Chin. Ann. Math.
36B(5), 2015, 737–762
DOI: 10.1007/s11401-015-0974-9

Chinese Annals of Mathematics, Series B

Identifiability and Stability of an Inverse Problem Involving a Fredholm Equation*

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(*In Honor of the Scientific Contributions of Professor Luc Tartar*)

Abstract The authors study a linear inverse problem with a biological interpretation, which is modelled by a Fredholm integral equation of the first kind, where the kernel is represented by step functions. Based on different assumptions, identifiability, stability and reconstruction results are obtained.

Keywords Inverse problems, Olfactory system, Kernel determination, Fredholm integral equation, Partial differential equations, Numerical reconstruction
2000 MR Subject Classification 35R20, 45B05, 45Q05, 65M32, 93B05

1 Introduction

In this paper, we study an integral inverse problem coming from the biology of the olfactory system. The transduction of an odor into an electrical signal is accomplished by a depolarizing influx of ions through cyclic-nucleotide-gated (CNG for short) channels in the membrane. Those channels, which form the lateral surface of the cilium, are activated by adenosine 3′, 5′-cyclic monophosphate (cAMP for short). The distribution of the channels should be crucial in determining the kinetics of the neuronal response.

Experimental procedures developed by Kleene and Flannery in the college of medicine (university of Cincinnati) produced data from which the distributions of CNG channels can be inferred by using mathematical and computational procedures were developed by Donald French et al. (see [4]). The techniques for the procedures were developed in [6–9]. We explore the hypothesis that CNG channel distributions can be derived from the experimental current data and known properties of the cilia (a biological inverse problem). To accomplish this, we consider a mathematical model of this experiment proposed by French and Groetsch [5].

French et al. [4] proposed a mathematical model for the dynamics of cAMP concentration in this experiment, consisting of two nonlinear differential equations and a constrained Fredholm

Manuscript received January 1, 2014.
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*This work was partially supported by the Basal-CMM Project, the Fondecyt Grant (No. 1130317, 1111012, 1140773), and “Agence Nationale de la Recherche” Project CISIFS (No. ANR-09-BLAN-0213-02). The first author was partially supported by ECOS-CONICYT C13E05 and Basal-CeBiB.

integral equation of the first kind. The unknowns of the problem are the concentration of cAMP, the membrane potential and the distribution ρ of CNG channels along the length of a cilium. A very natural issue is whether it is possible to recover the distribution of CNG channels along the length of a cilium by only measuring the electrical activity produced by the diffusion of cAMP into cilia. A simple numerical method to obtain estimates of channel distribution was proposed in [4]. Certain computations indicated that this mathematical problem was ill-conditioned.

Later, French and Edwards [3] studied the above inverse problem by using perturbation techniques. A simple perturbation approximation was derived and used to solve the inverse problem, and obtain estimates of the spatial distribution of CNG ion channels. A one-dimensional computer minimization and a special delay iteration were used with the perturbation formulas to obtain approximate channel distributions in the cases of simulated and experimental data. On the other hand, French and Groetsch [5] introduced some simplifications and approximations to the problem, leading to an analytical solution for the inverse problem. A numerical procedure was proposed for a class of integral equations suggested by this simplified model and numerical results were compared with laboratory data.

In this paper, we consider the linear problem proposed in [5], with an improved approximation of the kernel, along with studying the identifiability, stability and numerical reconstruction for the corresponding inverse problem. Precisely, the inverse problem which we are interested in this work consists in determining a positive function $\rho = \rho(x) > 0$ from the measurement of

$$I_m[\rho](t) = J_0 \int_0^L \rho(x) K_m(t,x)\mathrm{d}x \tag{1.1}$$

for $t \in \mathcal{I}$, where $\mathcal{I}$ is a time interval, ρ is the channel distribution, J_0 is a positive constant and the kernel $K_m(t,x)$ is defined by

$$K_m(t,x) = F_m(w(t,x)), \tag{1.2}$$

where $w(t,x)$, defined in (2.14), represents an approximation of the concentration of cAMP $c(t,x)$ defined in (2.3), while F_m, defined in (2.7), is a step function approximation of the Hill function F, given by

$$F(x) = \frac{x^n}{x^n + K_{\frac{1}{2}}^n}\cdot \tag{1.3}$$

In (1.3), the exponent n is an experimentally determined parameter and $K_{\frac{1}{2}} > 0$ is a constant which corresponds to the half-bulk concentration.

Under a strong assumption about the regularity of ρ (namely, ρ is analytic), we obtain in Theorem 3.1 an identifiability result for (1.1) with a single measurement of $I_m[\rho]$ on an arbitrary small interval around zero. The second identifiability result, Theorem 3.2, requires weaker regularity assumptions about ρ (namely, $\rho \in L^2(0,L)$), but it requires the measurement of $I_m[\rho]$ on a large time interval.

Furthermore, in Theorem 3.4, using appropriate weighted norms and Mellin transform (see [10]), we obtain a general stability result for the operator $I_m[\rho]$ for $\rho \in L^2(0,L)$. Using a non-regular mesh for the approximation of F_m, we develop a reconstruction procedure in Theorem 3.5 to recover ρ from I_m. Additionally, for this non-regular mesh, a general stability result for a large class of norms is rigorously established in Theorem 3.6.

2 Setting the Problem

In this section, we set the mathematical model related to the inverse problem arising in olfaction experimentation.

The starting point is the linear model introduced in [5]. As already mentioned, a nonlinear integral equation model was developed in [4] to determine the spatial distribution of ion channels along the length of frog olfactory cilia. The essential nonlinearity in the model arises from the binding of the channel activating ligand to the cyclic-nucleotide-gated ion channels as the ligand diffuses along the length of the cilium. We investigate a linear model for this process, in which the binding mechanism is neglected, leading to a particular type of linear Fredholm integral equations of the first kind with a diffusive kernel. The linear inverse problem consists in determining $\rho = \rho(x) > 0$ from the measurement of

$$I[\rho](t) = J_0 \int_0^L \rho(x) K(t,x) \mathrm{d}x, \quad t \geq 0, \tag{2.1}$$

where the kernel is

$$K(t,x) = F(c(t,x)), \tag{2.2}$$

with F being given by (1.3) and c denoting the concentration of cAMP, which is governed by the following initial boundary value problem:

$$\begin{aligned} &\frac{\partial c}{\partial t} - D\frac{\partial^2 c}{\partial x^2} = 0, \quad t > 0,\ x \in (0,L), \\ &c(t,0) = c_0, \quad t > 0, \\ &\frac{\partial c}{\partial x}(t,L) = 0, \quad t > 0, \\ &c(0,x) = 0, \quad x \in (0,L). \end{aligned} \tag{2.3}$$

The (unknown) function ρ is the ion channel density function, and c is the concentration of a channel activating ligand that is diffusing from left to right in a thin cylinder (the interior of the cilium) of length L with a diffusivity constant D. $I[\rho](t)$ is a given total transmembrane current, the constant J_0 has units of current/length, and c_0 is the maintained concentration of cAMP at the open end of the cylinder (while $x = L$ is considered as the closed end).

We note that (2.1) is a Fredholm integral equation of the first kind. The associated inverse problem is in general ill-posed. For instance, if K is sufficiently smooth, then the operator defined above is compact from $L^p(0,L)$ to $L^p(0,T)$ for $1 < p < \infty$. Even if the operator I is injective, its inverse will not be continuous. Indeed, if I is compact and I^{-1} is continuous, then it follows that the identity map in $L^p(0,L)$ is compact, a property which is clearly false.

In what follows, we consider a simplified version of the above problem under more general assumptions than those in [5]. More precisely, let us consider the constants J_0, c_0 and D introduced above and a fixed integer $m \in \mathbb{N}$. Then we introduce the approximate total current

$$I_m[\rho](t) = J_0 \int_0^L \rho(x) K_m(t,x) \mathrm{d}x, \quad t \geq 0, \tag{2.4}$$

where the kernel K_m is defined as

$$K_m(t,x) = F_m\Big(c_0 \operatorname{erfc}\Big(\frac{x}{2\sqrt{Dt}}\Big)\Big). \tag{2.5}$$

In (2.5), "erfc" denotes the complementary error function

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\tau^2)\mathrm{d}\tau. \tag{2.6}$$

We note that when L is large, $c_0 \operatorname{erfc}\big(\frac{x}{2\sqrt{Dt}}\big)$ provides an approximation of the solution of (2.3). The function F_m is a step function defined by

$$F_m(x) = F(c_0)\sum_{j=1}^{m} a_j H(x-\alpha_j), \quad \forall x \in [0, c_0] \tag{2.7}$$

with F as in (1.3). H is the Heaviside unit step function, that is,

$$H(u) = \begin{cases} 1, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0. \end{cases} \tag{2.8}$$

Finally, the positive constants $\{a_j\}_{j=1}^m$ and $\{\alpha_j\}_{j=1}^m$ satisfy

$$\sum_{j=1}^{m} a_j = 1, \quad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < c_0, \tag{2.9}$$

and hence $\{\alpha_j\}_{j=1}^m$ defines a partition of the interval $(0, c_0)$.

If we choose $\{a_j\}_{j=1}^m$ such that F_m is an approximation of Hill's function F on the interval $[0, c_0]$, i.e.,

$$F(x) \simeq F_m(x) = F(c_0)\sum_{j=1}^{m} a_j H(x-\alpha_j), \quad \forall x \in [0, c_0], \tag{2.10}$$

then

$$K_m \simeq K.$$

Therefore, we can view the functional I_m in (2.4) as an approximation of the functional I in (2.1).

Now, we introduce the operator (used thereafter)

$$\Phi_m[\varphi](t) = \sum_{j=1}^{m} a_j \varphi(h_j(t)), \quad \forall t \geq 0, \tag{2.11}$$

where $h_j(s) = \min\{L, \beta_j s\}$ with

$$\beta_j = 2\sqrt{D}\operatorname{erfc}^{-1}\Big(\frac{\alpha_j}{c_0}\Big) \quad \text{for } j = 1, \cdots, m. \tag{2.12}$$

Thus, we have the following useful relation:

$$I_m[\rho](t) = J_0 F(c_0)\Phi_m[\varphi](\sqrt{t}), \quad t \geq 0 \tag{2.13}$$

with

$$\varphi(x) = \int_0^x \rho(\tau)\mathrm{d}\tau.$$

Indeed, if we define

$$w(t,x) = c_0 \,\mathrm{erfc}\Big(\frac{x}{2\sqrt{Dt}}\Big), \tag{2.14}$$

and put together (2.4)–(2.5) and (2.7), we obtain

$$\begin{aligned} I_m[\rho](t) &= J_0 \int_0^L \rho(x) K_m(t,x)\mathrm{d}x \\ &= J_0 F(c_0) \sum_{j=1}^m a_j \int_0^L \rho(x) H(w(t,x) - \alpha_j)\mathrm{d}x \\ &= J_0 F(c_0) \sum_{j=1}^m a_j \int_{G_j(t)\cap(0,L)} \rho(x)\mathrm{d}x \end{aligned} \tag{2.15}$$

with $G_j(t) := \{x \in \mathbb{R} : w(t,x) \geq \alpha_j\}$. Since the "erfc" function is decreasing, we see that

$$G_j(t) = [0, \beta_j\sqrt{t}] \tag{2.16}$$

with $\{\beta_j\}_{j=1}^m$ as in (2.12) (note that $\beta_1 > \beta_2 > \cdots > \beta_m$). Thus, we have

$$I_m[\rho](t) = J_0 F(c_0)\Big(\sum_{j=1}^m a_j \int_0^{h_j(\sqrt{t})} \rho(x)\mathrm{d}x\Big). \tag{2.17}$$

Using the definition of Φ_m in (2.11), we obtain (2.13).

Clearly, Φ_m is linear, and it follows from (2.9) that $\Phi_m(1) = 1$, and that for any $f \in L^\infty(0,L)$ it holds that

$$\|\Phi_m[f]\|_{L^\infty(0,L/\beta_m)} \leq \|f\|_{L^\infty(0,L)}.$$

Furthermore, for any $f \in C^0([0,L])$ with $f(L) = 0$, we have

$$\|\Phi_m[f]\|_{L^p(0,L/\beta_m)} \leq \Big(\sum_{j=1}^m a_j \beta_j^{-\frac{1}{p}}\Big) \|f\|_{L^p(0,L)}, \quad 1 \leq p < \infty. \tag{2.18}$$

Note that the operator Φ_m is well defined on $C^0([0,L])$. Therefore, using (2.18) and the fact that the set $\{f \in C^0([0,L] : f(L) = 0\}$ is dense in $L^p(0,L)$, we can extend the operator Φ_m to $L^p(0,L)$ for all $1 \leq p < +\infty$.

Finally, we introduce some notations. We set

$$L_k = \frac{L}{\beta_k} \quad \text{for } k = 1, \cdots, m, \; L_0 = 0, \tag{2.19}$$

and for any $\gamma > 0$, we introduce the following weighted norms:

$$\begin{aligned} \|f\|_{0,\gamma,b} &= \|\sigma_\gamma f\|_{L^2(0,b)}, \\ \|f\|_{1,\gamma,b} &= \|\sigma_\gamma f\|_{H^1(0,b)}, \\ \|f\|_{-1,\gamma,b} &= \|\sigma_\gamma f\|_{H^{-1}(0,b)} \end{aligned}$$

with $\sigma_\gamma(x) = x^\gamma$.

3 Main Results

In this section, we present the main results in this paper. We begin with studying the functional Φ_m defined in (2.11). It is worth noticing with (2.13) that the identifiability for Φ_m is equivalent to the identifiability for I_m.

Firstly, we discuss some identifiability results for the operator Φ_m. We begin with the analytic case.

Theorem 3.1 (Identifiability for Analytic Functions) *Let* $\varphi : (-\varepsilon, L+\varepsilon) \to \mathbb{R}$ *be an analytic function satisfying*

$$\Phi_m[\varphi](t) = 0, \quad \forall t \in (0, \delta), \tag{3.1}$$

where Φ_m *is defined in* (2.11), *and* ε *and* δ *are some positive numbers. Then* $\varphi \equiv 0$ *in* $[0, L]$.

The second identifiability result requires less regularity for φ, provided that a measurement on a sufficiently large time interval is available.

Theorem 3.2 *Let* $\varphi : [0, L] \to \mathbb{R}$ *be a given continuous function satisfying*

$$\Phi_m[\varphi](t) = 0, \quad \forall t \in [0, L_m], \tag{3.2}$$

where Φ_m *is defined in* (2.11). *Then* $\varphi \equiv 0$ *in* $[0, L]$.

Remark 3.1 Theorem 3.2 is actually true for any function $\varphi : [0, L] \to \mathbb{R}$ satisfying (3.2).

The proof of Theorem 3.2, which is based only upon algebraic arguments, gives us an idea about how the kernel could be reconstructed and how one can envision a numerical algorithm.

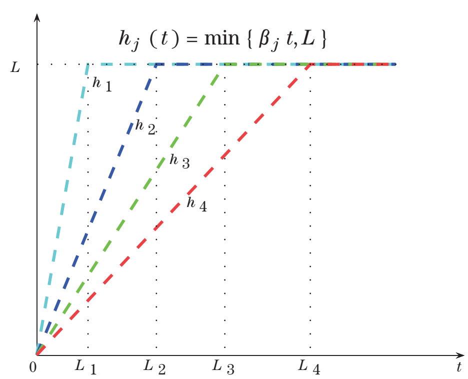

Figure 1 With $m = 4$, we plot the functions h_j on the interval $[0, L_m]$.

Let us give the main ideas in the proof of Theorem 3.2. Recall that the h_j's (see Figure 1) are the functions involved in the definition of Φ_m in (2.11). Note first that $\Phi_m[\varphi](0) = \varphi(0)$ and $\Phi_m[\varphi](t) = \varphi(L)$ for all $t \geq L_m$. Thus, using (3.2), we see that φ vanishes on $\{0, L\}$. Next, we observe that for $t \in [L_{m-1}, L_m)$, we have $\Phi_m[\varphi](t) = a_m\varphi(\beta_m t) + C\varphi(L)$, where $C = \sum\limits_{j=1}^{m-1} a_j$. It follows that φ vanishes in $[\lambda L, L] \cup \{0\}$, where $\lambda = \frac{\beta_m}{\beta_{m-1}} < 1$. Applying the same argument in

$[L_{m-1}, L_m)$, $[L_{m-2}, L_{m-1})$, etc., we can "increase" the set where φ is known to be zero. Note that, in general, it can not be done directly on $[L_{m-2}, L_{m-1})$.

Indeed, let us consider the case when $m = 4$ (see again Figure 1), and assume that φ vanishes on $[\lambda L, L] \cup \{0\}$, where $\lambda = \frac{\beta_4}{\beta_3} < 1$. For $t \in [L_2, L_3)$, we have

$$\Phi_4[\varphi](t) = a_4\varphi(\beta_4 t) + a_3\varphi(\beta_3 t) + C\varphi(L),$$

where $C = \sum_{j=1}^{2} a_j$, so that, using (2.11) and $\varphi(L) = 0$, we obtain

$$0 = a_4\varphi(\beta_4 t) + a_3\varphi(\beta_3 t), \quad \forall t \in [L_2, L_3),$$

i.e.,

$$0 = a_4\varphi(\lambda\tau) + a_3\varphi(\tau), \quad \forall \tau \in [\beta_3 L_2, L).$$

Therefore, if $\lambda_1 = \frac{\beta_3}{\beta_2} \geq \lambda$, the set $[\lambda_1 L, L)$ is contained in $[\lambda L, L] \cup \{0\}$, and we infer that φ vanishes in $[\beta_4 L_2, \lambda L) \cup [\lambda L, L] \cup \{0\}$. The same argument can be applied to the following interval, namely $[L_1, L_2)$. The above procedure suggests how the reconstruction process could be carried out, but under the condition

$$\frac{\beta_4}{\beta_3} \leq \frac{\beta_3}{\beta_2},$$

which is a restriction on the mesh defined in (2.9).

The corresponding identifiability results for the operator I_m are as follows.

Corollary 3.1 (Identifiability for Analytic Functions) *Let $\rho : (-\varepsilon, L+\varepsilon) \to \mathbb{R}$ be an analytic function satisfying*

$$I_m[\rho](t) = 0, \quad \forall t \in (0, \delta), \tag{3.3}$$

where I_m is defined in (2.4)*, and ε and δ are some positive numbers. Then $\rho \equiv 0$ in $[0, L]$.*

Corollary 3.2 *Let $\rho : [0, L] \to \mathbb{R}$ be a given function in $L^2(0, L)$ such that*

$$I_m[\rho](t) = 0, \quad \forall t \in [0, L_m^2]. \tag{3.4}$$

where I_m is defined in (2.4)*. Then $\rho \equiv 0$ in $[0, L]$.*

Corollaries 3.1–3.2 follow at once from Theorems 3.1–3.2 by letting

$$\varphi(x) = \int_0^x \rho(\tau)\mathrm{d}\tau.$$

Let us now proceed to the continuity and stability results.

Theorem 3.3 *Let $\varphi \in H^1(0, L)$ be a given function. Then there exists a constant $\widetilde{C}_1 > 0$ such that*

$$\|\Phi_m[\varphi]\|_{H^1(0,L_m)} \leq \widetilde{C}_1 \|\varphi\|_{H^1(0,L)}, \tag{3.5}$$

where $\widetilde{C}_1$ depends only on L, β_1, β_m and Φ_m given by (2.11)*.*

We are now in a position to state our first main result. Firstly, we define the function

$$\Lambda_m^\gamma(s) = \Big|\sum_{j=1}^{m} a_j \beta_j^{-(\frac{1}{2}+\gamma-\mathrm{i}s)}\Big|, \tag{3.6}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

Theorem 3.4 *Let $\varphi \in C([0, L])$ be a given function. Then there exists a constant $\gamma_0 \in \mathbb{R}$ such that for any $\gamma > \gamma_0$,*

$$C_\gamma \left\|\varphi(\cdot) - \varphi(L)\right\|_{0,\gamma,L} \le \left\|\Phi_m[\varphi](\cdot) - \Phi_m[\varphi](L_m)\right\|_{0,\gamma,L_m} \tag{3.7}$$

with

$$C_\gamma := \inf_{s\in\mathbb{R}} \Lambda_m^\gamma(s) > 0,$$

and Φ_m is given by (2.11).

It is worth noting that (3.7) can be viewed as an inverse inequality of (2.18) for $p = 2$ and functions $\varphi \in \{f \in C([0, L]); f(L) = 0\}$, and it can also be regarded as a stability estimate for the functional Φ_m. Its proof involves some properties of Mellin transform. Hereafter, we refer to γ_0 as the smallest number such that

$$C_\gamma > 0, \quad \forall \gamma > \gamma_0.$$

Next, we present a continuity result for the operator I_m.

Corollary 3.3 *Let $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$. Then, for $\gamma \ge \frac{3}{4}$, there exists a positive constant $C_1 > 0$, such that*

$$\left\|I_m[\rho]\right\|_{1,\gamma,L_m^2} \le C_1 \left\|\rho\right\|_{L^2(0,L)}, \tag{3.8}$$

where C_1 depends only on $L, \alpha_1, \alpha_{m-1}, \alpha_m, a_m$ and γ.

Besides, we present a stability result for the operator I_m.

Corollary 3.4 *Let $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$. Then, for any $\gamma > \max\left\{\gamma_0, \frac{3}{4}\right\}$, there exists a positive constant $C_2 > 0$ such that*

$$\left\|\rho\right\|_{-1,\gamma+1,L} \le C_2 \left\|I_m[\rho]\right\|_{1,\frac{\gamma}{2}-\frac{1}{4},L_m^2}, \tag{3.9}$$

where C_2 depends only on $L, C_\gamma > 0$ and γ.

Corollaries 3.3 and 3.4 are consequences of Theorems 3.3 and 3.4, respectively.

Even if the proof of Theorem 3.2 is provided for any choice of the partition $\{\alpha_j\}_{j=1}^m$ of $[0, c_0]$, its proof can be considerably simplified in the special case when

$$\alpha_j = c_0 \mathrm{erfc}\Big(\frac{\beta_0\beta^j}{2\sqrt{D}}\Big), \quad j = 1, \cdots, m, \tag{3.10}$$

with $\beta \in (0, 1)$ and $\beta_0 > 0$ being constants. Note that the corresponding mesh is non-regular.

In what follows, I_m and Φ_m are denoted by $\widetilde{I}_m$ and $\widetilde{\Phi}_m$, respectively, when α_j is given by (3.10).

For the reconstruction, we introduce the function

$$g(t) = \frac{\widetilde{I}_m[\rho]\big(\frac{t^2}{\beta_0^2}\big) - \widetilde{I}_m[\rho](L_m^2)}{J_0 F(c_0)}, \quad \forall t \in [0, \beta_0 L_m). \tag{3.11}$$

As mentioned in the introduction, we look for a reconstruction algorithm and a numerical scheme to recover function ρ from the measurement of $\widetilde{I}_m[\rho]$. We begin by recovering $\widetilde{\varphi}\colon [0, L] \to \mathbb{R}$, which satisfies

$$\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) = g(t), \quad \forall t \in [0, \beta_0 L_m). \tag{3.12}$$

Next, we define functions $\varphi_1, \varphi_2, \cdots, \varphi_m$ by means of the following induction formulae:

$$\varphi_1(x) = \begin{cases} \dfrac{1}{a_m} g\Big(\dfrac{x}{\beta^m}\Big), & \text{if } x \in [\beta L, L), \\ 0, & \text{otherwise} \end{cases} \tag{3.13}$$

and

$$\varphi_{k+1}(x) = \begin{cases} \dfrac{1}{a_m}\Big(g\Big(\dfrac{x}{\beta^m}\Big) - \displaystyle\sum_{j=1}^{k} a_{m-k-1+j}\varphi_j\Big(\dfrac{\beta^j x}{\beta^{k+1}}\Big)\Big), & x \in [\beta^{k+1}L, \beta^k L), \\ 0, & \text{otherwise} \end{cases} \tag{3.14}$$

for $k = 1, \cdots, m-1$. Furthermore, for $k \geq m$, we define

$$\varphi_{k+1}(x) = \begin{cases} \dfrac{1}{a_m}\Big(g\Big(\dfrac{x}{\beta^m}\Big) - \displaystyle\sum_{j=1}^{m-1} a_j \varphi_{j+k-m+1}\Big(\dfrac{\beta^j x}{\beta^m}\Big)\Big), & x \in [\beta^{k+1}L, \beta^k L), \\ 0, & \text{otherwise.} \end{cases} \tag{3.15}$$

With the above definitions, we have the following reconstruction result.

Theorem 3.5 *Let ρ be a function in $C^0([0, L])$, g be defined as in* (3.11)*, and $\{\varphi_j\}_{j\geq 1}$ be given by* (3.13)–(3.15)*. Then the function $\widetilde{\varphi}$ defined by*

$$\widetilde{\varphi}(x) = \begin{cases} \displaystyle\sum_{j=1}^{+\infty} \varphi_j(x), & \textit{if } x \in (0, L], \\ g(0), & \textit{if } x = 0 \end{cases} \tag{3.16}$$

is well defined and satisfies

$$\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) = g(t), \quad \forall t \in [0, \beta_0 L_m]. \tag{3.17}$$

Furthermore, ρ satisfies

$$\int_0^x \rho(z)\mathrm{d}z = \widetilde{\varphi}(x) + \frac{\widetilde{I}_m[\rho](L_m^2)}{J_0 F(c_0)}, \quad \forall x \in [0, L]. \tag{3.18}$$

Theorem 3.5 provides an explicit reconstruction procedure for both operators $\widetilde{\Phi}_m$ and $\widetilde{I}_m$, and therefore a numerical algorithm for the reconstruction.

The previous reconstruction procedure gives us the possibility to obtain a sharper stability result. We shall provide a stability result for $\widetilde{\Phi}_m$ in terms of a quite general norm.

We consider a family of norms $\|\cdot\|_{[a,b)}$ for functions $f : [a,b) \to \mathbb{R}$, where $0 \le a < b < \infty$, which enjoys the following properties:

(i) $\|f\|_{[a,b)} < \infty$ for any $f \in W^{1,1}(a,b)$;

(ii) if $[a_1,b_1) \subset [a,b)$, then

$$\|f\|_{[a_1,b_1)} \le \|f\|_{[a,b)}\,; \tag{3.19}$$

(iii) for any $\lambda > 0$, there exists a positive constant $C(\lambda)$ such that

$$\|g_\lambda\|_{[\lambda a,\lambda b)} \le C(\lambda)\,\|f\|_{[a,b)}\,, \tag{3.20}$$

where $g_\lambda(x) = f\left(\frac{x}{\lambda}\right)$, and $C(\cdot)$ is a nondecreasing function with $C(1) = 1$.

A natural family of norms fulfilling (i)–(iii) is that of L^p norms, where $1 \le p \le +\infty$. Indeed, (i)–(ii) are obvious, and (iii) holds with

$$C(\lambda) = \begin{cases} \lambda^{\frac{1}{p}}, & \text{if } p \in [1,+\infty), \\ 1, & \text{if } p = \infty. \end{cases}$$

Another family of norms fulfilling (i)–(iii) is the family of BV-norms

$$\|f\|_{BV(a,b)} = \|f\|_{L^\infty(a,b)} + \sup_{a \le x_1 < \cdots < x_k < b} \sum_{j=1}^{k} |f(x_k) - f(x_{k-1})|\,. \tag{3.21}$$

Here, we can pick $C(\lambda) = 1$ (note that $W^{1,1}(a,b) \subset BV(a,b)$ (see, e.g., [1])). These kinds of norms are adapted to functions with low regularity, as, e.g., step functions. The second main result in this paper is the following stability result.

Theorem 3.6 *Let $\rho \in C^0([0,L])$ be a function and let a family of norms satisfy conditions* (i)–(iii)*. Then, we have for all $k \ge 0$,*

$$\|\varphi(\cdot) - \varphi(L)\|_{[\beta^{k+1}L,\beta^k L)} \le C(\beta_0)\frac{C(\beta^m)}{a_m^{k+1}}\|\widetilde{\Phi}_m[\varphi](\cdot) - \widetilde{\Phi}_m[\varphi](L_m)\|_{[\beta^{k+1}L_m,L_m)}, \tag{3.22}$$

where $\varphi(x) = \int_0^x \rho(\tau)\mathrm{d}\tau$.

Theorem 3.6 shows in particular that the value of φ in the interval $[\beta^{k+1}L, \beta^k L)$ depends on the value of $\widetilde{\Phi}_m[\varphi]$ in the interval $[\beta^{k+1}L_m, L_m)$, a property which is closely related to the nature of the reconstruction procedure.

4 Proof of Identifiability Results

This section is devoted to proving the identifiability results for the operator Φ_m.

Proof of Theorem 3.1 Let φ be an analytic function such that

$$\Phi_m[\varphi](t) = \sum_{j=1}^{m} a_j \varphi(h_j(t)) = 0, \quad \forall t \in (0,\delta).$$

Then, taking $t \in (0, \min\{\delta, L_1\})$ and using the fact that

$$L_0 < L_1 < \cdots < L_m, \tag{4.1}$$

we see that $h_j(t) = \beta_j t$, $j = 1, \cdots, m$. Then, we have

$$\sum_{j=1}^{m} a_j \varphi(\beta_j t) = 0, \quad t \in (0, \min\{\delta, L_1\}).$$

If we derive the above expression and evaluate it at zero, we obtain

$$\varphi^{(k)}(0)\Big(\sum_{j=1}^{m} a_j (\beta_j)^k\Big) = 0, \quad \forall k \geq 0,$$

where $\varphi^{(k)}(0)$ denotes the k-th derivative of φ at zero. Since a_j, β_j are positive, we have that $\sum\limits_{j=1}^{m} a_j(\beta_j)^k > 0$. Therefore $\varphi^{(k)}(0) = 0$ for all $k \geq 0$, and hence $\varphi \equiv 0$. This proves the identifiability for Φ_m in the case of analytic functions.

To prove Theorem 3.2, we need some technical lemmas.

Lemma 4.1 *Let $f, g : [0, L] \to \mathbb{R}$ be functions, and let $s, \alpha_0 \in [0, 1)$ and $\lambda \in (0, 1)$ be numbers such that*

$$f(\tau) + g(\lambda\tau) = 0, \quad \forall \tau \in [sL, L) \tag{4.2}$$

and

$$f(\tau) = 0, \quad \forall \tau \in [\alpha_0 L, L). \tag{4.3}$$

Then

$$g(\tau) = 0, \quad \forall \tau \in [\alpha_1 L, \lambda L), \tag{4.4}$$

where $\alpha_1 = \lambda \max\{s, \alpha_0\}$.

Lemma 4.1 is a direct consequence of (4.2)–(4.3).

Lemma 4.2 *Let $f : [0, L] \to \mathbb{R}$ be a function, and let $s, \alpha_0 \in [0, 1)$ and $\lambda \in (0, 1)$ be some numbers such that*

$$f(\tau) = 0, \quad \forall \tau \in [\widetilde{\alpha}_k L, L), \ \forall k \geq 1, \tag{4.5}$$

where

$$\widetilde{\alpha}_k = \lambda \max\{s, \widetilde{\alpha}_{k-1}\}, \quad \forall k \geq 1 \tag{4.6}$$

with $\widetilde{\alpha}_0 = \alpha_0$.

Then, if $s > 0$,

$$f(\tau) = 0, \quad \forall \tau \in [s\lambda L, L),$$

and if $s = 0$,

$$f(\tau) = 0, \quad \forall \tau \in (0, L).$$

Proof To prove the above lemma, we need to consider the following two cases: $s = 0$ and $s > 0$.

If $s > 0$, we claim that there exists k_0 such that $\widetilde{\alpha}_{k_0} < s$. Otherwise, if $\widetilde{\alpha}_k \geq s,\ \forall k \geq 0$, replacing that in (4.6), we have

$$\widetilde{\alpha}_{k+1} = \lambda\widetilde{\alpha}_k,$$

and hence $\widetilde{\alpha}_k = \widetilde{\alpha}_0\lambda^k \to 0$, which is impossible, for $s > 0$.

Using (4.6), the desired result follows, since

$$\widetilde{\alpha}_k = \lambda s, \quad \forall k > k_0.$$

Now, if $s = 0$, replacing it in (4.6), we obtain

$$\widetilde{\alpha}_k = \alpha_0\lambda^k.$$

Then, using (4.5), we have

$$f(\tau) = 0, \quad \forall \tau \in (0, L),$$

which completes the proof.

Lemma 4.3 *Let $f : [0, L] \to \mathbb{R}$ be a function, and let $s, \alpha_0 \in [0, 1)$, $\lambda_1, \cdots, \lambda_n \in (0, 1)$ and $a_k > 0$, $k = 0, \cdots, n$ be some numbers such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n \geq \alpha_0$, and*

$$a_0 f(t) + \sum_{j=1}^{n} a_j f(\lambda_j t) = 0, \quad \forall t \in [sL, L) \tag{4.7}$$

and

$$f(\tau) = 0, \quad \forall \tau \in [\alpha_0 L, L). \tag{4.8}$$

Then

$$f(\tau) = 0, \quad \forall \tau \in [\overline{\alpha} L, L), \tag{4.9}$$

where $\overline{\alpha} = \lambda_n s$.

Proof We prove this result by induction on n.

Case $n = 1$ In this case, from (4.7), we have the following equations:

$$\begin{aligned} &a_0 f(t) + a_1 f(\lambda_1 t) = 0, \quad \forall t \in [sL, L), \\ &f(\tau) = 0, \quad \forall \tau \in [\alpha_0 L, L), \end{aligned} \tag{4.10}$$

and $\alpha_0 \leq \lambda_1$. Then, applying Lemma 4.1 with $g = f$, we get

$$f(\tau) = 0, \quad \forall \tau \in [\alpha_1 L, \lambda_1 L),$$

where $\alpha_1 = \lambda_1 \max\{s, \alpha_0\}$, and thus

$$f(\tau) = 0, \quad \forall \tau \in [\alpha_1 L, L)$$

for $\alpha_0 \leq \lambda_1$.

If $\alpha_0 = 0$, we obtain the desired result

$$f(\tau) = 0, \quad \forall \tau \in [\lambda_1 sL, L).$$

On the other hand, when $\alpha_0 > 0$, we can apply Lemma 4.1 again with α_0 replaced by α_1, since we have

$$\begin{aligned} &a_0 f(t) + a_1 f(\lambda_1 t) = 0, \quad \forall t \in [sL, L), \\ &f(\tau) = 0, \quad \forall \tau \in [\alpha_1 L, L) \end{aligned}$$

and $\alpha_1 \leq \lambda_1$. Thus, we get by induction on $k \geq 0$

$$f(\tau) = 0, \quad \forall \tau \in [\alpha_k L, L),\ \forall k \geq 1, \tag{4.11}$$

where

$$\alpha_k = \lambda_1 \max\{s, \alpha_{k-1}\}, \quad \forall k \geq 1. \tag{4.12}$$

Note that, if $s = 0$, letting $t = 0$ in (4.10) yields $f(0) = 0$. Using Lemma 4.2 with (4.11)–(4.12), we conclude that

$$f(\tau) = 0, \quad \forall \tau \in [\lambda_1 sL, L),$$

which completes the case $n = 1$.

Case $n + 1$ Assume that this lemma is proved up to the value n, and let us prove it for the value $n + 1$.

Assume a given function $f : [0, L] \to \mathbb{R}$ and some numbers $s, \alpha_0 \in [0, 1)$, $a_k > 0$ for $0 \leq k \leq n+1$, $\lambda_1, \cdots, \lambda_{n+1} \in (0, 1)$ with $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_{n+1} \geq \alpha_0$, such that

$$a_0 f(t) + \sum_{j=1}^{n+1} a_j f(\lambda_j t) = 0, \quad \forall t \in [sL, L) \tag{4.13}$$

and

$$f(\tau) \equiv 0, \quad \forall \tau \in [\alpha_0 L, L). \tag{4.14}$$

Then we aim to prove that

$$f(\tau) = 0, \quad \forall \tau \in [\lambda_{n+1} sL, L).$$

We introduce the function

$$\psi(\tau) = \sum_{j=1}^{n+1} a_j f\Big(\frac{\lambda_j}{\lambda_1}\tau\Big) = a_1 f(\tau) + \sum_{j=2}^{n+1} a_j f(\widetilde{\lambda}_j \tau),$$

where $\widetilde{\lambda}_j = \frac{\lambda_j}{\lambda_1}$, $j = 2, \cdots, n+1$.

Then, using (4.14), we have

$$\psi(\tau) = 0, \quad \forall \tau \in \Big[\lambda_1 \frac{\alpha_0}{\lambda_{n+1}} L, L\Big). \tag{4.15}$$

On the other hand, from (4.13), we have

$$a_0 f(\tau) + \psi(\lambda_1 \tau) = 0, \quad \forall \tau \in [sL, L).$$

Then, from (4.14) and Lemma 4.1 with $g = \psi$, we conclude

$$\psi(\tau) = 0, \quad \forall \tau \in [\lambda_1 \max\{\alpha_0, s\} L, \lambda_1 L).$$

Next, we set $s_1 = \lambda_1 \max\{\alpha_0, s\} \in [0, 1)$. Using (4.15), we have $\psi \equiv 0$ on $[s_1 L, \lambda_1 L) \cup [\lambda_1 \frac{\alpha_0}{\lambda_{n+1}} L, L)$. Therefore, with $\frac{\alpha_0}{\lambda_{n+1}} \leq 1$,

$$\psi(\tau) = a_1 f(\tau) + \sum_{i=2}^{n+1} a_i f(\widetilde{\lambda}_i \tau) = 0, \quad \forall \tau \in [s_1 L, L). \tag{4.16}$$

Note that $1 > \widetilde{\lambda}_2 > \widetilde{\lambda}_3 > \cdots > \widetilde{\lambda}_{n+1}$, and that $\alpha_0 \leq \lambda_{n+1} < \frac{\lambda_{n+1}}{\lambda_1} = \widetilde{\lambda}_{n+1}$. Then, by using the induction hypothesis with (4.16) and (4.14), we obtain

$$f(\tau) = 0, \quad \forall \tau \in [\alpha_1 L, L),$$

where $\widetilde{\alpha}_1 = s_1 \widetilde{\lambda}_{n+1} = \lambda_{n+1} \max\{s, \alpha_0\} < \lambda_{n+1}$. Then we can repeat the latter argument replacing α_0 by $\widetilde{\alpha}_1$, and we obtain

$$f(\tau) = 0, \quad \forall \tau \in [\widetilde{\alpha}_k L, L),\ \forall k \geq 1,$$

where

$$\widetilde{\alpha}_k = \lambda_{n+1} \max\{s, \widetilde{\alpha}_{k-1}\}, \quad \forall k \geq 1 \tag{4.17}$$

with $\widetilde{\alpha}_0 = \alpha_0$ given. If $s = 0$, letting $t = 0$ in (4.13) yields $f(0) = 0$. Using Lemma 4.2, we infer that

$$f(\tau) = 0, \quad \forall \tau \in [\overline{\alpha} L, L),$$

where $\overline{\alpha} = \lambda_{n+1} s$, which completes the proof.

Proof of Theorem 3.2 Let $\varphi : [0, L] \to \mathbb{R}$ be a function such that

$$\Phi_m[\varphi](t) = \sum_{j=1}^{m} a_j \varphi(h_j(t)) = 0, \quad \forall t \in [0, L_m].$$

Then, if $t = L_m$, we obtain

$$h_j(L_m) = L, \quad \forall j = 1, \cdots, m,$$

and hence

$$0 = \Phi_m[\varphi](L_m) = \varphi(L). \tag{4.18}$$

Next, for any $k \in \{1, \cdots, m\}$, we have

$$\sum_{j=k}^{m} a_j \varphi(\beta_j t) = 0, \quad \forall t \in [L_{k-1}, L_k],$$

which is equivalent to

$$a_k\varphi(t) + \sum_{j=k+1}^{m} a_j\varphi\Big(\frac{\beta_j}{\beta_k}t\Big) = 0, \quad \forall t \in [\beta_k L_{k-1}, \beta_k L_k] = [\beta_k L_{k-1}, L] \tag{4.19}$$

for $k = 1, 2, \cdots, m$. We aim to prove that

$$\varphi(\tau) = 0, \quad \forall \tau \in [\beta_m L_{k-1}, L]$$

for $k = 1, \cdots, m$. We proceed by induction on $i = m - k \in \{0, \cdots, m-1\}$.

Case $i = 0$ Letting $k = m$ in (4.19) yields

$$a_m\varphi(t) = 0, \quad \forall t \in [\beta_m L_{m-1}, L],$$

which implies

$$\varphi(\tau) = 0, \quad \forall \tau \in [\beta_m L_{m-1}, L], \tag{4.20}$$

which completes the case $i = 0$.

Case $i = 1$ Letting $k = m - 1$ in (4.19), we obtain

$$a_{m-1}\varphi(t) + a_m\varphi\Big(\frac{\beta_m}{\beta_{m-1}}t\Big) = 0, \quad \forall t \in [\beta_{m-1}L_{m-2}, L]. \tag{4.21}$$

We infer from Lemma 4.3 (applied with $\lambda_1 = \frac{\beta_m}{\beta_{m-1}}$, $s = \frac{\beta_{m-1}}{\beta_{m-2}}$ and $\alpha_0 = \frac{\beta_m}{\beta_{m-1}}$) that

$$\varphi(\tau) = 0, \quad \forall \tau \in [\beta_m L_{m-2}, L].$$

Case i Assume the property satisfied for $i - 1$, i.e.,

$$\varphi(\tau) = 0, \quad \forall \tau \in [\beta_m L_{m-i}, L]. \tag{4.22}$$

Replacing $k = m - i$ in (4.19), we obtain

$$a_{m-i}\varphi(t) + \sum_{j=m-i+1}^{m} a_j\varphi\Big(\frac{\beta_j}{\beta_{m-i}}t\Big) = 0, \quad \forall t \in [\beta_{m-i}L_{m-i-1}, L]. \tag{4.23}$$

Then, if we set $\lambda_j = \frac{\beta_j}{\beta_{m-i}} < 1$, for $j = m - i + 1, \cdots, m$,

$$s = \beta_{m-i}\frac{L_{m-i-1}}{L}$$

and $\alpha_0 = \dfrac{\beta_m}{\beta_{m-i}} = \lambda_m$, then we infer from Lemma 4.3 that

$$\varphi(\tau) = 0, \quad \forall \tau \in [\beta_m L_{m-i-1}, L].$$

Thus

$$\varphi(\tau) = 0, \quad \forall \tau \in [\beta_m L_{k-1}, L]$$

for $k = 1, \cdots, m$. This implies (with $k = 1$ and $L_0 = 0$)

$$\varphi(\tau) = 0, \quad \forall \tau \in [0, L].$$

The proof of Theorem 3.2 is complete.

5 Proofs of the Stability Results

We first prove Theorem 3.3.

Proof of Theorem 3.3 First, some estimates are established:

$$\begin{aligned}\|\varphi\circ h_j\|^2_{L^2(0,L_m)} &= \int_0^{L_m}\varphi^2(h_j(t))\mathrm{d}t\\ &= \int_0^{L_j}\varphi^2(\beta_j t)\mathrm{d}t+\varphi^2(L)L\Big(\frac{1}{\beta_m}-\frac{1}{\beta_j}\Big)\\ &\le \frac{1}{\beta_j}\int_0^{L}\varphi^2(t)\mathrm{d}t+\varphi^2(L)\frac{L}{\beta_m}\\ &\le \frac{1}{\beta_m}\{\|\varphi\|^2_{L^2(0,L)}+\varphi^2(L)L\}\\ &\le \frac{1}{\beta_m}(1+\|T_L\|^2L)\,\|\varphi\|^2_{H^1(0,L)}\,, \end{aligned}\tag{5.1}$$

where $T_L(u)=u(L)$ is the trace operator in $H^1(0,L)$.

Now, if we set

$$c_1=\frac{1}{\sqrt{\beta_m}}(1+\|T_L\|^2L)^{\frac12},$$

then using (5.1), we obtain

$$\|\Phi_m[\varphi]\|_{L^2(0,L_m)}\le\sum_{j=1}^m a_j\,\|\varphi\circ h_j\|_{L^2(0,L_m)}\le c_1\,\|\varphi\|_{H^1(0,L)}\,.\tag{5.2}$$

On the other hand, let ψ be any test function with compact support in $(0,L_m)$. Then

$$\begin{aligned}\int_0^{L_m}\Phi_m[\varphi](t)\psi'(t)\mathrm{d}t &= \sum_{j=1}^m a_j\Big\{\int_0^{L_j}\varphi(\beta_j t)\psi'(t)\mathrm{d}t+\varphi(L)\int_{L_j}^{L_m}\psi'(t)\mathrm{d}t\Big\}\\ &= -\sum_{j=1}^m a_j\beta_j\int_0^{L_j}\varphi'(\beta_j t)\psi(t)\mathrm{d}t\\ &= -\sum_{j=1}^m a_j\beta_j\int_0^{L_m}\varphi'(\beta_j t)\psi(t)(1-H(\beta_j t-L))\mathrm{d}t,\end{aligned}\tag{5.3}$$

where H denotes Heaviside's function. Thus

$$(\Phi_m[\varphi])'(t)=\sum_{j=1}^m a_j\beta_j\varphi'(\beta_j t)(1-H(\beta_j t-L)),\quad \forall t\in(0,L_m).\tag{5.4}$$

Therefore, for any $\varphi\in H^1(0,L)$, the function $\Phi_m[\varphi]$ belongs to $H^1(0,L_m)$. This, along with (5.4), yields

$$\|(\Phi_m[\varphi])'\|_{L^2(0,L_m)}\le\sum_{j=1}^m a_j\sqrt{\beta_j}\Big(\int_0^L(\varphi')^2(t)\mathrm{d}t\Big)^{\frac12}\le\sqrt{\beta_1}\,\|\varphi'\|_{L^2(0,L)}\,.\tag{5.5}$$

Combining (5.5) with (5.2), we obtain

$$\|\Phi_m[\varphi]\|_{1,0,L_m}\le\widetilde{C}_1\,\|\varphi\|_{1,0,L}\,,$$

where $\widetilde{C}_1 = \sqrt{(c_1)^2 + \beta_1}$. The proof of Theorem 3.3 is therefore complete.

Now we proceed to the proof of Theorem 3.4. Before establishing this stability result, we need to recall the well-known facts about Mellin transform (the reader is referred to [10, Chapter VIII] for details).

For any real numbers $\alpha < \beta$, let $\langle \alpha, \beta \rangle$ denote the open strip of complex numbers $s = \sigma + \mathrm{i}t$ ($\sigma, t \in \mathbb{R}$) such that $\alpha < \sigma < \beta$.

Definition 5.1 (Mellin Transform) *Let f be locally Lebesgue integrable over $(0, +\infty)$. The Mellin transform of f is defined by*

$$\mathcal{M}[f](s) = \int_0^{+\infty} f(x) x^{s-1} \mathrm{d}x, \quad \forall s \in \langle \alpha, \beta \rangle,$$

where $\langle \alpha, \beta \rangle$ is the largest open strip in which the integral converges (it is called the fundamental strip).

Lemma 5.1 *Let f be locally Lebesgue integrable over $(0, +\infty)$. Then the following properties hold true:*

(1) *Let $s_0 \in \mathbb{R}$. Then for all s such that $s + s_0 \in \langle \alpha, \beta \rangle$, we have*

$$\mathcal{M}[f(x)](s + s_0) = \mathcal{M}[x^{s_0} f(x)](s).$$

(2) *For any $\beta \in \mathbb{R}$, if $g(x) = f(\beta x)$, then*

$$\mathcal{M}[g](s) = \beta^{-s} \mathcal{M}[f](s), \quad \forall s \in \langle \alpha, \beta \rangle.$$

Definition 5.2 (Mellin Transform as an Operator in L^2) *For functions in $L^2(0, +\infty)$, we define a linear operator $\widetilde{\mathcal{M}}$ as*

$$\begin{aligned} \widetilde{\mathcal{M}} : \ & L^2(0, +\infty) \to L^2(-\infty, +\infty), \\ & f \to \widetilde{\mathcal{M}}[f](s) := \frac{1}{\sqrt{2\pi}} \mathcal{M}[f]\Big(\frac{1}{2} - \mathrm{i}s\Big). \end{aligned}$$

Theorem 5.1 (Mellin Inversion Theorem) *The operator $\widetilde{\mathcal{M}}$ is invertible with the inverse*

$$\begin{aligned} \widetilde{\mathcal{M}}^{-1} : \ & L^2(-\infty, +\infty) \to L^2(0, +\infty), \\ & \varphi \to \widetilde{\mathcal{M}}^{-1}[\varphi](x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^{-\frac{1}{2} - \mathrm{i}s} \varphi(s) \mathrm{d}s. \end{aligned}$$

Furthermore, this operator is an isometry, that is,

$$\|\widetilde{\mathcal{M}}[f]\|_{L^2(-\infty,\infty)} = \|f\|_{L^2(0,\infty)}, \quad \forall f \in L^2(0, +\infty).$$

Proof of Theorem 3.4 We note that for any function $f : [0, +\infty[\to \mathbb{R}$ such that $\mathrm{supp}(f) \subset [0, L)$, we have

$$f(h_j(t)) = f(\beta_j t).$$

Thus, we obtain

$$\Phi_m[f](t) = \sum_{j=1}^{m} a_j f(\beta_j t), \quad \forall t \geq 0, \tag{5.6}$$

where $\{\beta_j\}_{j=1}^m$ has been defined in (2.12).

Pick any $\varphi \in C([0, L])$ and let $g : [0, L_m] \to \mathbb{R}$ be such that

$$\Phi_m[\varphi](t) = g(t), \quad \forall t \in [0, L_m]. \tag{5.7}$$

Define the functions

$$\widetilde{g}(t) = \begin{cases} g(t) - g(L_m), & 0 \leq t \leq L_m, \\ 0, & t \geq L_m, \end{cases} \qquad \widetilde{\varphi}(t) = \begin{cases} \varphi(t) - \varphi(L), & 0 \leq t \leq L, \\ 0, & t \geq L. \end{cases} \tag{5.8}$$

If we replace t by L_m in (5.7), we have the following compatibility condition:

$$\varphi(L) = g(L_m).$$

Since $\Phi_m[1] = 1$, we infer that

$$\Phi_m[\widetilde{\varphi}](t) = \widetilde{g}(t), \quad \forall t \geq 0. \tag{5.9}$$

Letting $f = \widetilde{\varphi}$ in (5.6) yields

$$\Phi_m[\widetilde{\varphi}](t) = \sum_{j=1}^{m} a_j \widetilde{\varphi}(\beta_j t), \quad \forall t \geq 0.$$

It follows from Lemma 5.1 that

$$\mathcal{M}[\Phi_m[\widetilde{\varphi}]](s) = \Big(\sum_{j=1}^{m} a_j \beta_j^{-s}\Big) \mathcal{M}[\widetilde{\varphi}](s), \quad \forall s \in \langle \alpha, \beta \rangle, \tag{5.10}$$

where $\langle \alpha, \beta \rangle$ is the fundamental strip associated with $\widetilde{\varphi}$.

Let $\gamma > 0$ be a fixed constant. Using (5.10) and Lemma 5.1, we obtain

$$\Lambda_m^\gamma(s) |\widetilde{\mathcal{M}}[x^\gamma \widetilde{\varphi}(x)](s)| = |\widetilde{\mathcal{M}}[x^\gamma \Phi_m[\widetilde{\varphi}](x)](s)|, \quad \forall s \in \mathbb{R}, \tag{5.11}$$

where Λ_m^γ has been defined in (3.6). On the other hand,

$$\begin{aligned} \Lambda_m^\gamma(s) &\geq a_m \beta_m^{-\gamma-\frac{1}{2}} - \Big| \sum_{j=1}^{m-1} a_j \beta_j^{-(\gamma+\frac{1}{2}-\mathrm{i}s)} \Big| \\ &\geq a_m \beta_m^{-\gamma-\frac{1}{2}} - \sum_{j=1}^{m-1} a_j \beta_j^{-(\gamma+\frac{1}{2})} \\ &\geq a_m \beta_m^{-\gamma-\frac{1}{2}} - \beta_{m-1}^{-(\gamma+\frac{1}{2})} \\ &= \beta_m^{-\gamma-\frac{1}{2}} \Big(a_m - \Big(\frac{\beta_{m-1}}{\beta_m}\Big)^{-(\gamma+\frac{1}{2})} \Big). \end{aligned} \tag{5.12}$$

Therefore, if we choose

$$\gamma > \frac{\ln(a_m)}{\ln(\frac{\beta_m}{\beta_{m-1}})} - \frac{1}{2},$$

then

$$\Lambda_m^\gamma(s) \geq \beta_m^{-\gamma-\frac{1}{2}}\Big(a_m - \Big(\frac{\beta_{m-1}}{\beta_m}\Big)^{-(\gamma+\frac{1}{2})}\Big) > 0, \quad \forall s \in \mathbb{R}.$$

Thus, there exists γ_0 such that

$$C_\gamma = \inf_{s\in\mathbb{R}} \Lambda_m^\gamma(s) > 0, \quad \forall \gamma > \gamma_0.$$

Therefore, using the fact that $\widetilde{\mathcal{M}}$ is an isometry and (5.11), we obtain

$$C_\gamma \|\widetilde{\varphi}\|_{0,\gamma,L} \leq \|\Phi_m[\widetilde{\varphi}]\|_{0,\gamma,L_m}, \tag{5.13}$$

which completes the proof of Theorem 3.4.

We are now in a position to prove Theorems 3.3–3.4.

Proof of Theorem 3.3 Let us fix any $\gamma > 0$ and let $\rho : [0, L] \to \mathbb{R}$ be a function in $L^2(0, L)$. From (2.13), we have

$$\begin{aligned}(x^\gamma I_m[\rho](x))' &= \gamma x^{\gamma-1} I_m[\rho](x) + x^\gamma (I_m[\rho](x))' \\ &= \gamma x^{\gamma-1} I_m[\rho](x) + \frac{x^{\gamma-\frac{1}{2}} J_0 F(c_0)}{2} (\Phi_m[\varphi])'(\sqrt{x}),\end{aligned} \tag{5.14}$$

where $\varphi(x) = \int_0^x \rho(\tau)\mathrm{d}\tau$ (note that $\varphi \in H^1(0, L)$). Since

$$\begin{aligned}\int_0^{L_m^2} x^{2\gamma-1}((\Phi_m[\varphi])'(\sqrt{x}))^2 \mathrm{d}x &= 2\int_0^{L_m} \tau^{4\gamma-1}((\Phi_m[\varphi])'(\tau))^2 \mathrm{d}\tau \\ &= 2\,\|(\Phi_m[\varphi])'\|^2_{0,2\gamma-\frac{1}{2},L_m},\end{aligned} \tag{5.15}$$

we have

$$\begin{aligned}\|I_m[\rho]\|^2_{1,\gamma,L_m^2} &\leq \|I_m[\rho]\|^2_{0,\gamma,L_m^2} + \Big(\gamma\,\|I_m[\rho]\|_{0,\gamma-1,L_m^2} + \frac{|J_0F(c_0)|}{\sqrt{2}}\,\|(\Phi_m[\varphi])'\|_{0,2\gamma-\frac{1}{2},L_m}\Big)^2 \\ &\leq \|I_m[\rho]\|^2_{0,\gamma,L_m^2} + 2\gamma^2\,\|I_m[\rho]\|^2_{0,\gamma-1,L_m^2} + (J_0F(c_0))^2\,\|(\Phi_m[\varphi])'\|^2_{0,2\gamma-\frac{1}{2},L_m} \\ &\leq (L^2 + 2\gamma^2)\,\|I_m[\rho]\|^2_{0,\gamma-1,L_m^2} + (J_0F(c_0))^2\,\|(\Phi_m[\varphi])'\|^2_{0,2\gamma-\frac{1}{2},L_m}.\end{aligned} \tag{5.16}$$

On the other hand, using (2.13) and the change of variable $\tau = x^2$, we have

$$\begin{aligned}\|\Phi_m[\varphi]\|^2_{0,2\gamma-\frac{3}{2},L_m} &= \frac{1}{(F(c_0)J_0)^2}\int_0^{L_m} x^{4\gamma-3}(I_m[\rho](x^2))^2 \mathrm{d}x \\ &= \frac{1}{2(F(c_0)J_0)^2}\,\|I_m[\rho]\|^2_{0,\gamma-1,L_m^2}.\end{aligned} \tag{5.17}$$

By replacing (5.17) in (5.16), we obtain

$$\begin{aligned}\|I_m[\rho]\|^2_{1,\gamma,L_m^2} \leq{}& (L^2 + 2\gamma^2)2(F(c_0)J_0)^2\,\|\Phi_m[\varphi]\|^2_{0,2\gamma-\frac{3}{2},L_m} \\ &+ (F(c_0)J_0)^2\,\|(\Phi_m[\varphi])'\|^2_{0,2\gamma-\frac{1}{2},L_m},\end{aligned} \tag{5.18}$$

and assuming that $\gamma \geq \frac{3}{4}$, from Theorem 3.3, we have

$$\begin{aligned}\|I_m[\rho]\|_{1,\gamma,L_m^2} &\leq \sqrt{3L^2+4\gamma^2}J_0F(c_0)L^{2\gamma-\frac{3}{2}}\,\|\Phi_m[\varphi]\|_{H^1(0,L_m)}\\ &\leq \sqrt{3L^2+4\gamma^2}J_0F(c_0)L^{2\gamma-\frac{3}{2}}\widetilde{C}_1\,\|\varphi\|_{H^1(0,L)}\,. \end{aligned}\tag{5.19}$$

But, from the Cauchy-Schwarz inequality, we have $|\varphi(x)| \leq \sqrt{L}\,\|\rho\|_{L^2(0,L)}$, and hence

$$\|\varphi\|^2_{H^1(0,L)} = \|\varphi\|^2_{L^2(0,L)} + \|\rho\|^2_{L^2(0,L)} \leq (L^2+1)\,\|\rho\|^2_{L^2(0,L)}\,.$$

Therefore, for any $\gamma \geq \frac{3}{4}$, we have

$$\|I_m[\rho]\|_{1,\gamma,L_m^2} \leq C_1\,\|\rho\|_{L^2(0,L)}\,,$$

where

$$C_1 = \sqrt{3L^2+4\gamma^2}J_0F(c_0)L^{2\gamma-\frac{3}{2}}\widetilde{C}_1(L^2+1)^{\frac{1}{2}},$$

and the proof of Theorem 3.3 is therefore finished.

Proof of Theorem 3.4 Let ψ be any test function compactly supported in $(0,L)$, and let γ be a positive constant. Set

$$g_\gamma(x) = x^\gamma\rho(x), \quad \varphi(x) = \int_0^x \rho(\tau)\mathrm{d}\tau$$

and

$$\widetilde{\varphi}(t) = \varphi(x) - \varphi(L).$$

It follows that

$$(x^{\gamma+1}\widetilde{\varphi}(x))' = (\gamma+1)x^\gamma\widetilde{\varphi}(x) + g_{\gamma+1}(x),$$

and hence,

$$\begin{aligned}\langle g_{\gamma+1},\psi\rangle &= \int_0^L g_{\gamma+1}(x)\psi(x)\mathrm{d}x\\ &= \int_0^L ((x^{\gamma+1}\widetilde{\varphi}(x))' - (\gamma+1)x^\gamma\widetilde{\varphi}(x))\psi(x)\mathrm{d}x\\ &= -\int_0^L (x^{\gamma+1}\widetilde{\varphi}(x)\psi'(x) + (\gamma+1)x^\gamma\widetilde{\varphi}(x)\psi(x))\mathrm{d}x.\end{aligned}$$

Then, we have

$$\begin{aligned}|\langle g_{\gamma+1},\psi\rangle| &\leq (\|\widetilde{\varphi}\|_{0,\gamma+1,L} + (\gamma+1)\,\|\widetilde{\varphi}\|_{0,\gamma,L})\,\|\psi\|_{H^1(0,L)}\\ &\leq (L+\gamma+1)\,\|\widetilde{\varphi}\|_{0,\gamma,L}\,\|\psi\|_{H^1(0,L)}\,.\end{aligned}$$

Therefore,

$$\|g_{\gamma+1}\|_{H^{-1}(0,L)} \leq (L+\gamma+1)\,\|\widetilde{\varphi}\|_{0,\gamma,L}\,. \tag{5.20}$$

Thus, using Theorem 3.4, we have that for any $\gamma > \max\{\gamma_0, \frac{3}{4}\}$, there exists a constant $C_\gamma > 0$ such that

$$\begin{aligned}\|\rho\|_{-1,\gamma+1,L} &= \|g_{\gamma+1}\|_{H^{-1}(0,L)} \\ &\le (L+\gamma+1)C_\gamma^{-1}\Big\{ \|\Phi_m[\varphi]\|_{0,\gamma,L_m} + \frac{L_m^{\gamma+\frac{1}{2}}}{\sqrt{2\gamma+1}}|\Phi_m[\varphi](L_m)|\Big\}. \end{aligned} \tag{5.21}$$

Using (2.13), we have

$$\Phi_m[\varphi](L_m) = \frac{1}{F(c_0)J_0} I_m[\rho](L_m^2). \tag{5.22}$$

Replacing (5.22) in (5.21) and using (5.17), with $2\gamma - \frac{3}{2}$ replaced by γ, we obtain

$$\|\rho\|_{-1,\gamma+1,L} \le \frac{(L+\gamma+1)}{\sqrt{2}|J_0F(c_0)|}C_\gamma^{-1}\Big\{1+\sqrt{2}\frac{L_m}{\sqrt{2\gamma+1}}\|T_{L_m^2}\|\Big\}\,\|I_m[\rho]\|_{1,\frac{\gamma}{2}-\frac{1}{4},L_m^2}\,.$$

Therefore, setting

$$C_2 = \frac{(L+\gamma+1)}{\sqrt{2}|J_0F(c_0)|}C_\gamma^{-1}\Big\{1+\sqrt{2}\frac{L_m}{\sqrt{2\gamma+1}}\|T_{L_m^2}\|\Big\},$$

we obtain (3.9). The proof of Theorem 3.4 is achieved.

6 Numerical Reconstruction Results

This section is devoted to the proof of Theorems 3.5–3.6.

Proof of Theorem 3.5 Let ρ be a function in $C^0([0,L])$, and let us consider the functions $\{\varphi_j\}_{j\ge1}$ defined in (3.13)–(3.15).

First, we note that for all $k \ge 1$, we have

$$\varphi_k(x) = 0, \quad \forall x \notin [\beta^k L, \beta^{k-1}L). \tag{6.1}$$

Then, we can define the sequence $\{\psi_p\}_{p\in\mathbb{N}^*}$ as

$$\psi_p(x) = \sum_{j=1}^{p}\varphi_j(x), \quad \forall x \in \mathbb{R}.$$

Using (6.1), we have that for all $x \in \mathbb{R}\setminus(0,L)$,

$$\psi_p(x) = 0, \quad \forall p \in \mathbb{N}^*,$$

and hence,

$$\lim_{p\to+\infty}\psi_p(x) = 0, \quad \forall x \in \mathbb{R}\setminus(0,L).$$

Besides that, we consider the ceiling function

$$\lceil x\rceil = \min\{k \in \mathbb{Z} \,\big|\, k \ge x\},$$

i.e., $\lceil x\rceil$ is the smallest integer not less than x.

Next, we define

$$k^*(x) = \left\lceil \frac{\ln\left(\frac{x}{L}\right)}{\ln(\beta)} \right\rceil, \quad \forall x \in (0, L). \tag{6.2}$$

Then, we have

$$x \in [\beta^{k^*(x)} L, \beta^{k^*(x)-1} L), \quad \forall x \in (0, L).$$

Therefore, we obtain for $x \in (0, L)$

$$\psi_p(x) = \varphi_{k^*(x)}(x), \quad \forall p \geq k^*(x),$$

and hence,

$$\lim_{p \to +\infty} \psi_p(x) = \varphi_{k^*(x)}(x), \quad \forall x \in (0, L). \tag{6.3}$$

Thus, the series in (3.16) is convergent, i.e., the function $\widetilde{\varphi}$ is well defined.

On the other hand, by replacing (3.10) in (2.12) we obtain

$$\beta_j = \beta_0 \beta^j, \quad j = 1, \cdots, m.$$

By using (6.1), we have

$$\widetilde{\varphi}(x) = 0, \quad \forall x \in \mathbb{R} \setminus [0, L).$$

Combining with (5.6), we get

$$\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) = \sum_{j=1}^{m} a_j \widetilde{\varphi}(\beta^j t), \quad \forall t \geq 0. \tag{6.4}$$

By replacing $t = 0$ and $t = \beta_0 L_m$ in (6.4), and using (3.11), (3.13), and (3.16), we obtain

$$\widetilde{\Phi}_m[\widetilde{\varphi}](0) = g(0), \quad \widetilde{\Phi}_m[\widetilde{\varphi}](L_m) = g(\beta_0 L_m) = 0.$$

Now, if we take $t \in (0, \beta_0 L_m) = \big(0, \frac{L}{\beta^m}\big)$, we have

$$\beta^j t \in (0, \beta^{j-m} L) \quad \text{for } j \in \{1, 2, \cdots, m\}.$$

We need to consider the following two cases: $t < L$ and $t \geq L$.

Case $t < L$ In this case, we have

$$\beta^j t \in (0, L) \quad \text{for } j \in \{1, 2, \cdots, m\}$$

and

$$k^*(\beta^j t) = j + k^*(t).$$

Thus, replacing (6.3) in (6.4), we obtain

$$\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) = \sum_{j=1}^{m} a_j \varphi_{j+k^*(t)}(\beta^j t),$$

and hence, using (3.15) with $k+1 = m + k^*(t)$, we obtain

$$\begin{aligned}\widetilde{\Phi}_m[\widetilde{\varphi}]\big(\frac{t}{\beta_0}\big) &= \sum_{j=1}^{m-1} a_j \varphi_{j+k^*(t)}(\beta^j t) + a_m \varphi_{m+k^*(t)}(\beta^m t) \\ &= \sum_{j=1}^{m-1} a_j \varphi_{j+k^*(t)}(\beta^j t) + \Big(g(t) - \sum_{j=1}^{m-1} a_j \varphi_{j+k^*(t)}(\beta^j t)\Big) \\ &= g(t).\end{aligned}$$

Case $t \geq L$ Let us set

$$k_*(x) = \Big\lfloor \frac{\ln\big(\frac{x}{L}\big)}{\ln\big(\frac{1}{\beta}\big)} \Big\rfloor, \quad \forall x \geq L,$$

where

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$$

is the floor function, i.e., it is the largest integer not greater than x. Thus, we have

$$\beta^{k_*(x)+1} x < L \leq \beta^{k_*(x)} x, \quad \forall x \geq L \tag{6.5}$$

and

$$k^*(\beta^{k_*(t)+1} t) = 1. \tag{6.6}$$

Then, we infer from (6.5) that

$$\begin{aligned}&\beta^j t \geq L, \quad \forall j \leq k_*(t), \\ &\beta^j t < L, \quad \forall j \geq k_*(t)+1.\end{aligned}$$

By using (6.3)–(6.4), it follows that

$$\begin{aligned}\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) &= \sum_{j=k_*(t)+1}^{m} a_j \varphi_{k^*(\beta^j t)}(\beta^j t) \\ &= \sum_{j=1}^{m-k_*(t)} a_{j+k_*(t)} \varphi_{k^*(\beta^{j+k_*(t)} t)}(\beta^{j+k_*(t)} t).\end{aligned}$$

From (6.6), we have

$$k^*(\beta^{j+k_*(t)} t) = j, \quad \forall j \geq 1,$$

and hence, from (3.13)–(3.14), with $k+1 = m - k_*(t)$, we obtain

$$\begin{aligned}\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) &= \sum_{j=1}^{m-k_*(t)} a_{j+k_*(t)} \varphi_j(\beta^{j+k_*(t)} t) \\ &= \sum_{j=1}^{m-k_*(t)-1} a_{j+k_*(t)} \varphi_j(\beta^{j+k_*(t)} t) + a_m \varphi_{m-k_*(t)}(\beta^m t) \\ &= \sum_{j=1}^{m-k_*(t)-1} a_{j+k_*(t)} \varphi_j(\beta^{j+k_*(t)} t) + \Big(g(t) - \sum_{j=1}^{m-k_*(t)-1} a_{k_*(t)+j} \varphi_j(\beta^{j+k_*(t)} t)\Big) \\ &= g(t).\end{aligned}$$

It remains to prove (3.18). Replacing (3.11) in (3.17) and using (2.13), we get

$$\begin{aligned}\widetilde{\Phi}_m[\widetilde{\varphi}]\Big(\frac{t}{\beta_0}\Big) &= \frac{\widetilde{I}_m[\rho]\Big(\frac{t^2}{\beta_0^2}\Big) - \widetilde{I}_m[\rho](L_m^2)}{J_0F(c_0)} \\ &= \widetilde{\Phi}_m[\varphi]\big(\frac{t}{\beta_0}\big) - \frac{\widetilde{I}_m[\rho](L_m^2)}{J_0F(c_0)}, \quad \forall t \in [0, \beta_0 L_m], \end{aligned} \tag{6.7}$$

where $\varphi(x) = \int_0^x \rho(\tau)\mathrm{d}\tau$. Using $\widetilde{\Phi}_m[1] = 1$ and Theorem 3.2, we obtain (3.18), i.e.,

$$\widetilde{\varphi}(x) = \varphi(x) - \frac{\widetilde{I}_m[\rho](L_m^2)}{J_0F(c_0)}, \quad \forall x \in [0, L].$$

This completes the proof of Theorem 3.5.

Proof of Theorem 3.6 Let ρ be a function in $C^0([0,L])$, and let $\{\varphi_j\}_{j\geq 1}$ be defined as in (3.13)–(3.15). Using (3.18), we obtain

$$\widetilde{\varphi}(x) = \varphi(x) - \varphi(L), \quad \forall x \in [0, L], \tag{6.8}$$

where $\varphi = \int_0^x \rho(\tau)\mathrm{d}\tau$ and $\widetilde{\varphi}$ has been defined in (3.16).

Recall that the family of norms $\|\cdot\|_{[a,b)}$ (for $0 \leq a < b < \infty$) satisfies (3.19)–(3.20). Using (3.16), (6.2)–(6.3) and (6.8), we obtain

$$\|\varphi(\cdot) - \varphi(L)\|_{[\beta^{k+1}L, \beta^k L)} = \|\varphi_{k+1}\|_{[\beta^{k+1}L, \beta^k L)}. \tag{6.9}$$

Let us prove that for any $k \geq 0$, we have

$$\|\varphi_{k+1}\|_{[\beta^{k+1}L, \beta^k L)} \leq \frac{C(\beta^m)}{a_m^{k+1}} \|g\|_{[\beta^{k+1}\beta_0 L_m, \beta_0 L_m)}. \tag{6.10}$$

The proof of (6.10) is done by induction on k.

Case $\boldsymbol{k = 0}$ Using (3.13) and (3.19)–(3.20), we have

$$\|\varphi_1\|_{[\beta L, L)} \leq \frac{C(\beta^m)}{a_m} \|g\|_{[\beta\beta_0 L_m, \beta_0 L_m)},$$

as desired.

Assume now that for all $j = 1, \cdots, k$ (with $k \geq 1$), we have

$$\|\varphi_j\|_{[\beta^j L, \beta^{j-1} L)} \leq \frac{C(\beta^m)}{a_m^j} \|g\|_{[\beta^j \beta_0 L_m, \beta_0 L_m)}, \tag{6.11}$$

and let us prove (6.10).

Case $\boldsymbol{k+1 \leq m}$ Using (3.14) and (3.19)–(3.20), we obtain

$$\begin{aligned}\|\varphi_{k+1}\|_{[\beta^{k+1}L, \beta^k L)} \leq \frac{1}{a_m}\Big(& C(\beta^m)\|g\|_{[\beta^{k+1}\beta_0 L_m, \beta^k \beta_0 L_m)} \\ &+ \sum_{j=1}^{k} a_{m-k-1+j} C\Big(\frac{\beta^{k+1}}{\beta^j}\Big)\|\varphi_j\|_{[\beta^j L, \beta^{j-1}L)}\Big).\end{aligned}$$

Using the induction hypothesis (6.11), we have

$$\begin{aligned}\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} &\le \frac{1}{a_m}\Big(C(\beta^m)\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)}\\ &\quad+\sum_{j=1}^{k} a_{m-k-1+j}C\Big(\frac{\beta^{k+1}}{\beta^j}\Big)\frac{C(\beta^m)}{a_m^j}\,\|g\|_{[\beta^j\beta_0 L_m,\beta_0 L_m)}\Big)\\ &\le \frac{C(\beta^m)}{a_m^{k+1}}\Big(a_m^k\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)}\\ &\quad+\sum_{j=1}^{k} a_{m-k-1+j}C\Big(\frac{\beta^{k+1}}{\beta^j}\Big)\,\|g\|_{[\beta^j\beta_0 L_m,\beta_0 L_m)}\Big)\\ &\le \frac{C(\beta^m)}{a_m^{k+1}}\Big(a_m^k+\sum_{j=1}^{k} a_{m-k-1+j}C\Big(\frac{\beta^{k+1}}{\beta^j}\Big)\Big)\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)}\,. \end{aligned}\tag{6.12}$$

Note that

$$C(u)\le 1,\quad \forall u\in(0,1),\tag{6.13}$$

where $C(\cdot)$ is nondecreasing and $C(1)=1$. Therefore,

$$C\Big(\frac{\beta^{k+1}}{\beta^j}\Big)\le 1,\quad \forall j\in\{1,\cdots,k\}.$$

Thus, we obtain

$$\|\varphi_{k+1}\|_{([\beta^{k+1},\beta^k L)}\le \frac{C(\beta^m)}{a_m^{k+1}}\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)}\,.$$

This proves (6.10) for all $k=\{0,\cdots,m-1\}$.

Case $k+1>m$ Replacing φ_{k+1} by the expression in (3.15) and using (3.19)–(3.20) and the induction hypothesis, we obtain

$$\begin{aligned}\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} &\le \frac{1}{a_m}\Big(C(\beta^m)\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)}\\ &\quad+\sum_{j=1}^{m-1} a_j C\Big(\frac{\beta^m}{\beta^j}\Big)\frac{C(\beta^m)}{a_m^{j+k-m+1}}\,\|g\|_{[\beta^{j+k-m+1}\beta_0 L_m,\beta_0 L_m)}\Big)\\ &= \frac{C(\beta^m)}{a_m^{k+1}}\Big(a_m^k\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta^k\beta_0 L_m)}\\ &\quad+\sum_{j=1}^{m-1} a_j C\Big(\frac{\beta^m}{\beta^j}\Big)\,\|g\|_{[\beta^{j+k-m+1}\beta_0 L_m,\beta_0 L_m)}\Big)\\ &\le \frac{C(\beta^m)}{a_m^{k+1}}\Big(a_m^k+\sum_{j=1}^{m-1} a_j C\Big(\frac{\beta^m}{\beta^j}\Big)\Big)\,\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)}\,, \end{aligned}\tag{6.14}$$

and with (6.13) we infer that $C(\beta^m/\beta^j)\le 1$ for all $j\in\{1,\cdots,m-1\}$. This completes the proof of (6.10).

On the other hand, using (3.17) and (3.20), we obtain

$$\|g\|_{[\beta^{k+1}\beta_0 L_m,\beta_0 L_m)}\le C(\beta_0)\|\widetilde{\Phi}_m[\widetilde{\varphi}]\|_{[\beta^{k+1}L_m,L_m)}.$$

By replacing (6.8) in (6.10), we obtain

$$\|\varphi_{k+1}\|_{[\beta^{k+1}L,\beta^k L)} \leq C(\beta_0)\frac{C(\beta^m)}{a_m^{k+1}}\|\widetilde{\Phi}_m[\varphi](\cdot) - \widetilde{\Phi}_m[\varphi](L_m)\|_{[\beta^{k+1}L_m,L_m)},$$

and by replacing it in (6.9), we obtain (3.22). This completes the proof of Theorem 3.6.

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