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Global Exact Boundary Controllability for General First-Order Quasilinear Hyperbolic Systems*

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Abstract For general first-order quasilinear hyperbolic systems, based on the analysis of simple wave solutions along characteristic trajectories, the global two-sided exact boundary controllability is achieved in a relatively short controlling time.

Keywords Global exact boundary controllability, General quasilinear hyperbolic system, Simple wave solution, Characteristic trajectory, Short controlling time

2000 MR Subject Classification 35L50, 49J20, 93B05, 93C20

1 Introduction

Consider the following first-order quasilinear hyperbolic system with one space variable:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \quad 0 < x < L, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}} \in \mathcal{U}$ is the unknown vector function of (t, x), taking values in a bounded and connected domain $\mathcal{U} \subset \mathbb{R}^n$ (for convenience, all equations hold for $u \in \overline{\mathcal{U}}$ throughout this paper, unless otherwise indicated), and $A(u) = (a_{ij}(u))$ is a C^2 smooth $n \times n$ matrix function. By hyperbolicity, the coefficient matrix $A(u)$ possesses n real eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ a complete set of left eigenvectors

$$l_i(u) = (l_{i1}(u), \cdots, l_{in}(u)), \quad i = 1, \cdots, n$$

satisfying

$$l_i(u)A(u) = \lambda_i(u)l_i(u), \quad i = 1, \cdots, n \tag{1.2}$$

and

$$\det(l_{ij}(u)) \neq 0, \tag{1.3}$$

and a complete set of right eigenvectors

$$r_i(u) = (r_{1i}(u), \cdots, r_{ni}(u))^{\mathrm{T}}, \quad i = 1, \cdots, n$$

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satisfying

$$A(u)r_i(u) = \lambda_i(u)r_i(u), \quad i = 1, \cdots, n \tag{1.4}$$

and

$$\det(r_{ij}(u)) \neq 0. \tag{1.5}$$

Without loss of generality, one may assume that

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad i, j = 1, \cdots, n, \tag{1.6}$$

where δ_{ij} is the Kronecker symbol. Suppose that $\lambda_i(u), l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ are also C^2 smooth with respect to u. Since, generally speaking, first-order quasilinear hyperbolic systems with zero eigenvalues do not have exact boundary controllability (see [12]), we assume that the system possesses no zero eigenvalue, namely,

$$\lambda_r(u) < 0 < \lambda_s(u), \quad r = 1, \cdots, m, \ s = m+1, \cdots, n. \tag{1.7}$$

Besides, for the simplicity of analyzing simple wave solutions, we assume the following condition for each eigenvalue:

$$\nabla\lambda_i(u) \cdot r_i(u) \geq 0, \quad \forall u \in \mathcal{U}, \ i = 1, \cdots, n. \tag{1.8}$$

Remark 1.1 For each eigenvalue $\lambda_i(u)$ of the system, if either it is linearly degenerate, namely,

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 0, \quad \forall u \in \mathcal{U},$$

or it is genuinely nonlinear, namely,

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 1, \quad \forall u \in \mathcal{U},$$

then the assumption (1.8) obviously holds.

We consider the mixed initial-boundary value problem for the system (1.1) with the initial condition

$$t = 0: \ u = \phi(x) \in C^1([0, L]; \mathcal{U}) \tag{1.9}$$

and the following boundary conditions:

$$x = 0: \ H_s(t, u) = h_s(t), \quad s = m+1, \cdots, n, \tag{1.10}$$

$$x = L: H_r(t, u) = h_r(t), \quad r = 1, \cdots, m, \tag{1.11}$$

where $H_i(\ i = 1, \cdots, n)$ are C^1 functions and satisfy the following solvability conditions:

$$\det(\nabla_u H_s(t, u) \cdot r_{s'}(u))_{s,s'=m+1}^n \neq 0, \quad \forall t \geq 0, \ \forall u \in \mathcal{U}, \tag{1.12}$$

$$\det(\nabla_u H_r(t, u) \cdot r_{r'}(u))_{r,r'=1}^m \neq 0, \qquad \forall t \geq 0, \ \forall u \in \mathcal{U}, \tag{1.13}$$

and $h_i(t) \in C^1(\ i = 1, \cdots, n)$ will be taken as boundary controls.

This mixed initial-boundary value problem (1.1) and (1.9)–(1.11) admits a unique local C^1 solution $u = u(t, x)$ for any given initial data satisfying suitable C^1 compatibility conditions (see [13]). Many papers, including the present one, consider the following problem of exact boundary controllability: For any given initial data $\phi(x)$ and final data $\psi(x) \in C^1([0, L]; \mathcal{U})$, under certain reasonable assumptions, it is asked to find a controlling time $T_0 > 0$ and n boundary controls

$$h_i(t) \in C^1[0, T_0], \quad i = 1, \cdots, n,$$

such that the mixed initial-boundary value problem (1.1) and (1.9)–(1.11) admits a unique C^1 solution $u = u(t, x)$ on the domain $[0, T_0] \times [0, L]$, which verifies the following final condition:

$$t = T_0 : \ u = \psi(x). \tag{1.14}$$

For linear hyperbolic systems, the exact boundary controllability has a complete theory (see [5, 14]). For semilinear hyperbolic systems, one can refer to [4] and [19–20]. For quasilinear hyperbolic systems, the theory of local exact boundary controllability has been established (see [1, 7, 9–10]). Roughly speaking, if the initial data $\phi(x)$ and the final data $\psi(x)$ are both small C^1 perturbations around one constant equilibrium of the system, then one can find a sharp controlling time T_0 and use some or all of the boundary functions $h_i(t)$ ($i = 1, \cdots, n$) as controls to achieve the exact boundary controllability in the framework of C^1 classical solutions. Since the whole controlling process is achieved around one point in the phase space, this result is called to be the local controllability.

Now it is natural to ask that for general initial data and final data, whether or not one has the global controllability. Since, in the general situation, the classical solutions may blow-up in a finite time, the general global exact boundary controllability is hard to be built in the framework of classical solutions. However, for linearly degenerate hyperbolic systems of the diagonal form, since the blowup can be prevented for classical solutions, [18] presents the corresponding results on the global exact boundary controllability.

On the other hand, we may consider the global exact boundary controllability in a slightly different sense as follows. If the initial data and the final data are small C^1 perturbations of two distinct constant equilibria u_* and u_{**} of the system, namely,

$$\|\phi(x) - u_*\|_{(C^1[0,L])^n} \ll 1, \tag{1.15}$$

$$\|\psi(x) - u_{**}\|_{(C^1[0,L])^n} \ll 1, \tag{1.16}$$

we wish to get the corresponding exact boundary controllability. For the results in this aspect, one may refer to [2, 8, 11, 15] for hyperbolic systems of the diagonal form and wave equations, and [3] for the Saint-Venant equations with slope and friction, which is a special hyperbolic system of the diagonal form with source terms. The main method of these works is successively using exact boundary controllable neighborhoods of finitely many constant equilibria of the system to cover a curve connecting u_* and u_{**} composed of equilibria of the system, and then using the local exact boundary controllability to move the solution step by step from the controllable neighborhood of u_* to the one of u_{**}. In this way, the total controlling time of the global exact boundary controllability might be quite long. We point out that up to now, all the known results on the global exact boundary controllability are only restricted to quasilinear hyperbolic systems of the diagonal form, but not on the general quasilinear hyperbolic systems.

In this paper, the general quasilinear hyperbolic system (1.1) (not necessarily of the diagonal form) is concerned with the initial data $\phi(x)$ and the final data $\psi(x)$ as C^1 perturbations of two distinct constant equilibria u_* and u_{**}, respectively. Besides, we hope to reduce the controlling time of the global exact boundary controllability. Our strategy is listed as follows. The first step is connecting those two constant equilibria with a set of characteristic trajectories of the system, and along these characteristic trajectories we can construct the simple wave solutions of the system which take values on them, respectively. By this analysis, a special solution $u = \overline{u}(t, x)$ to system (1.1) with

$$\overline{u}(T', x) \equiv u_* \quad \text{and} \quad \overline{u}(T'', x) \equiv u_{**}$$

can be constructed with a possibly large C^1 norm, which gives the exact boundary controllability connecting those two equilibria. Based on this, the desired global exact boundary controllability can then be achieved by applying the local exact boundary controllability near those two equilibria, respectively. Since the simple wave solutions used in this process may possess a large C^1 norm, the values taken by this solution may change rapidly in the phase space, which overcomes the long-time consumption of the original pointwise extension control method. By means of characteristic trajectories, we are able to develop this simple wave method for the general quasilinear hyperbolic systems.

In §2, special solutions with monotone initial data are analyzed for the transport equation. Based on this, in §3, simple wave solutions to system (1.1) are constructed along the corresponding characteristic trajectories and are combined to form a special solution to system (1.1) on the domain $[0,T_0]\times[0,L]$ with u_* as its initial data and u_{**} as its final data. This special solution, together with the local exact boundary controllability, then leads to the global exact boundary controllability. In §4, the quasilinear hyperbolic system of the diagonal form is analyzed as an example to show the reduction of the controlling time when using this new method.

2 Special Solutions to the Transport Equation

In this section, we consider the following transport equation:

$$\frac{\partial z}{\partial t}+\lambda(z)\frac{\partial z}{\partial x}=0,\quad x\in\mathbb{R},\ t\geq 0, \tag{2.1}$$

where z is the unknown function of (t,x), taking its values on a closed interval $I\subset\mathbb{R}$, and $\lambda(z)\in C^1(I;\mathbb{R})$ satisfies

$$\frac{\mathrm{d}\lambda(z)}{\mathrm{d}z}\geq 0,\quad \forall z\in I. \tag{2.2}$$

Consider the Cauchy problem of (2.1) with the following initial condition:

$$t=0:\ z=\varphi(x)\in C^1(\mathbb{R};I). \tag{2.3}$$

From Theorem 1.1 of [6], we have the following lemma.

Lemma 2.1 *Under the hypothesis* (2.2)*, if*

$$\varphi'(x)\geq 0,\quad \forall x\in\mathbb{R},$$

then the Cauchy problem (2.1) *and* (2.3) *admits a unique global C^1 solution*

$$z=z(t,x)\in C^1([0,\infty)\times\mathbb{R};I)$$

on $t\geq 0$.

Using this result, some C^1 solutions with specific properties to the equation (2.1) can be constructed as follows.

Lemma 2.2 *Under the hypothesis* (2.2) *and*

$$\lambda(z)\neq 0, \tag{2.4}$$

if z^* and z^{**} are two given points on I, then for

$$T_1=\max_{z\in I}\frac{L}{|\lambda(z)|} \tag{2.5}$$

and any given $\delta > 0$*, there exists a* C^1 *solution*

$$z = z(t,x) \in C^1([0, T_1 + \delta] \times [0, L]; I)$$

to the equation (2.1)*, satisfying*

$$z(0,x) = z^*, \quad \forall x \in [0, L] \tag{2.6}$$

and

$$z(T_1 + \delta, x) = z^{**}, \quad \forall x \in [0, L]. \tag{2.7}$$

Proof Set

$$\theta = \delta \min_{z \in I} |\lambda(z)|.$$

We construct $z(t,x)$ according to the following different cases.

Case 1 (Forward rarefaction waves) If $\lambda(z) > 0$ and $z^{**} \le z^*$, one may choose a monotone function $\varphi \in C^1(\mathbb{R}; I)$ satisfying

$$\begin{aligned} \varphi'(x) &\ge 0, \quad && \forall x \in \mathbb{R}, \\ \varphi(x) &\equiv z^*, && \forall x \ge -\frac{1}{3}\theta, \\ \varphi(x) &\equiv z^{**}, && \forall x \le -\frac{2}{3}\theta. \end{aligned}$$

Then by Lemma 2.1, the Cauchy problem (2.1) and (2.3) admits a unique C^1 solution $z = z(t,x)$ on $t \ge 0$. Moreover, by the method of characteristics, one has

$$\begin{aligned} z(t,x) &\equiv z^* \quad && \text{for } x - \lambda(z^*)t \ge -\frac{1}{3}\theta, \\ z(t,x) &\equiv z^{**} && \text{for } x - \lambda(z^{**})t \le -\frac{2}{3}\theta, \end{aligned}$$

which imply that $z = z(t,x)$ satisfies all the requirements of Lemma 2.2 on the domain $[0, T_1 + \delta] \times [0, L]$ (see Fig. 1).

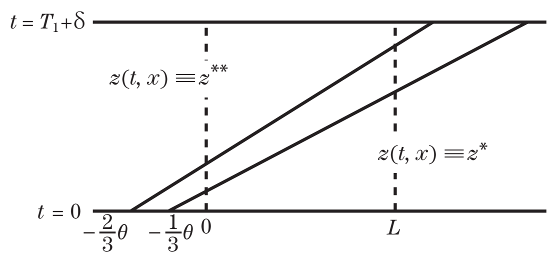

Figure 1 Forward rarefaction waves in the case $\lambda(z) > 0$ and $z^{**} \le z^*$

Case 2 (Backward rarefaction waves) If $\lambda(z) < 0$ and $z^{**} \ge z^*$, one may choose a monotone function $\varphi \in C^1(\mathbb{R}; I)$ satisfying

$$\begin{aligned} \varphi'(x) &\ge 0, \quad && \forall x \in \mathbb{R}, \\ \varphi(x) &\equiv z^{**}, && \forall x \ge L + \frac{2}{3}\theta, \\ \varphi(x) &\equiv z^*, && \forall x \le L + \frac{1}{3}\theta. \end{aligned}$$

Then by Lemma 2.1, the Cauchy problem (2.1) and (2.3) admits a unique C^1 solution $z = z(t, x)$ on $t \geq 0$. Moreover, by the method of characteristics, it is easy to show that $z = z(t, x)$ fulfills all the requirements of Lemma 2.2 on the domain $[0, T_1 + \delta] \times [0, L]$ (see Fig. 2).

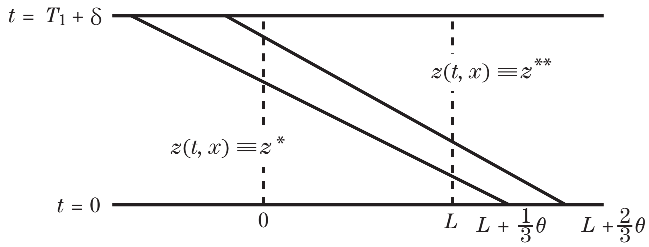

Figure 2 Backward rarefaction waves in the case $\lambda(z) < 0$ and $z^{**} \geq z^*$

Case 3 (Compression waves) If $\lambda(z) > 0$ and $z^{**} > z^*$, or $\lambda(z) < 0$ and $z^{**} < z^*$, using the results in Cases 1–2, one can get a solution

$$z = \widehat{z}(t, x) \in C^1([0, T_1 + \delta] \times [0, L]; I)$$

to the equation (2.1), satisfying

$$\widehat{z}(0, x) \equiv z^{**}, \quad \forall x \in [0, L]$$

and

$$\widehat{z}(T_1 + \delta, x) \equiv z^*, \quad \forall x \in [0, L].$$

Setting

$$z(t, x) = \widehat{z}(T_1 + \delta - t, L - x), \quad t \in [0, T_1 + \delta], \ x \in [0, L],$$

it is easy to show that $z = z(t, x)$ is a C^1 solution to the equation (2.1) on the domain $[0, T_1 + \delta] \times [0, L]$, which fulfills all the requirements of Lemma 2.2 (see Fig. 3).

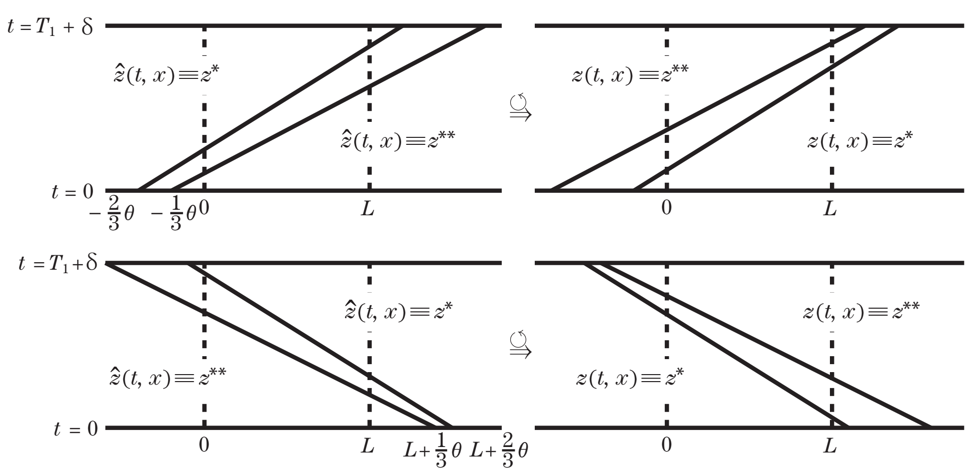

Figure 3 Compression waves

3 Simple Wave Solutions to Quasilinear Hyperbolic Systems and Global Exact Boundary Controllability

In this section, simple wave solutions to the quasilinear hyperbolic system (1.1), suitably constructed based on the result of §2, are combined on the stripe $x \in [0, L]$ to form a desired special solution, and then the global exact boundary controllability can be realized.

For $i = 1, \cdots, n$, the curve $u = u^{(i)}(s, u^B)$ defined in $\mathcal{U}$ by the following initial value problem of ODE:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} u^{(i)}(s, u^B) = r_i(u^{(i)}(s, u^B)), \\ u^{(i)}(0, u^B) = u^B \end{cases} \tag{3.1}$$

is called the ith characteristic trajectory passing through the point $u = u^B$ (see [6]). By hyperbolicity of the system, there are n distinct characteristic trajectories passing through any given point $u^B \in \mathcal{U}$, and they form a set of local curved coordinates in the neighborhood of u^B. Therefore, for any given points u_* and u_{**} in $\mathcal{U}$, one can find a set of (finitely many) characteristic trajectories to connect them successively. In other words, there exist K characteristic trajectories $u^{(i_k)}(s, u^{\langle k-1 \rangle})$ and K real numbers $s_k \in \mathbb{R}$ ($k = 1, \cdots, K$), such that

$$u^{\langle k \rangle} = u^{(i_k)}(s_k, u^{\langle k-1 \rangle}), \quad 1 \le k \le K, \tag{3.2}$$

and

$$u^{\langle K \rangle} = u_{**},$$
$$u^{\langle 0 \rangle} = u_*.$$

Obviously, the selection of these characteristic trajectories is not unique. In applications, we may choose the one with the least number of characteristic trajectories.

On each given characteristic trajectory, we have the following lemma.

Lemma 3.1 *Suppose that $u = u^{(i)}(s, u^B)$ is the ith characteristic trajectory of the system* (1.1) *passing through $u = u^B$, $i = 1, \cdots, n$, and $z = z(t, x)$ is a C^1 solution to the following transport equation:*

$$\frac{\partial z}{\partial t} + \lambda_i(u^{(i)}(z, u^B)) \frac{\partial z}{\partial x} = 0 \tag{3.3}$$

on the domain $[0, T] \times [0, L]$. Then $u = u^{(i)}(z(t, x), u^B)$ is a C^1 solution to the system (1.1) *on the domain $[0, T] \times [0, L]$.*

Proof Due to the smoothness assumption of the right eigenvector $r_i(u)$, $u = u^{(i)}(z(t, x), u^B)$ is a C^1 function on the domain $[0, T] \times [0, L]$. Substituting it into the equation (1.1) and noting (3.1) and (1.4), we have

$$\begin{aligned} \frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} &= \Big(\frac{\partial z}{\partial t} r_i(u) \Big) + A(u) \Big(\frac{\partial z}{\partial x} r_i(u) \Big) \\ &= \Big(\frac{\partial z}{\partial t} + \lambda_i(u^{(i)}(z, u^B)) \frac{\partial z}{\partial x} \Big) r_i(u) \\ &= 0. \end{aligned}$$

Note that the solution $u = u^{(i)}(z(t,x), u^B)$ given above depends on one scalar function $z = z(t,x)$, which means it is a simple wave solution. Moreover, it is easy to show that for $i = 1, \cdots, n$, if (1.8) holds for the ith eigenvalue $\lambda_i(u)$ of the system (1.1), then the function $\lambda(z) = \lambda_i(u^{(i)}(z, u^B))$ satisfies (2.2). Then, using Lemmas 2.2 and 3.1, we get the following corollary.

Corollary 3.1 *Under hypotheses* (1.7)–(1.8)*, if $u^E \in \mathcal{U}$ is located on the ith characteristic trajectory passing through $u^B \in \mathcal{U}$, namely,*

$$u^E = u^{(i)}(s^E, u^B)$$

for $s^E \in \mathbb{R}$, then for

$$T_{i,*} = \max_{\substack{0 \le s \le s^E \\ \text{or } s^E \le s \le 0}} \frac{L}{|\lambda_i(u^{(i)}(s, u^B))|} \tag{3.4}$$

and any given $\delta > 0$, there exists a C^1 solution $u = u(t,x) \in C^1([0, T_{i,*} + \delta] \times [0, L]; \mathcal{U})$ to the system (1.1)*, satisfying*

$$u(0,x) \equiv u^B, \quad \forall x \in [0, L] \tag{3.5}$$

and

$$u(T_{i,*} + \delta, x) \equiv u^E, \quad \forall x \in [0, L]. \tag{3.6}$$

By this result, for $k = 1, \cdots, K$, on each characteristic trajectory $u^{(i_k)}(s, u^{\langle k-1 \rangle})$ a simple wave solution can be constructed to get a C^1 solution developing from $u^{\langle k-1 \rangle}$ to $u^{\langle k \rangle}$. After suitable translations with respect to t, these solutions can be combined to get the following result.

Proposition 3.1 *Under hypotheses* (1.7)–(1.8)*, for any given points u_* and u_{**} in $\mathcal{U}$, if they can be connected by K characteristic trajectories of the system* (1.1)*, then for*

$$T_* = \max_{1 \le i \le n} \sup_{u \in \mathcal{U}} \frac{L}{|\lambda_i(u)|} \tag{3.7}$$

and any given $T_0 > KT_*$, there exists a C^1 solution $u = \overline{u}(t,x) \in C^1([0, T_0] \times [0, L]; \mathcal{U})$ to the system (1.1) *on the domain* $[0, T_0] \times [0, L]$*, satisfying*

$$\overline{u}(0,x) \equiv u_*, \quad \forall x \in [0, L] \tag{3.8}$$

and

$$\overline{u}(T_0, x) \equiv u_{**}, \quad \forall x \in [0, L]. \tag{3.9}$$

Remark 3.1 If we choose $\varphi(x) \in C^k(\mathbb{R})$ $(k \ge 1)$ in the above process, $\overline{u}(t,x)$ may have higher regularity: $\overline{u}(t,x) \in C^k([0, T_0] \times [0, L]; \mathcal{U})$.

Now the global exact boundary controllability can be precisely presented and proved by means of Proposition 3.1 and the local exact boundary controllability.

Theorem 3.1 *Under hypotheses* (1.7)–(1.8) *and* (1.12)–(1.13)*, for any given initial data* $\phi(x)$ *and final data* $\psi(x)$ *satisfying* (1.15)–(1.16)*, where* u_* *and* u_{**} *are constant equilibria of the system* (1.1)*, being connected by* K *characteristic trajectories, let* T_* *be defined by* (3.7)*. Then for any given* $T_0 > (K+2)T_*$*, there exist* C^1 *controls* $h_i(t)$ $(i = 1, \cdots, n)$ *on* $[0, T_0]$*, such that the mixed initial-boundary value problem* (1.1) *and* (1.9)–(1.11) *admits a unique* C^1 *solution* $u = u(t, x)$ *on the domain* $[0, T_0] \times [0, L]$*, which verifies exactly the final condition* (1.14).

Proof By the local exact boundary controllability given in [7, 10], a local control can be performed to get a C^1 solution $u = u(t, x)$ on the domain $[0, T_* + \delta] \times [0, L]$ to the system (1.1), satisfying

$$u(0, x) = \phi(x), \quad u(T_* + \delta, x) = u_*.$$

Then by Proposition 3.1, a C^1 solution $u = u(t, x)$ to the system (1.1) can be constructed on the domain $[T_* + \delta, (K + 1)(T_* + \delta)] \times [0, L]$, satisfying

$$u(T_* + \delta, x) = u_*, \quad u((K + 1)(T_* + \delta), x) = u_{**}.$$

Finally, another local control can be performed to get a C^1 solution $u = u(t, x)$ to the system (1.1) on the domain $[(K + 1)(T_* + \delta), (K + 2)(T_* + \delta)] \times [0, L]$, satisfying

$$u((K + 1)(T_* + \delta), x) = u_{**}, \quad u((K + 2)(T_* + \delta), x) = \psi(x).$$

Now a special solution $u(t, x) \in C^1([0, T_0] \times [0, L]; \mathcal{U})$ to the system (1.1), which possesses the initial data (1.9) and the final data (1.14), can be constructed just by combining the above three solutions. Substituting this solution into boundary conditions (1.10)–(1.11), one can obtain the C^1 boundary controls

$$h_s(t) = H_s(t, u(t, 0)), \quad s = m + 1, \cdots, n, \tag{3.10}$$

$$h_r(t) = H_r(t, u(t, L)), \quad r = 1, \cdots, m. \tag{3.11}$$

Obviously, for these boundary controls, the aforementioned solution $u = u(t, x)$ is the unique C^1 solution to the corresponding mixed initial-boundary value problem (1.1) and (1.9)–(1.11). Moreover, $u = u(t, x)$ satisfies the final condition (1.14).

4 Further Discussions

First, we take the quasilinear hyperbolic system of the diagonal form as an example to show the validity of our methods as well as the reduction of the controlling time.

Consider the following quasilinear hyperbolic system of the diagonal form:

$$\frac{\partial u_i}{\partial t} + \lambda_i(u) \frac{\partial u_i}{\partial x} = 0, \quad i = 1, \cdots, n, \tag{4.1}$$

where $\lambda_i(u)$ $(i=1,\cdots,n)$ are smooth and bounded, satisfying

$$\lambda_r(u)<0<\lambda_s(u),\quad \forall u\in\mathbb{R}^n,\ r=1,\cdots,m,\ s=m+1,\cdots,n \tag{4.2}$$

and

$$\frac{\partial\lambda_i(u)}{\partial u_i}\geq 0,\quad \forall u\in\mathbb{R}^n,\ i=1,\cdots,n. \tag{4.3}$$

For the following boundary conditions given at $x=0$ and $x=L$:

$$x=0: u_s=G_s(t,u_1,\cdots,u_m)+h_s(t),\qquad s=m+1,\cdots,n, \tag{4.4}$$

$$x=L: u_r=G_r(t,u_{m+1},\cdots,u_n)+h_r(t),\quad r=1,\cdots,m, \tag{4.5}$$

we hope to find suitable boundary controls $h_i(t)$ $(i=1,\cdots,n)$ such that there exists a C^1 solution $u=u(t,x)$ which develops from the initial data

$$t=0: u=\phi(x) \tag{4.6}$$

in a neighborhood of one given constant equilibrium to the final data

$$t=T_0: u=\psi(x) \tag{4.7}$$

in a neighborhood of another given constant equilibrium. Since the ith characteristic trajectory of (4.1) passing through any given point is a straight line parallel to the u_i axis, it is possible to give a much clearer estimate on the controlling time. Actually, from Theorem 3.1 we have the following result.

Corollary 4.1 *Under hypotheses* (4.2)–(4.3), *for any given points u_* and u_{**} in $\mathbb{R}^n$, any given controlling time*

$$T_0>(n+2)\max_{1\leq i\leq n}\sup_{u\in\mathcal{U}}\frac{L}{|\lambda_i(u)|}$$

and any given initial data $\phi(x)$ and final data $\psi(x)$ satisfying

$$\|\phi-u_*\|_{(C^1[0,L])^n}\ll 1\quad \textit{and}\quad \|\psi-u_{**}\|_{(C^1[0,L])^n}\ll 1,$$

there exist boundary controls $h_i(t)\in C^1[0,T_0]$ $(i=1,\cdots,n)$ such that the mixed initial-boundary value problem (4.1) *and* (4.4)–(4.6) *admits a unique C^1 solution $u=u(t,x)$ on the domain $[0,T_0]\times[0,L]$, which verifies exactly the final data* (4.7). *Here $\mathcal{U}$ can be chosen as any rectangular domain containing u_* and u_{**}.*

Roughly speaking, the global controllability time given in Corollary 4.1 is $(n+2)$ times of the time of two-sided local exact boundary controllability (see [7]), which, in general, is much shorter than that consumed by the method of pointwise extension control.

Compared with the method given in previous results, the process shown in this paper has the following advantages: First, the global exact boundary controllability is established for general hyperbolic systems, but not just for systems of the diagonal form. Secondly, the controlling time can be significantly reduced. However, the result given in this paper is only feasible to homogeneous systems, since the constant equilibria of an inhomogeneous system generally form a complicated manifold in the phase space, and there may not exist a set of finite characteristic trajectories, composed of equilibria, that connect any two given equilibria. Moreover, the global exact boundary controllability is established only with two-sided boundary controls. In general, since the simple waves do not satisfy the uncontrolled boundary conditions, the global one-sided exact boundary controllability or the two-sided one with less controls (see [7] for the corresponding theories of local controllability) can not be achieved using the above method of simple waves.

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