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Long-Time Turbulence Model Deduced from the Navier-Stokes Equations*

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(*In Honor of the Scientific Contributions of Professor Luc Tartar*)

Abstract The author shows the existence of long-time averages to turbulent solutions of the Navier-Stokes equations and determines the equations satisfied by them, involving a Reynolds stress that is shown to be dissipative.

Keywords Navier-Stokes equations, Weak solutions, Turbulence modeling, Reynolds stress
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1 Introduction

This paper aims to report results that have been exposed during a talk given at the "International Conference on Nonlinear and Multiscale Partial Differential Equations: Theory, Numerics and Applications, Fudan University, Shanghai" in China September 16–20, 2013 in honor of Luc Tartar. These results were first obtained in [5].

Turbulent flows are chaotic systems, highly sensitive to small changes in data (see [15]), which means that any tiny change in body forces, any external action and/or initial data, might give rise almost instantly to significant changes in the flow features.

To be more specific, let us consider an experiment which measures the velocity (or one of its components) of a turbulent flow N times at a given point. Each measurement is carried out under the same conditions (the same initial data, constant temperature, and the same source). Although advanced technologies allow measurements to be made to high precision, the experiment will yield N different results, because in reality infinitesimal changes occur during each measurement that cannot be controlled.

Moreover, because of the structure of the turbulence, any code using the Navier-Stokes equations (NSE for short)

$$\begin{cases}\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0,\end{cases} \tag{1.1}$$

that specify flow motions (see [2, 5]), would be very complex and would require too many computational resources in order to run the simulation. In the equations above, $\mathbf{v} = (v_1, v_2, v_3) = \mathbf{v}(t, \mathbf{x})$ denotes the Eulerian velocity of the fluid, $p = p(t, \mathbf{x})$ denotes its pressure, $(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega$

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for some bounded domain $\Omega \subset \mathbb{R}^3$, $\nu > 0$ is the kinematic viscosity and $\mathbf{f}$ is a given external force. Throughout this paper, we assume that $\mathbf{v}$ satisfies the no slip boundary condition, i.e., $\mathbf{v}|_\Gamma = 0$, and that $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x}) = \mathbf{v}(0, \mathbf{x})$ is a given initial data.

A long time ago, Reynolds [14], Stokes [17], Boussinesq [3] and Prandtl [13] suggested to decompose the flow field as the sum of a mean field and a fluctuation, i.e.,

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad p = \overline{p} + p'. \tag{1.2}$$

In those works, the means $\overline{\mathbf{v}}$ and $\overline{p}$ were formally expressed by long time averages

$$\overline{\mathbf{v}}(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{v}(t, \mathbf{x})\mathrm{d}t, \quad \overline{p}(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t, \mathbf{x})\mathrm{d}t. \tag{1.3}$$

Later, Taylor [20] and then Kolmogorov [7] considered statiscal means instead of long-time averages (see details also in [5]).

We focus in this paper on the long-time average (1.3), and in particular:

(i) We show that the long-time average $(\overline{\mathbf{v}}, \overline{p})$ is well-defined in some Sobolev spaces for global turbulent solutions of the NSE (1.1), when the domain Ω is smooth enough, and under appropriate assumptions on the source term $\mathbf{f}$ and the initial data $\mathbf{v}_0$.

(ii) We show that $(\overline{\mathbf{v}}, \overline{p})$ satisfy the steady-state NSE, with an additional source term of the form $-\nabla\sigma^{(\mathrm{R})}$, where $\sigma^{(\mathrm{R})}$ is a Reynolds stress. Finally, We show that $\sigma^{(\mathrm{R})}$ is dissipative.

We mention that recently Layton [10] showed that, for smooth solutions of the NSE that satisfy the energy equality, the Reynolds stress is also dissipative when considering ensemble averages.

This paper is organised as follows. Section 2 is devoted to outlining the functional framework which we shall use, to recalling the basic Leray-Hopf result (see [6, 11]) that states the existence of turbulent solutions of the NSE, and to deriving from the energy inequality long-time estimates. We then proceed with the programme set out above in Section 3.

2 Framework and Basic Results

2.1 Functional spaces

We assume in this section that Γ is of class C^1 for simplicity.[1] For given q, p, s, we set

$$\mathbf{L}^q(\Omega) = \{\mathbf{w} = (w_1, w_2, w_3);\, w_i \in L^q(\Omega),\ i = 1, 2, 3\}, \tag{2.1}$$

$$\mathbf{W}^{s,p}(\Omega) = \{\mathbf{w} = (w_1, w_2, w_3);\, w_i \in W^{s,p}(\Omega),\ i = 1, 2, 3\}. \tag{2.2}$$

We denote by $\|\cdot\|_{q,p,\Omega}$ the standard $\mathbf{W}^{s,p}(\Omega)$ norm. For any $s > \frac{1}{2}$, we consider the spaces

$$\mathbf{H}^s(\Omega) = \{\mathbf{w} = (w_1, w_2, w_3);\, w_i \in H^s(\Omega),\ i = 1, 2, 3\}, \tag{2.3}$$

$$\mathbf{H}^s_0(\Omega) = \{\mathbf{w} \in \mathbf{H}^s(\Omega);\, \gamma_0\mathbf{w} = 0 \text{ on } \Gamma\}. \tag{2.4}$$

In the above definition, γ_0 is the trace operator, which is defined by

$$\gamma_0\varphi = \varphi|_\Gamma, \quad \forall\, \varphi \in C^\infty(\overline{\Omega}),$$

that can be extended to $H^s(\Omega)$, when $s > \frac{1}{2}$, in a continuous operator with values in the space $H^{s-\frac{1}{2}}(\Gamma)$. When no risk of confusion occurs, we also denote $\gamma_0\mathbf{w} = \mathbf{w}$. The space $\mathbf{H}^1_0(\Omega)$ is equipped with its standard norm

$$\|\mathbf{w}\|_{H^1_0(\Omega)} = \|\nabla\mathbf{w}\|_{0,2,\Omega},$$

[1] Many results reported in this section also hold for Lipchitz domains (see for instance [18]).

which is a norm equivalent to the $\|\cdot\|_{1,2,\Omega}$ norm, due to the Poincaré's inequality. Details about Sobolev spaces can be found in [19]. We shall also make use of the following spaces:

$$\mathcal{V}_{\rm div}(\Omega) = \{\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3),\ \varphi_i \in \mathcal{D}(\Omega),\ \nabla\cdot\boldsymbol{\varphi} = 0\}, \tag{2.5}$$

$$\mathbf{V}_{\rm div}(\Omega) = \{\mathbf{w} \in \mathbf{H}^1_0(\Omega),\ \nabla\cdot\mathbf{w} = 0\}, \tag{2.6}$$

$$\mathbf{L}^2_{\rm div,0}(\Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega),\ \gamma_n\mathbf{w} = 0 \text{ on } \Gamma,\ \nabla\cdot\mathbf{w} = 0\}. \tag{2.7}$$

In the definition above, γ_n is the normal trace operator, which is defined by

$$\gamma_n\boldsymbol{\varphi} = \boldsymbol{\varphi}\cdot\mathbf{n}|_\Gamma, \quad \forall\,\boldsymbol{\varphi} \in C^\infty(\overline{\Omega})^3,$$

the vector $\mathbf{n}$ being the outward-pointing unit normal vector to Γ. We know that this operator can be extended to $\mathbf{L}^2_{\rm div}(\Omega)$, in a continuous operator with values in the space $H^{-\frac{1}{2}}(\Gamma)$ (see [8]), where

$$\mathbf{L}^2_{\rm div}(\Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega);\ \nabla\cdot\mathbf{w} \in L^2(\Omega)\}.$$

2.2 Variational formulation of the NSE

For simplicity, we denote by (u, v) the duality pairing $\langle L^{p'}(\Omega), L^p(\Omega)\rangle$,

$$(u, v)_\Omega = \int_\Omega u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x},$$

and we define the diffusion and transport operators by

$$a(\mathbf{v}, \mathbf{w}) = \nu(\nabla\mathbf{v}, \nabla\mathbf{w})_\Omega, \quad b(\mathbf{z}; \mathbf{v}, \mathbf{w}) = ((\mathbf{z}\cdot\nabla)\,\mathbf{v}, \mathbf{w})_\Omega, \tag{2.8}$$

respectively. We know that these multilinear forms are continuous over $\mathbf{H}^1(\Omega)$ (see [5]). Moreover, we also know that, $\forall\,\mathbf{z}, \mathbf{v} \in \mathbf{V}_{\rm div}(\Omega)$, $\forall\,p \in L^2(\Omega)$,

$$b(\mathbf{z}; \mathbf{v}, \mathbf{v}) = 0, \quad \langle\nabla p, \mathbf{v}\rangle = -(p, \nabla\cdot\mathbf{v}) = 0. \tag{2.9}$$

We assume from now on that

$$\mathbf{v}_0 \in \mathbf{L}^2_{\rm div,0}(\Omega), \quad \mathbf{f} \in L^2_{\rm loc}(\mathbb{R}_+, \mathbf{V}_{\rm div}(\Omega)'). \tag{2.10}$$

Following [6, 11], we say that $\mathbf{v}$ is a turbulent solution of the NSE (1.1) if and only if $\forall\,T > 0$,

$$\begin{cases}\mathbf{v} \in L^2([0,T], \mathbf{V}_{\rm div}(\Omega)) \cap C_w([0,T], \mathbf{L}^2_{\rm div,0}(\Omega)),\\ \partial_t\mathbf{v} \in L^{\frac{4}{3}}[0,T], \mathbf{V}_{\rm div}(\Omega)'),\end{cases} \tag{2.11}$$

$$\lim_{t\to 0}\|\mathbf{v}(t,\cdot) - \mathbf{v}_0(\cdot)\|_{0,2,\Omega} = 0, \tag{2.12}$$

and $\forall\,\mathbf{w} \in \mathbf{V}_{\rm div}(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}, \mathbf{w})_\Omega + b(\mathbf{z}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) = \langle\mathbf{f}, \mathbf{w}\rangle \quad \text{in } \mathcal{D}'([0,T]). \tag{2.13}$$

Remark 2.1 According to the definition of the space $L^p([0,T], E)$ through the Bochner integral, where E is any given Banach space (see [16]), (2.12) can be replaced by

$$\begin{aligned}&\int_0^T\langle\partial_t\mathbf{v}, \mathbf{w}\rangle\mathrm{d}t + \int_0^T\int_\Omega((\mathbf{v}\cdot\nabla)\,\mathbf{v})(t,\mathbf{x})\cdot\mathbf{w}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &+\nu\int_0^T\int_\Omega\nabla\mathbf{v}(t,\mathbf{x}) : \nabla\mathbf{w}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ =&\int_0^T\langle\mathbf{f}, \mathbf{w}\rangle\mathrm{d}t, \quad \forall\,\mathbf{w} \in L^4([0,T], \mathbf{V}_{\rm div}(\Omega))\end{aligned} \tag{2.14}$$

(see [5] for instance).

The following existence result is standard (see [6, 11]).

Theorem 2.1 *The NSE* (1.1) *has a turbulent solution which satisfies the energy inequality at every* $t \in [0,T]$, *i.e.,*

$$\frac{\mathrm{d}}{2\mathrm{d}t}\|\mathbf{v}(t,\cdot)\|^2_{0,2,\Omega} + \nu\|\nabla\mathbf{v}(t,\cdot)\|^2_{0,2,\Omega} \le \langle \mathbf{f}, \mathbf{v}\rangle \quad \text{in } \mathcal{D}'([0,T]). \tag{2.15}$$

The uniqueness of this solution is still an open problem at the time of writing this paper. Similarly, we do not know if the energy inequality (2.15) is an equality. The energy inequality (2.15) also yields

$$\frac{1}{2}\|\mathbf{v}(t,\cdot)\|^2_{0,2,\Omega} + \nu\int_0^t \|\nabla\mathbf{v}\|^2_{0,2,\Omega} \le \frac{1}{2}\|\mathbf{v}_0\|^2_{0,2,\Omega} + \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle \tag{2.16}$$

for all $t > 0$. The pressure is recovered from the De Rham's theorem, leading to the following statement (see for instance [9, 12, 18, 21]).

Lemma 2.1 *There exists* $p \in \mathcal{D}'([0,T], L^2_0(\Omega))$, *such that* $(\mathbf{v}, p)$ *is a solution of the NSE* (1.1) *in the sense of distributions.*

In the statement above,

$$L^2_0(\Omega) = \Big\{ q \in L^2(\Omega);\ \int_\Omega q(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \Big\}.$$

The pressure p is considered as a constraint in this kind of formulation. Therefore, p is called a Lagrange multiplier. It can also be proved that $p \in L^{\frac{5}{4}}(Q)$, $Q = [0,T] \times \Omega$ (see for instance [4]).

2.3 Long-time estimate

From now on and until the end of the report, we assume that the source term $\mathbf{f} \in \mathbf{H}^{-1}(\Omega) \subset \mathbf{V}_{\mathrm{div}}(\Omega)'$ does not depend on t, and we set $F = \|\mathbf{f}\|_{-1,2,\Omega}$.

The real number μ denotes the best constant in the Poincaré's inequality, written as

$$C\|\mathbf{v}\|_{0,2,\Omega} \le \|\nabla\mathbf{v}\|_{0,2,\Omega}, \quad \forall\, \mathbf{v} \in H^1_0(\Omega).$$

The energy inequality (2.16) yields that $\|\mathbf{v}(t,\cdot)\|_{0,2,\Omega}$ is bounded uniformly in t. To be more specific, we prove the following proposition.

Proposition 2.1 *Let* $\mathbf{v}$ *be any turbulent solution to the NSE. Then we have*

$$\|\mathbf{v}(t,\cdot)\|^2_{0,2,\Omega} \le \|\mathbf{v}_0\|^2_{0,2,\Omega}\,\mathrm{e}^{-\nu\mu t} + \frac{F^2}{\nu^2\mu}(1 - \mathrm{e}^{-\nu\mu t}) \tag{2.17}$$

for all $t > 0$.

Proof Set

$$W(t) = \|\mathbf{v}(t,\cdot)\|^2_{0,2,\Omega}, \quad W(0) = \|\mathbf{v}_0\|^2_{0,2,\Omega}. \tag{2.18}$$

Energy inequality (2.15) yields

$$\frac{1}{2}W'(t) + \nu \int_\Omega |\nabla \mathbf{v}|^2 \le \langle \mathbf{f}, \mathbf{v} \rangle \le \frac{F^2}{2\nu} + \frac{\nu}{2} \int_\Omega |\nabla \mathbf{v}|^2. \tag{2.19}$$

We apply Poincaré's inequality in the second term of the left-hand side of (2.19), leading to

$$W'(t) + \nu\mu W(t) \le \frac{F^2}{\nu}. \tag{2.20}$$

Therefore, W is a subsolution of the ordinary differential equation

$$\begin{cases} \lambda'(t) + \nu\mu\lambda(t) = \dfrac{F^2}{\nu}, \\ \lambda(0) = W(0), \end{cases} \tag{2.21}$$

the solution of which is

$$\lambda(t) = W(0)\mathrm{e}^{-\nu\mu t} + \frac{F^2}{\nu^2\mu}(1 - \mathrm{e}^{-\nu\mu t}), \tag{2.22}$$

and hence (2.17) holds.

As a consequence, we deduce that the turbulent solution is well-defined all over $\mathbb{R}_+$, and hence can be extended to $L^\infty(\mathbb{R}_+, \mathbf{L}^2_{\mathrm{div}}(\Omega))$ as a global time solution. In particular, we have

$$\sup_{t\ge 0} \|\mathbf{v}(t,\cdot)\|^2_{0,2,\Omega} \le \max_{t \ge 0} K(t) = E_\infty, \tag{2.23}$$

where

$$K(t) = \|\mathbf{v}_0\|^2_{0,2,\Omega}\, \mathrm{e}^{-\nu\mu t} + \frac{F^2}{\nu^2\mu}(1 - \mathrm{e}^{-\nu\mu t}). \tag{2.24}$$

We also deduce from (2.19) combined with (2.23), the following inequality:

$$\frac{1}{t} \int\int_{Q_t} |\nabla \mathbf{v}(s, \mathbf{x})|^2 \mathrm{d}\mathbf{x}\mathrm{d}s \le \frac{F^2}{\nu^2} + \frac{\|\mathbf{v}_0\|^2_{0,2,\Omega}}{\nu t}, \quad \forall\, t > 0. \tag{2.25}$$

Moreover, from standard interpolation inequalites (see [5]), we infer that

$$\|\mathbf{v}\|_{0,\frac{10}{3},Q_t} \le C_1 E_\infty^{\frac{1}{5}} \|\nabla \mathbf{v}\|^{\frac{3}{5}}_{0,2,Q_t}, \quad \mathbf{v} \in \mathbf{L}^{\frac{10}{3}}(Q_t),\ \forall\, t > 0, \tag{2.26}$$

leading to

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} \in L^{\frac{5}{4}}(Q_t), \quad \|(\mathbf{v}\cdot\nabla)\,\mathbf{v}\|_{0,\frac{5}{4},Q_t} \le C_1 E_\infty^{\frac{1}{5}} \|\nabla \mathbf{v}\|^{\frac{8}{5}}_{0,2,Q_t}. \tag{2.27}$$

3 Main Results

3.1 Long-time average operator

We start with the study of the mean operator M_t over $[0, t]$, for a given fixed time $t > 0$, expressed by

$$M_t(\boldsymbol{\psi}) = \frac{1}{t} \int_0^t \boldsymbol{\psi}(s, \mathbf{x})\, \mathrm{d}s, \tag{3.1}$$

$\boldsymbol{\psi} = \boldsymbol{\psi}(t, \mathbf{x})$ being any given field.

Lemma 3.1 *Let $t > 0$, $Q_t = [0,t] \times \Omega$. Assume $\boldsymbol{\psi} \in \mathbf{L}^p(Q_t)$. Then $M_t(\boldsymbol{\psi}) \in \mathbf{L}^p(\Omega)$ and one has*

$$\|M_t(\boldsymbol{\psi})\|_{0,p,\Omega} \leq \frac{1}{t^{\frac{1}{p}}} \|\boldsymbol{\psi}\|_{0,p,Q_t}. \tag{3.2}$$

Proof By the Hölder's inequality, we have

$$\Big| \frac{1}{t} \int_0^t \boldsymbol{\psi}(s,\mathbf{x}) \, \mathrm{d}s \Big| \leq \frac{1}{t} \int_0^t |\boldsymbol{\psi}(s,\mathbf{x})|^p \, \mathrm{d}s. \tag{3.3}$$

Thus (3.2) follows by Fubini's theorem.

We study the effect of M_t on $(\mathbf{v}, p)$ in defining

$$\mathbf{V}_t(\mathbf{x}) = M_t(\mathbf{v})(\mathbf{x}), \quad P_t(\mathbf{x}) = M_t(p)(\mathbf{x}). \tag{3.4}$$

We deduce from the NSE that $(\mathbf{V}_t, P_t)$ is the solution of the following Stokes problem, at least in the sense of distributions,

$$\begin{cases} -\nu \Delta \mathbf{V}_t + \nabla P_t = -M_t((\mathbf{v} \cdot \nabla)\, \mathbf{v}) + \mathbf{f} + \boldsymbol{\varepsilon}_t \quad \text{in } Q, \\ \nabla \cdot \mathbf{V}_t = 0 \quad \text{in } Q, \\ \mathbf{V}_t = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.5}$$

In (3.5),

$$\boldsymbol{\varepsilon}_t(\mathbf{x}) = \frac{\mathbf{v}_0(\mathbf{x}) - \mathbf{v}(t,\mathbf{x})}{t}, \tag{3.6}$$

which goes to zero in $\mathbf{L}^2(\Omega)$ when $t \to +\infty$, according to (2.23).

3.2 Existence of velocity-pressure long-time averages

In addition to the previous assumptions, we assume now that the domain Ω is of class $C^{\frac{9}{4},1}$, and $\mathbf{f} \in \mathbf{L}^{\frac{5}{4}}(\Omega) \cap \mathbf{H}^{-1}(\Omega)$ does not depend on t, $\mathbf{v}_0 \in \mathbf{L}^2_{\mathrm{div},0}(\Omega)$.

Theorem 3.1 *There exists*
(i) *a sequence $(t_n)_{n \in \mathbf{N}}$ that goes to $+\infty$ when $n \to +\infty$,*
(ii) $(\overline{\mathbf{v}}, \overline{p}) \in \mathbf{W}^{2,\frac{5}{4}}(\Omega) \times \mathbf{W}^{1,\frac{5}{4}}(\Omega)/\mathbb{R}$,
(iii) $\mathbf{F} \in \mathbf{L}^{\frac{5}{4}}(\Omega)$,
such that $(\mathbf{V}_{t_n}, P_{t_n})_{n \in \mathbf{N}}$ converges to $(\overline{\mathbf{v}}, \overline{p})$, weakly in $\mathbf{W}^{2,\frac{5}{4}}(\Omega) \times \mathbf{W}^{1,\frac{5}{4}}(\Omega)/\mathbb{R}$, that satisfies

$$\begin{cases} (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = -\mathbf{F} + \mathbf{f} \quad \textit{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 \quad \textit{in } \Omega, \\ \overline{\mathbf{v}} = 0 \quad \textit{on } \Gamma, \end{cases} \tag{3.7}$$

in the sense of distributions.

Proof The proof is divided into 3 steps. We first find estimates and extract convergent subsequences. We then take the limit in the equations, firstly in the conservation equation, and then in the momentum equation.

Step 1 We first show that the nonlinear term $-M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})$ is bounded in $\mathbf{L}^{\frac{5}{4}}(\Omega)$. By (3.2), we have

$$\|M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})\|_{0,\frac{5}{4},\Omega} \le \frac{1}{t^{\frac{4}{5}}}\|(\mathbf{v}\cdot\nabla)\,\mathbf{v}\|_{0,\frac{5}{4},Q_t}, \tag{3.8}$$

where $Q_t = [0,t]\times\Omega$. Combining this inequality with (2.23) and (2.27), we find

$$\|M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})\|^{\frac{5}{4}}_{0,\frac{5}{4},\Omega} \le C_1^{\frac{5}{4}} E_\infty^{\frac{1}{4}}\Big(\frac{1}{t}\int_0^t\int_\Omega |\nabla\mathbf{v}(s,\mathbf{x})|^2\mathrm{d}\mathbf{x}\mathrm{d}s\Big), \tag{3.9}$$

and hence $(M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}))_{t>0}$ is bounded in $\mathbf{L}^{\frac{5}{4}}(\Omega)$, uniformly in t due to (2.25). Since Ω is of class $C^{1+\frac{5}{4},1} = C^{\frac{9}{4},1}$, $\mathbf{f}\in\mathbf{L}^{\frac{5}{4}}(\Omega)$ and

$$(M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}))_{t>0} \text{ and } (\boldsymbol{\varepsilon}_t)_{t>0} \text{ are bounded in } \mathbf{L}^{\frac{5}{4}}(\Omega), \tag{3.10}$$

the results in [1] apply, that is, there exists a unique solution (V_t, P_t) of (3.5) that satisfies

$$\|\mathbf{V}_t\|_{2,\frac{5}{4},\Omega} + \|P_t\|_{W^{1,\frac{5}{4}}(\Omega)\mathbb{R}} \le \|M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})\|_{0,\frac{5}{4},\Omega} + \|\mathbf{f}\|_{0,\frac{5}{4},\Omega} + \|\boldsymbol{\varepsilon}_t\|_{0,\frac{5}{4},\Omega}. \tag{3.11}$$

Because of uniqueness, this solution (V_t, P_t) is indeed that defined by (3.4). Statement (3.10) combined with estimate (3.11) ensures that

$$\begin{cases}(\mathbf{V}_t)_{t>0} \text{ is bounded in } \mathbf{W}^{2,\frac{5}{4}}(\Omega),\\ (P_t)_{t>0} \text{ is bounded in } W^{1,\frac{5}{4}}(\Omega)/\mathbb{R}.\end{cases} \tag{3.12}$$

Therefore, there exist

$$\overline{\mathbf{v}}\in\mathbf{W}^{2,\frac{5}{4}}(\Omega), \quad \overline{p}\in W^{1,\frac{5}{4}}(\Omega)/\mathbb{R}, \quad \mathbf{B}\in\mathbf{L}^{\frac{5}{4}}(\Omega),$$

and a sequence $(t_n)_{n\in\mathbf{N}}$ which goes to ∞ as $n\to\infty$, such that

$$\lim_{n\to\infty}\mathbf{V}_{t_n} = \overline{\mathbf{v}} \quad \text{weakly in } \mathbf{W}^{2,\frac{5}{4}}(\Omega), \tag{3.13}$$

$$\lim_{n\to\infty} M_{t_n} = \overline{p} \quad \text{weakly in } W^{1,\frac{5}{4}}(\Omega)/\mathbb{R}, \tag{3.14}$$

$$\lim_{n\to\infty} M_{t_n}((\mathbf{v}\cdot\nabla)\,\mathbf{v}) = \mathbf{B} \quad \text{weakly in } L^{\frac{5}{4}}(\Omega)^9. \tag{3.15}$$

Moreover, $W^{2,\frac{5}{4}}(\Omega)\hookrightarrow W^{1,\frac{15}{7}}(\Omega)$, the injection being compact. Then,

$$(\mathbf{V}_{t_n})_{n\in\mathbf{N}} \text{ converges to } \overline{\mathbf{v}} \text{ strongly in } \mathbf{W}^{1,\frac{15}{7}}(\Omega). \tag{3.16}$$

Step 2 We check that $\nabla\cdot\overline{\mathbf{v}} = 0$ in an appropriate Lebesgue space. To do so, we first prove that $\nabla\cdot V_t = 0$ in $\mathcal{D}'(Q_T)$ regardless of $T>0$. For any given $\varphi\in\mathcal{D}(Q_T)$, we have

$$\begin{aligned}\langle\nabla\cdot\mathbf{V}_t,\varphi\rangle &= \iint_Q \nabla\cdot\Big(\frac{1}{t}\int_0^t \mathbf{v}(s,\mathbf{x})\mathrm{d}s\Big)\varphi(t,\mathbf{x})\,\mathrm{d}x\mathrm{d}t\\ &= -\iint_Q \Big(\int_0^t \mathbf{v}(s,\mathbf{x})\mathrm{d}s\Big)\cdot\frac{1}{t}\nabla\varphi(t,\mathbf{x})\,\mathrm{d}x\mathrm{d}t\\ &= \iint_Q\int_0^t \mathbf{v}(t,\mathbf{x})\cdot\Big(\int_0^t\frac{1}{s}\nabla\varphi(s,\mathbf{x})\mathrm{d}s\Big)\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.17}$$

which holds because $\varphi \in \mathcal{D}(Q_T)$. Moreover, since $\varphi \in \mathcal{D}(Q_T)$,

$$\int_0^t \frac{1}{s} \nabla \varphi(s, \mathbf{x}) \mathrm{d}s = \nabla \int_0^t \frac{\varphi(s, \mathbf{x})}{s} \mathrm{d}s = \nabla \psi(t, \mathbf{x}), \quad \forall\, t \in [0, T]. \tag{3.18}$$

Therefore, we deduce from (3.17)–(3.18) that

$$\langle \nabla \cdot \mathbf{V}_t, \varphi \rangle = \langle \mathbf{v}, \nabla \psi \rangle = -\langle \nabla \cdot \mathbf{v}, \psi \rangle = 0. \tag{3.19}$$

Because $\nabla \cdot \mathbf{v} = 0$, we have $\langle \nabla \cdot \mathbf{V}_t, \varphi \rangle = 0$. Then,

$$\nabla \cdot \mathbf{V}_t = 0 \quad \text{in } \mathcal{D}'(Q_T),\ \forall\, T > 0. \tag{3.20}$$

Furthermore, by setting $\mathbf{V}_0 = \mathbf{v}_0$, we get $\mathbf{V}_t \in C([0, T], \mathbf{L}^2(\Omega))$, so that (3.20) becomes

$$\nabla \cdot \mathbf{V}_t = 0 \quad \text{in } \mathbf{H}^{-1}(\Omega),\ \forall\, t \in [0, T],$$

and in reality in $\mathbf{L}^{\frac{15}{7}}(\Omega)$ by (3.16), and regardless of $T > 0$, which allows us to take the limit as $t_n \to \infty$, leading to $\nabla \cdot \overline{\mathbf{v}} = 0$ in $\mathbf{L}^{\frac{15}{7}}(\Omega)$.

Step 3 We now take the limit in the momentum equation. Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Since $\boldsymbol{\varphi}, \nabla \boldsymbol{\varphi}, \Delta \boldsymbol{\varphi} \in \mathbf{L}^5(\Omega)$, we deduce from (3.13)–(3.15) and the convergence to zero of $(\boldsymbol{\varepsilon}_{t_n})_{n \in \mathbf{N}}$ in all $\mathbf{L}^p(\Omega)$, $p \leq 2$, on the one hand,

$$\lim_{n \to \infty} \langle M_{t_n}((\mathbf{v} \cdot \nabla)\, \mathbf{v}), \boldsymbol{\varphi} \rangle = \lim_{n \to \infty} (M_{t_n}((\mathbf{v} \cdot \nabla)\, \mathbf{v}), \boldsymbol{\varphi})_\Omega = (\mathbf{B}, \boldsymbol{\varphi})_\Omega = \langle \mathbf{B}, \boldsymbol{\varphi} \rangle, \tag{3.21}$$

and on the other hand,

$$\begin{aligned} \lim_{n \to \infty} \langle \boldsymbol{\varepsilon}_{t_n}, \boldsymbol{\varphi} \rangle &= \lim_{n \to \infty} (\boldsymbol{\varepsilon}_{t_n}, \boldsymbol{\varphi})_\Omega = 0, \\ \lim_{n \to \infty} \langle -\Delta \mathbf{V}_{t_n}, \boldsymbol{\varphi} \rangle &= \lim_{n \to \infty} (\mathbf{V}_{t_n}, -\Delta \boldsymbol{\varphi})_\Omega = (\overline{\mathbf{v}}, -\Delta \boldsymbol{\varphi})_\Omega = (-\Delta \overline{\mathbf{v}}, \boldsymbol{\varphi})_\Omega, \\ \lim_{n \to \infty} \langle \nabla P_{t_n}, \boldsymbol{\varphi} \rangle &= -\lim_{n \to \infty} (P_{t_n}, \nabla \cdot \boldsymbol{\varphi})_\Omega = -(\overline{p}, \nabla \cdot \boldsymbol{\varphi})_\Omega = \langle \nabla \overline{p}, \boldsymbol{\varphi} \rangle, \end{aligned}$$

which shows by (3.5) that $(\overline{\mathbf{v}}, \overline{p})$ satisfies in $\mathcal{D}'(\Omega)$,

$$\begin{cases} -\nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = -\mathbf{B} + \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 \quad \text{in } \Omega, \\ \overline{\mathbf{v}} = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.22}$$

Let $\mathbf{F}$ denote the tensor defined by

$$\mathbf{F} = \mathbf{B} - (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} = \mathbf{B} - \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}). \tag{3.23}$$

As $W^{2, \frac{5}{4}}(\Omega) \hookrightarrow L^{\frac{15}{2}}(\Omega)$ and $W^{2, \frac{5}{4}}(\Omega) \hookrightarrow W^{1, \frac{15}{7}}(\Omega)$, we get

$$\nabla \overline{\mathbf{v}} \in \mathbf{L}^{\frac{15}{7}}(\Omega)^3 \ \text{ and } \ \overline{\mathbf{v}} \in \mathbf{L}^{\frac{15}{2}}(\Omega), \ \text{ so then } (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} \in \mathbf{L}^{\frac{15}{9}}(\Omega) \hookrightarrow \mathbf{L}^{\frac{5}{4}}(\Omega),$$

and we deduce that $\mathbf{F} \in \mathbf{L}^{\frac{5}{4}}(\Omega)$. Hence $(\overline{\mathbf{v}}, \overline{p})$ satisfies (3.7) in the sense of distributions.

Corollary 3.1 *The long-time velocity* $\overline{\mathbf{v}}$ *is a solution of the following variational problem:*

$$b(\overline{\mathbf{v}}; \overline{\mathbf{v}}, \mathbf{w}) + a(\overline{\mathbf{v}}, \mathbf{w}) = -(\mathbf{F}, \mathbf{w})_\Omega + (\mathbf{f}, \mathbf{w})_\Omega, \quad \forall\, \mathbf{w} \in \mathbf{W}^{1,5}_{\mathrm{div}}(\Omega), \tag{3.24}$$

the operators a *and* b *being defined by* (2.8).

Remark 3.1 The proof of Theorem 3.1 contains the proof of the general identity, $\forall\, p \geq 1$, $\forall\, T > 0$, $\forall\, t \in [0, T]$,

$$\nabla \cdot M_t(\boldsymbol{\varphi}) = M_t(\nabla \cdot \boldsymbol{\varphi}), \quad \forall\, \boldsymbol{\varphi} \in L^1([0,T], \mathbf{W}^{1,p}(\Omega)). \tag{3.25}$$

Furthermore, the same reasoning also yields

$$\nabla M_t(\varphi) = M_t(\nabla \varphi), \tag{3.26}$$

which is called the Reynolds rule.

3.3 Reynolds decomposition

We aim to identify the source term $\mathbf{F}$ that appears in (3.7), to link the results of Theorem 3.1 with the usual approach to modelling turbulence, by introducing the Reynolds decomposition and the Reynolds stress.

Let $\mathbf{v}$ be a given turbulent solution of the NSE, and p be its associated pressure. We respect the conditions for the application of Theorem 3.1, which ensures that we can split $(\mathbf{v}, p)$ into

$$\mathbf{v}(t, \mathbf{x}) = \overline{\mathbf{v}}(\mathbf{x}) + \mathbf{v}'(t, \mathbf{x}), \tag{3.27}$$

$$p(t, \mathbf{x}) = \overline{p}(\mathbf{x}) + p'(t, \mathbf{x}), \tag{3.28}$$

where $(\mathbf{v}', p')$ stands for the fluctuations around the mean field $(\overline{\mathbf{v}}, \overline{p})$. We call the decomposition (3.27)–(3.28) a Reynolds decomposition.

To identify the source term $\mathbf{F}$ in (3.7), we start from (3.5) and notice that, according to the Reynolds rule (3.26),

$$M_t((\mathbf{v} \cdot \nabla)\, \mathbf{v}) = M_t(\nabla \cdot (\mathbf{v} \otimes \mathbf{v})) = \nabla \cdot M_t(\mathbf{v} \otimes \mathbf{v}).$$

We shall find out from the Reynolds decomposition that it suffices to study the convergence of

$$M_t(\mathbf{v}' \otimes \mathbf{v}')(\mathbf{x}) = \frac{1}{t} \int_0^t \mathbf{v}'(s, \mathbf{x}) \otimes \mathbf{v}'(s, \mathbf{x})\, \mathrm{d}s \tag{3.29}$$

as $t \to \infty$, which yields what we call a Reynolds stress, denoted by $\sigma^{(\mathrm{R})}$.

Remark 3.2 The definition of $(\overline{\mathbf{v}}, \overline{p})$, and hence the Reynolds decomposition (3.27)–(3.28) and the Reynolds stress that we shall find, depend on the sequence $(t_n)_{n \in \mathbf{N}}$ that appears in Theorem 3.1, and we do not know if the limit of $(\mathbf{V}_t, P_t)_{t>0}$ is solely defined when $t \to \infty$. As a result, we do not know if $\mathbf{F}$ is solely defined either, and even if it were, it is not known if (3.7) has a unique solution. All of these imply that without any further information, this analysis will not provide means and decomposition that are intrinsically defined.

3.4 Reynolds stress

Theorem 3.2 *Let* $(t_n)_{n \in \mathbf{N}}$ *be as in Theorem* 3.1 *and* $\mathbf{F}$ *as in* (3.7). *Then there exists* $\sigma^{(\mathrm{R})} \in \mathbf{L}^{\frac{5}{3}}(\Omega)^3$ *such that*

(i) *we can extract from* $(M_{t_n}(\mathbf{v}' \otimes \mathbf{v}'))_{n \in \mathbf{N}}$ *a subsequence, that we denote by* $(M_{t_n}(\mathbf{v}' \otimes \mathbf{v}'))_{n \in \mathbf{N}}$, *which converges to* $\sigma^{(\mathrm{R})}$ *weakly in* $\mathbf{L}^{\frac{5}{3}}(\Omega)$;

(ii) $\mathbf{F} = \nabla \cdot \sigma^{(\mathrm{R})}$ *in* $\mathcal{D}'(\Omega)$;

(iii) *the following energy balance holds:*

$$\nu\|\nabla\overline{\mathbf{v}}\|^2_{0,2,\Omega} + \langle \mathbf{F}, \overline{\mathbf{v}}\rangle = (\mathbf{f}, \overline{\mathbf{v}})_\Omega; \tag{3.30}$$

(iv) $\mathbf{F}$ *is dissipative, in the sense*

$$\langle \mathbf{F}, \overline{\mathbf{v}}\rangle \geq 0. \tag{3.31}$$

Proof Remember that M_t is defined by (3.1). We derive from (3.27)–(3.28) that

$$V_{t_n} = \overline{\mathbf{v}} + M_{t_n}(\mathbf{v}'), \quad P_{t_n} = \overline{p} + M_{t_n}(p'). \tag{3.32}$$

Therefore we deduce

$$\overline{\mathbf{v}'} = \lim_{n\to\infty} M_{t_n}(\mathbf{v}') = 0, \quad \overline{p'} = \lim_{n\to\infty} M_{t_n}(p') = 0, \tag{3.33}$$

the limit being weak in $\mathbf{W}^{2,\frac{5}{4}}(\Omega)$ and $\mathbf{W}^{1,\frac{5}{4}}(\Omega)/\mathbb{R}$, respectively. In addition $(t_n)_{n\in\mathbf{N}}$ can be chosen such that the convergence of $(M_{t_n}(\mathbf{v}'))_{n\in\mathbf{N}}$ toward 0 is strong in $\mathbf{L}^{\frac{15}{2}}(\Omega)$ because the injection

$$W^{2,\frac{5}{4}}(\Omega) \hookrightarrow L^{\frac{15}{2}}(\Omega)$$

is compact. We now demonstrate each item of the above statement.

Proof of (i) By using decomposition (3.27), we write

$$\mathbf{v}\otimes\mathbf{v} = \overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + \mathbf{v}'\otimes\overline{\mathbf{v}} + \overline{\mathbf{v}}\otimes\mathbf{v}' + \mathbf{v}'\otimes\mathbf{v}', \tag{3.34}$$

leading to

$$M_t(\mathbf{v}\otimes\mathbf{v}) = \overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + M_t(\mathbf{v}')\otimes\overline{\mathbf{v}} + \overline{\mathbf{v}}\otimes M_t(\mathbf{v}') + M_t(\mathbf{v}'\otimes\mathbf{v}') \tag{3.35}$$

for each $t > 0$. As both $\overline{\mathbf{v}}$ and $M_t(\mathbf{v}') \in \mathbf{L}^{\frac{15}{2}}(\Omega)$, we obtain from Hölder's inequality,

$$M_t(\mathbf{v}')\otimes\overline{\mathbf{v}} \text{ and } \overline{\mathbf{v}}\otimes M_t(\mathbf{v}') \in L^{1\frac{5}{4}}(\Omega)^9 \hookrightarrow L^{\frac{5}{3}}(\Omega)^9.$$

In particular, (3.33) yields

$$\lim_{n\to\infty} M_{t_n}(\mathbf{v}')\otimes\overline{\mathbf{v}} = \lim_{n\to\infty}\overline{\mathbf{v}}\otimes M_{t_n}(\mathbf{v}') = 0, \tag{3.36}$$

strongly in $L^{\frac{5}{3}}(\Omega)^9$. Moreover, we infer from (3.2), combined with (2.23) and (2.26), that

$$\|M_t(\mathbf{v}\otimes\mathbf{v})\|_{0,\frac{5}{3},\Omega} \leq C_1^{\frac{10}{3}} E_\infty^{\frac{2}{3}}\Big(\frac{1}{t}\int_0^t\int_\Omega |\nabla\mathbf{v}|^2 \mathrm{d}\mathbf{x}\mathrm{d}s\Big). \tag{3.37}$$

We are led to rewrite (3.35) in the form of the asymptotic expansion, which holds in $L^{\frac{5}{3}}(\Omega)^9$,

$$M_{t_n}(\mathbf{v}\otimes\mathbf{v}) = \overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + M_{t_n}(\mathbf{v}'\otimes\mathbf{v}') + o(1). \tag{3.38}$$

We deduce from the estimate (3.37) that $(M_{t_n}(\mathbf{v}\otimes\mathbf{v}))_{n\in\mathbf{N}}$ is bounded in $\mathbf{L}^{\frac{5}{3}}(\Omega)$. Therefore, we can extract a subsequence (written likewise), which converges weakly in $\mathbf{L}^{\frac{5}{3}}(\Omega)$ to some $\boldsymbol{\vartheta} \in L^{\frac{5}{3}}(\Omega)^9$. The expansion (3.38) shows that the sequence $(M_{t_n}(\mathbf{v}'\otimes\mathbf{v}'))_{n\in\mathbf{N}}$ weakly converges to $\sigma^{(\mathrm{R})} \in L^{\frac{5}{3}}(\Omega)^9$, linked to $\boldsymbol{\vartheta}$ by the relation

$$\sigma^{(\mathrm{R})} = \boldsymbol{\vartheta} - \overline{\mathbf{v}}\otimes\overline{\mathbf{v}}, \tag{3.39}$$

which proves (i).

Proof of (ii) According to (3.15) and the Reynolds rule (3.26), we note that $\nabla\cdot\vartheta=\mathbf{B}\in L^{\frac{5}{4}}(\Omega)^9$, and therefore (3.23) combined with (3.39) yields $\mathbf{F}=\nabla\cdot\sigma^{(\mathrm{R})}$.

Proof of (iii) As already quoted, $\overline{\mathbf{v}}\in\mathbf{W}^{2,\frac{5}{4}}(\Omega)\hookrightarrow\mathbf{W}^{1,\frac{15}{7}}(\Omega)\hookrightarrow\mathbf{H}^1(\Omega)$. Moreover, since $\overline{\mathbf{v}}=0$ on Γ and $\nabla\cdot\overline{\mathbf{v}}=0$, we have $\overline{\mathbf{v}}\in\mathbf{V}_{\mathrm{div}}(\Omega)$. Consequently, we can take $\overline{\mathbf{v}}$ as a test in formulation (2.13), which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v},\overline{\mathbf{v}})_\Omega+b(\mathbf{v};\mathbf{v},\overline{\mathbf{v}})+a(\mathbf{v},\overline{\mathbf{v}})=(\mathbf{f},\overline{\mathbf{v}})_\Omega. \tag{3.40}$$

We integrate (3.40) over $[0,t]$ and divide the result by t, leading to

$$\frac{1}{t}(\mathbf{v}(t,\cdot)-\mathbf{v}_0(\cdot),\overline{\mathbf{v}}(\cdot))_\Omega+(M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}),\overline{\mathbf{v}})_\Omega+\nu(\nabla\mathbf{V}_t,\nabla\overline{\mathbf{v}})_\Omega=(\mathbf{f},\overline{\mathbf{v}})_\Omega. \tag{3.41}$$

We take the limit of each term in (3.41). Firstly

$$\frac{1}{t}|(\mathbf{v}(t,\cdot)-\mathbf{v}_0(\cdot),\overline{\mathbf{v}}(\cdot))_\Omega|\le\frac{1}{t}\|\mathbf{v}(t,\cdot)-\mathbf{v}_0(\cdot)\|_{0,2,\Omega}\|\overline{\mathbf{v}}\|_{0,2,\Omega}, \tag{3.42}$$

which goes to zero when $t\to\infty$, due to the L^2 uniform bound (2.23). We also have $\overline{\mathbf{v}}\in\mathbf{L}^{\frac{15}{2}}(\Omega)$, and $M_{t_n}((\mathbf{v}\cdot\nabla)\,\mathbf{v})$ converges to $\mathbf{B}$ in $L^{\frac{5}{4}}(\Omega)^9$. Fortunately, we observe that $\frac{2}{15}+\frac{4}{5}=\frac{14}{15}<1$, and thus, according to (3.23),

$$\lim_{n\to\infty}(M_{t_n}((\mathbf{v}\cdot\nabla)\,\mathbf{v}),\overline{\mathbf{v}})_\Omega=(\mathbf{B},\overline{\mathbf{v}})_\Omega=(\mathbf{F},\overline{\mathbf{v}})_\Omega+((\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}},\overline{\mathbf{v}})_\Omega=(\mathbf{F},\overline{\mathbf{v}})_\Omega, \tag{3.43}$$

since it is easily verified from $\nabla\cdot\overline{\mathbf{v}}=0$ that $((\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}},\overline{\mathbf{v}})_\Omega=0$.

Finally, we deduce from Theorem 3.1 and Sobolev embeddings that $(\nabla\mathbf{V}_{t_n})_{n\in\mathbf{N}}$ converges strongly to $\nabla\overline{\mathbf{v}}$ in $\mathbf{L}^q(\Omega)$ for all $q<\frac{15}{2}$, and in particular, for $q=2$, leading to

$$\lim_{n\to\infty}(\nabla\mathbf{V}_{t_n},\nabla\overline{\mathbf{v}})_\Omega=(\nabla\overline{\mathbf{v}},\nabla\overline{\mathbf{v}})_\Omega=\|\nabla\overline{\mathbf{v}}\|^2_{0,2,\Omega}, \tag{3.44}$$

so the energy balance (3.30) follows from (3.41)–(3.44).

Proof of (iv) We start from the energy inequality (2.16) that we divide by t_n, and we let n go to infinity. Using again the strong convergence of $(\nabla\mathbf{V}_{t_n})_{n\in\mathbf{N}}$ to $\nabla\overline{\mathbf{v}}$ in $\mathbf{L}^2(\Omega)$ and the L^2 uniform bound as above, we obtain

$$\nu\|\nabla\overline{\mathbf{v}}\|_{0,2,\Omega}\le(\mathbf{f},\overline{\mathbf{v}})_\Omega, \tag{3.45}$$

which combined with (3.30) yields (3.31) and concludes the proof.

In summary, $(\overline{\mathbf{v}},\overline{p})\in\mathbf{W}^{2,\frac{5}{4}}(\Omega)\times\mathbf{W}^{1,\frac{5}{4}}(\Omega)/\mathbb{R}$ satisfies

$$\begin{cases}(\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}}-\nu\Delta\overline{\mathbf{v}}+\nabla\overline{p}=-\nabla\cdot\sigma^{(\mathrm{R})}+\mathbf{f}\quad\text{in }\Omega,\\ \nabla\cdot\overline{\mathbf{v}}=0\quad\text{in }\Omega,\\ \overline{\mathbf{v}}=0\quad\text{on }\Gamma,\end{cases} \tag{3.46}$$

in the sense of distributions, where in addition, $(\nabla\cdot\sigma^{(\mathrm{R})},\overline{\mathbf{v}})_\Omega\ge0$.

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