}{3}$. The inviscid limit of the viscous shallow water equations to the Saint-Venant system was also established in [20].

2.4 Spherically symmetric solutions to the multidimensional homentropic Euler equations

The homentropic Euler equations for multidimensional compressible fluids take the following form:

$$\begin{cases} \rho_t + \nabla_{\mathbf{x}}(\rho\mathbf{v}) = 0, \\ (\rho\mathbf{v})_t + \nabla_{\mathbf{x}}(\rho\mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}}p = 0, \end{cases} \tag{2.18}$$

where $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\nabla_{\mathbf{x}}$ is the gradient with respect to $\mathbf{x} \in \mathbb{R}^d$, and $\mathbf{v} = (v_1, \cdots, v_d) \in \mathbb{R}^d$ is the velocity. The pressure-density constitutive relation (by scaling) satisfies (2.9).

We seek the spherically symmetric solutions of the following form:

$$\rho(t,\mathbf{x}) = \rho(t,r), \quad \mathbf{v}(t,\mathbf{x}) = u(t,r)\frac{\mathbf{x}}{r}, \quad r = |\mathbf{x}|. \tag{2.19}$$

Then the functions $(\rho, m) = (\rho, \rho u)$ are governed by

$$\begin{cases} \rho_t + m_r + \dfrac{d-1}{r}m = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_r + \dfrac{d-1}{r}\dfrac{m^2}{\rho} = 0. \end{cases} \tag{2.20}$$

For the defocusing case, the existence of expanding spherically symmetric solutions with the following bounds:

$$0 \leq \rho(t,r)^{\frac{\gamma-1}{2}} \leq u(t,r) \leq C < \infty,$$

has been constructed, provided that the initial functions have the same bounds in [9].

For the focusing case, the singularity of imploding self-similar spherically symmetric solutions was discussed in [26, 44, 79, 93]. It is indicated indeed in [78] that there is no BV or L^∞ bound for the imploding solutions in general.

A longstanding open problem is whether the concentration phenomenon occurs at the origin, that is, whether the density ρ develops a measure at the origin. In [21], we developed a method of vanishing artificial viscosity to prove that the vanishing viscosity limit solution does not form concentration at the origin, but has a bounded total energy. More precisely, we construct a sequence of vanishing viscosity solutions to the following initial-boundary problem:

$$
\begin{cases}
\rho_t + m_r + \dfrac{d-1}{r}m = \varepsilon\Big(\rho_{rr} + \dfrac{d-1}{r}\rho_r\Big), \\[2mm]
m_t + \Big(\dfrac{m^2}{\rho} + p_\delta(\rho)\Big)_r + \dfrac{d-1}{r}\dfrac{m^2}{\rho} = \varepsilon\Big(m_{rr} + \dfrac{d-1}{r}m\Big)_r,
\end{cases}
\tag{2.21}
$$

with the appropriate approximate initial data

$$
(\rho, m)|_{t=0} = (\rho_0^\varepsilon(r), m_0^\varepsilon(r)) \to (\rho_0, m_0) \quad \text{a.e. as } \varepsilon \to 0,
\tag{2.22}
$$

and the boundary condition

$$
(\rho_r, m)|_{r=a(\varepsilon)} = (0, 0), \quad (\rho, m)|_{r=b(\varepsilon)} = (\overline{\rho}(\varepsilon), 0),
\tag{2.23}
$$

where (ρ_0, m_0) is the initial data for the spherical symmetric solution to (2.20), $p_\delta(\rho) = \delta\rho^2 + \kappa\rho^\gamma$ with $\delta = \delta(\varepsilon)$, and $a(\varepsilon), b(\varepsilon), \overline{\rho}(\varepsilon)$ and $\delta(\varepsilon)$ are positive with $a(\varepsilon) \to 0$, $b(\varepsilon) \to \infty$, $(\overline{\rho}(\varepsilon), \delta(\varepsilon)) \to (0, 0)$, as well as certain combinations of $(b(\varepsilon), \overline{\rho}(\varepsilon), \delta(\varepsilon))$ tending to 0, as $\varepsilon \to 0$ (see [21]). Then we have the following theorem.

Theorem 2.4 (Chen-Perepelitsa [21]) *Let the initial functions (ρ_0, m_0) for (2.20) satisfy the finite-energy conditions. Then*

(i) *for sufficiently small fixed $\varepsilon > 0$, there exists a global viscous solution $(\rho^\varepsilon, m^\varepsilon)$ to the initial-boundary value problem (2.21)–(2.23) satisfying that, for any compact set $K \subset \mathbb{R}_+$ and $T > 0$, there exists $C_T > 0$ independent of $\varepsilon > 0$ such that, for any $0 < t \le T$,*

$$
\int_{a(\varepsilon)}^{b(\varepsilon)} \Big(\frac{1}{2}\rho^\varepsilon(u^\varepsilon)^2 + \rho^\varepsilon e(\rho^\varepsilon)\Big)(t, r)\, r^{d-1}\mathrm{d}r
$$

$$
+ \varepsilon \int_0^t \int_{a(\varepsilon)}^{b(\varepsilon)} \Big((\rho^\varepsilon)^{\gamma-2}|\rho_r^\varepsilon|^2 + \rho^\varepsilon|u_r^\varepsilon|^2 + \frac{\rho^\varepsilon(u^\varepsilon)^2}{2r^2}\Big) r^{d-1}\mathrm{d}r\mathrm{d}t \le C_T,
$$

$$
\int_0^t \int_K (\rho^\varepsilon|u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta} + (\rho^\varepsilon)^{\gamma+1})\, r^{d-1}\mathrm{d}r\mathrm{d}\tau \le C_T;
$$

(ii) *when $\varepsilon \to 0$, there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ that converges strongly almost everywhere to a finite-energy spherically symmetric solution (ρ, m) to (2.20) for any $\gamma > 1$ with the initial data (ρ_0, m_0).*

The key ingredients are the uniform a priori estimates in (i) in combination with the reduction of the corresponding Young measure discussed in Subsection 2.3.

Recently, we have also solved and/or made progress on several fundamental problems in nonlinear partial differential equations by employing the viscosity method. These include the vanishing viscosity approximation for transonic flow in Chen-Slemrod-Wang [22] (see also Morawetz [67, 69]), and the subsonic-sonic limit of exact/approximate solutions to the full Euler equations for multidimensional steady compressible fluids in Chen-Huang-Wang [13].

3 Weak Continuity and Rigidity of the Gauss-Codazzi-Ricci System and Corresponding Isometric Embeddings

The isometric embedding problem is a longstanding fundamental problem in differential geometry. As is well-known from differential geometry, given a surface, we can compute its metric $\{g_{ij}\}$ and the associated first fundamental form

$$\mathrm{I} = \sum g_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

and its curvatures are determined by the second fundamental form

$$\mathrm{II} = \sum h_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

Then a natural mathematical question is as follows:

Isometric Embedding Problem Given a metric $\{g_{ij}\}$, can we find a surface in the Euclidean space with the given metric $\{g_{ij}\}$?

In other words, we seek a map $\mathbf{r} : \Omega \to \mathbb{R}^N$ such that

$$\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = \sum_{i,j=1}^{N} g_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

in the local coordinates, that is, $\partial_{x^i}\mathbf{r} \cdot \partial_{x^j}\mathbf{r} = g_{ij}$ so that $(\partial_{x^i}\mathbf{r}, \partial_{x^j}\mathbf{r})$, $i \neq j$, in $\mathbb{R}^d$ are linearly independent.

This is an inverse problem, which is a realization question for a given abstract metric $\{g_{ij}\}$. A further question is whether we can produce even more sophisticated surfaces or thin sheets for applications. These questions are truly fundamental, not only in mathematics such as differential geometry and topology, but also in many applications such as the understanding evolution of sophisticated shapes of surfaces or thin sheets in nature including elastic materials, protein folding in biology and algorithmic origami, as well as design and visual arts, among others.

The mathematical study of this problem has a long history, including the early important works by Schlaefli (1873), Darboux (1894), Hilbert (1901), Weyl (1916), Janet (1926–1927) and Cartan (1926–1927) (see also [47] and the references therein). In particular, Nash [75] established the Nash isometric embedding theorem (also called the C^k-embedding theorem, $k \geq 3$) as follows.

Theorem 3.1 *Every n-dimensional Riemannian manifold (analytic or C^k, $k \geq 3$) can be C^k-isometrically imbedded in the Euclidean space $\mathbb{R}^d$ with $d = 2s_n + 4n$ for the compact case*

and $d = (n+1)(3s_n + 4n)$ for the noncompact case, where $s_n = \frac{n(n+1)}{2}$ is the Jenet dimension (see [53]).

The results were further improved with lowerer target dimensions by Gromov [43] with $d = s_n + 2n + 3$ and Günther [46] with $d = \max\{s_n + 2n, s_n + n + 5\}$.

The following further problems are important for applications:

(i) Rigidity of isometric embeddings: Is a weak limit of a sequence of isometric embeddings in some topology still an isometric embedding?

(ii) Lowerest target dimension for global isometric embeddings, which is expected to be the Janet dimension $d = s_n$.

(iii) Optimal or assigned regularity such as $C^{1,1}$, $W^{2,p}$ and BV^1.

The regularity issue is quite sensitive. For example, Efimov's example in [38] indicates that there is no C^2-isometric embedding when $n = 2$ and $d = s_n = 3$.

For $n = 2$ and $d = 3$, the fundamental theorem in differential geometry indicates the following result.

Theorem 3.2 *There exists a surface in $\mathbb{R}^3$ whose first and second fundamental forms are* I *and* II*, if the coefficients $\{g_{ij}\}$ and $\{h_{ij}\}$ of the two given quadratic forms* I *and* II*,* I *being positive definite, satisfy the Gauss-Codazzi system. That is, given $\{g_{ij}\}$, the second fundamental coefficients $\{h_{ij}\}$ are determined by the Codazzi equations (compatibility) as follows:*

$$\begin{cases} \partial_x M - \partial_y L = L\Gamma_{22}^{(2)} - 2M\Gamma_{12}^{(2)} + N\Gamma_{11}^{(2)}, \\ \partial_x N - \partial_y M = -L\Gamma_{22}^{(1)} + 2M\Gamma_{12}^{(1)} - N\Gamma_{11}^{(1)}, \end{cases} \tag{3.1}$$

subject to the following Gauss equation (i.e., the Monge-Ampère type constraint):

$$LN - M^2 = K, \tag{3.2}$$

where

$$L = \frac{h_{11}}{\sqrt{|g|}}, \quad M = \frac{h_{12}}{\sqrt{|g|}}, \quad N = \frac{h_{22}}{\sqrt{|g|}}, \quad |g| = g_{11}g_{22} - g_{12}^2,$$

$\Gamma_{ij}^{(k)}$ *are the Christoffel symbols, depending on g_{ij} up to their first derivatives, and $K(x,y)$ is the Gauss curvature, determined by g_{ij} up to their second derivatives.*

This theorem holds even when $h_{ij} \in L^p$ (see [65–66]). Note that (3.1)–(3.2) is a system of nonlinear PDEs of the mixed elliptic-hyperbolic type, which is determined by the sign of the Gauss curvature K. Surfaces with Gauss curvature of changing signs are very normal in geometry, including tori such as toroidal shells or doughnut surfaces.

Fluid Dynamics Formalism for Isometric Embedding (Chen-Slemrod-Wang [23]). Set $L = \rho v^2 + p, M = -\rho uv, N = \rho u^2 + p$ and $q^2 = u^2 + v^2$. Choose p as the Chaplygin type gas as follows: $p = -\frac{1}{\rho}$.

The Codazzi equations (3.1) become the following balance laws of momentum equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -(\rho v^2 + p)\Gamma_{22}^{(2)} - 2\rho uv\Gamma_{12}^{(2)} - (\rho u^2 + p)\Gamma_{11}^{(2)}, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -(\rho v^2 + p)\Gamma_{22}^{(1)} - 2\rho uv\Gamma_{12}^{(1)} - (\rho u^2 + p)\Gamma_{11}^{(1)}, \end{cases}$$

and the Gauss equation becomes the Bernoulli relation

$$p = -\sqrt{q^2 + K}.$$

Define the sound speed $c^2 = p'(\rho)$. Then $c^2 = \frac{1}{\rho^2} = q^2 + K$.

$c^2 > q^2$ and the "flow" is subsonic when $K > 0$.

$c^2 < q^2$ and the "flow" is supersonic when $K < 0$.

$c^2 = q^2$ and the "flow" is sonic when $K = 0$.

Based on this connection, the existence and continuity of isometric embeddings via compensated compactness and entropy analysis were first addressed in [23].

For higher-dimensional cases, the isometric embeddings of n-dimensional Riemannian manifolds ($n \geq 3$) into $\mathbb{R}^d$ are described by the following Gauss-Codazzi-Ricci system:

Gauss equations

$$h_{ji}^a h_{kl}^a - h_{ki}^a h_{jl}^a = R_{ijkl}; \tag{3.3}$$

Codazzi equations

$$\frac{\partial h_{lj}^a}{\partial x^k} - \frac{\partial h_{kj}^a}{\partial x^l} + \Gamma_{lj}^m h_{km}^a - \Gamma_{kj}^m h_{lm}^a + \kappa_{kb}^a h_{lj}^b - \kappa_{lb}^a h_{kj}^b = 0; \tag{3.4}$$

Ricci equations

$$\frac{\partial \kappa_{lb}^a}{\partial x^k} - \frac{\partial \kappa_{kb}^a}{\partial x^l} - g^{mn}\left(h_{ml}^a h_{kn}^b - h_{mk}^a h_{ln}^b\right) + \kappa_{kc}^a \kappa_{lb}^c - \kappa_{lc}^a \kappa_{kb}^c = 0, \tag{3.5}$$

where $\{R_{ijkl}\}$ is the Riemann curvature tensor, and $\kappa_{kb}^a = -\kappa_{ka}^b$ are the coefficients of the connection form (torsion coefficients) on the normal bundle; the indices a, b, c run from 1 to N, and i, j, k, l, m, n run from 1 to $d \geq 3$.

The Gauss-Codazzi-Ricci system (3.3)–(3.5) has no type, neither purely hyperbolic nor purely elliptic for the general Riemann curvature tensor R_{ijkl} (see [7]). However, we have established the following weak continuity and rigidity of (3.3)–(3.5) and the corresponding embedded surfaces.

Theorem 3.3 (Chen-Slemrod-Wang [24]) *Consider the Gauss-Codazzi-Ricci system* (3.3)–(3.5).

(i) *Let $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ be a sequence of solutions to* (3.3)–(3.5), *which is uniformly bounded in L^p, $p > 2$. Then the weak limit vector field $(h_{ij}^a, \kappa_{lb}^a)$ of the sequence $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ in L^p is still a solution to* (3.3)–(3.5).

(ii) *There exists a minimizer $(h_{ij}^a, \kappa_{lb}^a)$ for the following minimization problem:*

$$\min_S \|(h,\kappa)\|_{L^p(\Omega)}^p := \min_S \int_\Omega \sqrt{|g|}(|h_{ij}h_{ij}|^{\frac{p}{2}} + |\kappa_{lb}\kappa_{lb}|^{\frac{p}{2}}) \mathrm{d}x,$$

where S is the set of weak solutions to (3.3)–(3.5).

This weak continuity and rigidity theorem is a reminiscence of the polyconvexity theory in nonlinear elasticity by Ball [2], for which the rigidity of elastic bodies can be achieved.

The proof of this theorem is based on the following observations on the div-curl structure of the Gauss-Codazzi-Ricci system:

$$\text{div}\,(\overbrace{0,\cdots,0,\underbrace{h_{lj}^{a,\varepsilon}}_{},0,\cdots,-h_{kj}^{a,\varepsilon}}^{k},0,\cdots,0) = R_1,$$

$$\text{curl}\,(h_{1i}^{a,\varepsilon},h_{2i}^{a,\varepsilon},\cdots,h_{di}^{a,\varepsilon}) = R_2,$$

$$\text{div}\,(\overbrace{0,\cdots,0,\underbrace{\kappa_{lb}^{a,\varepsilon}}_{},0,\cdots,-\kappa_{kb}^{a,\varepsilon}}^{k},0,\cdots,0) = R_3,$$

$$\text{curl}\,(\kappa_{1b}^{a,\varepsilon},\kappa_{2b}^{a,\varepsilon},\cdots,\kappa_{db}^{a,\varepsilon}) = R_4,$$

$$\text{div}\,(\overbrace{0,\cdots,0,\underbrace{h_{li}^{b,\varepsilon}}_{},0,\cdots,-h_{ki}^{b,\varepsilon}}^{k},0,\cdots,0) = R_5,$$

$$\text{curl}\,(h_{1i}^{b,\varepsilon},h_{2i}^{b,\varepsilon},\cdots,h_{di}^{b,\varepsilon}) = R_6,$$

$$\text{div}\,(\overbrace{0,\cdots,0,\underbrace{\kappa_{lc}^{b,\varepsilon}}_{},0,\cdots,-\kappa_{kc}^{b,\varepsilon}}^{k},0,\cdots,0) = R_7,$$

$$\text{curl}\,(\kappa_{1c}^{b,\varepsilon},\kappa_{2c}^{b,\varepsilon},\cdots,\kappa_{dc}^{b,\varepsilon}) = R_8,$$

and $R_j,\ j = 1, 2, \cdots, 8$, are confined in a compact set in $H_{\text{loc}}^{-1}(\Omega)$.

Then employing the Murat-Tartar's div-curl lemma directly yields

$$h_{lj}^{a,\varepsilon}h_{ki}^{a,\varepsilon} - h_{kj}^{a,\varepsilon}h_{li}^{a,\varepsilon} \rightharpoonup h_{lj}^{a}h_{ki}^{a} - h_{kj}^{a}h_{li}^{a},$$

$$h_{lj}^{a,\varepsilon}h_{ki}^{b,\varepsilon} - h_{kj}^{a,\varepsilon}h_{li}^{b,\varepsilon} \rightharpoonup h_{lj}^{a}h_{ki}^{b} - h_{kj}^{a}h_{li}^{b},$$

$$\kappa_{kb}^{a,\varepsilon}\kappa_{lc}^{b,\varepsilon} - \kappa_{lb}^{a,\varepsilon}\kappa_{kc}^{b,\varepsilon} \rightharpoonup \kappa_{kb}^{a}\kappa_{lc}^{b} - \kappa_{lb}^{a}\kappa_{kc}^{b},$$

$$\kappa_{kb}^{a,\varepsilon}h_{li}^{b,\varepsilon} - \kappa_{lb}^{a,\varepsilon}h_{ki}^{b,\varepsilon} \rightharpoonup \kappa_{kb}^{a}h_{li}^{b} - \kappa_{lb}^{a}h_{ki}^{b}$$

in the sense of distributions as $\varepsilon \to 0$, which implies the weak continuity and rigidity of (3.3)–(3.5) and the corresponding isometric embeddings.

A compactness framework for the Gauss-Codazzi-Ricci system (3.3)–(3.5) was established in [24] as follows: Given any sequence of approximate solutions to this system which is uniformly bounded in L^2 and has reasonable bounds on the errors made in the approximation (the errors are confined in a compact subset of H_{loc}^{-1}), the approximating sequence has a weakly convergent subsequence whose limit is still a solution to (3.3)–(3.5).

These results indicate that the weak limit of isometrically embedded surfaces is still an isometrically embedded surface in $\mathbb{R}^d$ for any Riemann curvature tensor $\{R_{ijkl}\}$ without restriction, which is the rigidity property of embedded surfaces in geometry.

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