

Cyclic Covers over Strongly Lifiable Schemes*

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Abstract A smooth scheme X over a field k of positive characteristic is said to be strongly liftable over $W_2(k)$, if X and all prime divisors on X can be lifted simultaneously over $W_2(k)$. In this paper, the author gives a criterion for those cyclic covers over strongly liftable schemes that are still strongly liftable. As a corollary, cyclic covers over projective spaces of dimension at least three are strongly liftable over $W_2(k)$.

Keywords Cyclic cover, Strongly liftable scheme, Positive characteristic
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1 Introduction

Throughout this paper, we always work over an algebraically closed field k of characteristic $p > 0$ unless otherwise stated. A smooth scheme X is said to be strongly liftable over $W_2(k)$, if X and all prime divisors on X can be lifted simultaneously over $W_2(k)$. This notion was first introduced in [5] to study the Kawamata-Viehweg vanishing theorem in positive characteristic, and furthermore, many examples and properties of strongly liftable schemes were given in [5–7].

Before stating the main theorem, let us fix some notations and assumptions.

Let X be a smooth projective variety, and \mathcal{L} be an invertible sheaf on X . Let N be a positive integer prime to p , $0 \neq s \in H^0(X, \mathcal{L}^N)$, and $D = \text{div}_0(s)$ be the effective divisor of zeros of s . Let $\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}(\lfloor \frac{iD}{N} \rfloor)$, $Y = \mathbf{Spec} \mathcal{A}$, and $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of s .

Assume that X is strongly liftable over $W_2(k)$, $H^1(X, \mathcal{L}^N) = 0$ and $\text{Sing}(D_{\text{red}}) = \emptyset$. By [6, Theorem 4.1 and Corollary 4.3], X has a lifting \tilde{X} over $W_2(k)$, \mathcal{L} has a lifting $\tilde{\mathcal{L}}$ on \tilde{X} , s has a lifting $\tilde{s} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^N)$, and Y is a smooth projective scheme which is liftable over $W_2(k)$.

In this paper, we give a criterion for those cyclic covers over strongly liftable schemes that are still strongly liftable (see Sections 3–4 for more details).

Theorem 1.1 *With the same notations, assumptions and liftings \tilde{X} , $\tilde{\mathcal{L}}$ and \tilde{s} as above, assume further that for any prime divisor E on X which is not contained in $\text{Supp}(D)$, there exists a lifting $\tilde{E} \subset \tilde{X}$ of $E \subset X$, such that $\tilde{s}|_{\tilde{E}} \in H^0(\tilde{E}, \tilde{\mathcal{L}}^N|_{\tilde{E}})$ is a divisible lifting of $s|_E \in H^0(E, \mathcal{L}^N|_E)$. Then Y is strongly liftable over $W_2(k)$.*

As a consequence of Theorem 1.1, we have the following corollaries.

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Corollary 1.1 *Let X be a smooth projective variety satisfying the H^i -vanishing condition for $i = 1, 2$. Then X is strongly liftable over $W_2(k)$. Let \mathcal{L} be an invertible sheaf on X , N be a positive integer prime to p , and D be an effective divisor on X with $\mathcal{L}^N = \mathcal{O}_X(D)$ and $\text{Sing}(D_{\text{red}}) = \emptyset$. Let $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of D . Then Y is a smooth projective scheme which is strongly liftable over $W_2(k)$.*

Corollary 1.2 *Let $X = \mathbb{P}_k^n$ with $n \geq 3$, and \mathcal{L} be an invertible sheaf on X . Let N be a positive integer prime to p , and D be an effective divisor on X with $\mathcal{L}^N = \mathcal{O}_X(D)$ and $\text{Sing}(D_{\text{red}}) = \emptyset$. Let $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of D . Then Y is a smooth projective scheme which is strongly liftable over $W_2(k)$.*

In Section 2, we recall some definitions and preliminary results of strongly liftable schemes. In Section 3, we give some preliminary results of cyclic covers. The main theorem will be proved in Section 4. For the necessary notions and results on the cyclic cover trick, we refer the reader to [2].

Notation 1.1 We use $[B] = \sum [b_i]B_i$ (resp. $\lceil B \rceil = \sum \lceil b_i \rceil B_i$, $\langle B \rangle = \sum \langle b_i \rangle B_i$) to denote the round-down (resp. round-up, fractional part) of a \mathbb{Q} -divisor $B = \sum b_i B_i$, where for a real number b , $[b] := \max\{n \in \mathbb{Z} \mid n \leq b\}$, $\lceil b \rceil := -[-b]$ and $\langle b \rangle := b - [b]$. We use $\text{Sing}(D_{\text{red}})$ (resp. $\text{Supp}(D)$) to denote the singular locus of the reduced part (resp. the support) of a divisor D .

2 Preliminaries on Strongly Liftable Schemes

Definition 2.1 *Let $W_2(k)$ be the ring of Witt vectors of length two of k . Then $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$, and $W_2(k) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p = k$. The following definition (see [2, Definition 8.11]) generalizes the definition in [1, Subsection 1.6] of liftings of k -schemes over $W_2(k)$.*

Let X be a Noetherian scheme over k , and $D = \sum D_i$ be a reduced Cartier divisor on X . A lifting of (X, D) over $W_2(k)$ consists of a scheme \tilde{X} and closed subschemes $\tilde{D}_i \subset \tilde{X}$, all defined and flat over $W_2(k)$, such that $X = \tilde{X} \times_{\text{Spec } W_2(k)} \text{Spec } k$ and $D_i = \tilde{D}_i \times_{\text{Spec } W_2(k)} \text{Spec } k$. We write $\tilde{D} = \sum \tilde{D}_i$ and say that (\tilde{X}, \tilde{D}) is a lifting of (X, D) over $W_2(k)$, if no confusion is possible.

Let \mathcal{L} be an invertible sheaf on X . A lifting of (X, \mathcal{L}) consists of a lifting \tilde{X} of X over $W_2(k)$ and an invertible sheaf $\tilde{\mathcal{L}}$ on \tilde{X} , such that $\tilde{\mathcal{L}}|_X = \mathcal{L}$. For simplicity, we say that $\tilde{\mathcal{L}}$ is a lifting of \mathcal{L} on \tilde{X} , if no confusion is possible.

Let \tilde{X} be a lifting of X over $W_2(k)$. Then $\mathcal{O}_{\tilde{X}}$ is flat over $W_2(k)$, and hence flat over $\mathbb{Z}/p^2\mathbb{Z}$. Note that there is an exact sequence of $\mathbb{Z}/p^2\mathbb{Z}$ -modules

$$0 \longrightarrow p \cdot \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{r} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

and a $\mathbb{Z}/p^2\mathbb{Z}$ -module isomorphism $p : \mathbb{Z}/p\mathbb{Z} \rightarrow p \cdot \mathbb{Z}/p^2\mathbb{Z}$. Tensoring the above by $\mathcal{O}_{\tilde{X}}$, we obtain an exact sequence of $\mathcal{O}_{\tilde{X}}$ -modules

$$0 \longrightarrow p \cdot \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{r} \mathcal{O}_X \longrightarrow 0 \quad (2.1)$$

and an $\mathcal{O}_{\tilde{X}}$ -module isomorphism

$$p : \mathcal{O}_X \longrightarrow p \cdot \mathcal{O}_{\tilde{X}}, \quad (2.2)$$

where r is the reduction modulo p satisfying $p(x) = p\tilde{x}$, $r(\tilde{x}) = x$ for $x \in \mathcal{O}_X$, $\tilde{x} \in \mathcal{O}_{\tilde{X}}$.

Definition 2.2 *Let X be a smooth scheme over k . X is said to be strongly liftable over $W_2(k)$, if there is a lifting \tilde{X} of X over $W_2(k)$, such that for any prime divisor D on X , (X, D) has a lifting (\tilde{X}, \tilde{D}) over $W_2(k)$ as in Definition 2.1, where \tilde{X} is fixed for all liftings \tilde{D} .*

Let X be a smooth scheme over k , \tilde{X} be a lifting of X over $W_2(k)$, D be a prime divisor on X , and $\mathcal{L}_D = \mathcal{O}_X(D)$ be the associated invertible sheaf on X . Then there is an exact sequence of abelian sheaves

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{q} \mathcal{O}_{\tilde{X}}^* \xrightarrow{r} \mathcal{O}_X^* \longrightarrow 1, \quad (2.3)$$

where $q(x) = p(x) + 1$ for $x \in \mathcal{O}_X$, $p : \mathcal{O}_X \longrightarrow p \cdot \mathcal{O}_{\tilde{X}}$ is the isomorphism (2.2) and r is the reduction modulo p . The exact sequence (2.3) gives rise to an exact sequence of cohomology groups

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) \xrightarrow{r} H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathcal{O}_X). \quad (2.4)$$

If $r : H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) \longrightarrow H^1(X, \mathcal{O}_X^*)$ is surjective, then \mathcal{L}_D has a lifting $\tilde{\mathcal{L}}_D$. We combine (2.1) and (2.2) to obtain an exact sequence of $\mathcal{O}_{\tilde{X}}$ -modules

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{p} \mathcal{O}_{\tilde{X}} \xrightarrow{r} \mathcal{O}_X \longrightarrow 0. \quad (2.5)$$

Tensoring (2.5) by $\tilde{\mathcal{L}}_D$, we have an exact sequence of $\mathcal{O}_{\tilde{X}}$ -modules

$$0 \longrightarrow \mathcal{L}_D \xrightarrow{p} \tilde{\mathcal{L}}_D \xrightarrow{r} \mathcal{L}_D \longrightarrow 0,$$

which gives rise to an exact sequence of cohomology groups

$$H^0(\tilde{X}, \tilde{\mathcal{L}}_D) \xrightarrow{r} H^0(X, \mathcal{L}_D) \longrightarrow H^1(X, \mathcal{L}_D). \quad (2.6)$$

There is a criterion for strong liftability over $W_2(k)$ (see [6, Proposition 2.5]).

Proposition 2.1 *Let X be a smooth scheme over k , and \tilde{X} be a lifting of X over $W_2(k)$. If for any prime divisor D on X , there is a lifting $\tilde{\mathcal{L}}_D$ of $\mathcal{L}_D = \mathcal{O}_X(D)$ on \tilde{X} , such that the natural map $r : H^0(\tilde{X}, \tilde{\mathcal{L}}_D) \longrightarrow H^0(X, \mathcal{L}_D)$ is surjective, then X is strongly liftable over $W_2(k)$.*

3 Preliminaries on Cyclic Covers

For convenience of citation, we recall the following result (see [6, Theorem 4.1 and Corollary 4.3]) with a sketch of the proof.

Theorem 3.1 *Let X be a smooth projective variety, and \mathcal{L} be an invertible sheaf on X . Let N be a positive integer prime to p , $0 \neq s \in H^0(X, \mathcal{L}^N)$, and $D = \text{div}_0(s)$ be the divisor of zeros of s . Let $\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i} \left(\left[\frac{iD}{N} \right] \right)$, $Y = \mathbf{Spec} \mathcal{A}$, and $\pi : Y \longrightarrow X$ be the cyclic cover obtained by taking the N -th root out of s . Assume that X is strongly liftable over $W_2(k)$, $H^1(X, \mathcal{L}^N) = 0$ and $\text{Sing}(D_{\text{red}}) = \emptyset$. Then X has a lifting \tilde{X} over $W_2(k)$, \mathcal{L} has a lifting $\tilde{\mathcal{L}}$ on \tilde{X} , s has a lifting $\tilde{s} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^N)$, and Y is a smooth projective scheme, which is liftable over $W_2(k)$.*

Proof Since X is strongly liftable over $W_2(k)$, there is a lifting \tilde{X} of X and a lifting $\tilde{\mathcal{L}}$ of \mathcal{L} on \tilde{X} . Since $H^1(X, \mathcal{L}^N) = 0$, the exact sequence (2.6) gives rise to a surjection $H^0(\tilde{X}, \tilde{\mathcal{L}}^N) \xrightarrow{r} H^0(X, \mathcal{L}^N)$, and hence s has a lifting $\tilde{s} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^N)$. Let $\tilde{D} = \text{div}_0(\tilde{s})$. Then \tilde{D} is a lifting of D . Let $\tilde{\mathcal{A}} = \bigoplus_{i=0}^{N-1} \tilde{\mathcal{L}}^{-i}(\lfloor \frac{i\tilde{D}}{N} \rfloor)$ and $\tilde{Y} = \mathbf{Spec} \tilde{\mathcal{A}}$. Then \tilde{Y} is a lifting of Y . Thus Y is a smooth projective scheme, which is liftable over $W_2(k)$.

The above result says that cyclic covers over strongly liftable schemes are liftable over $W_2(k)$ under certain conditions, however, in general, they are not strongly liftable over $W_2(k)$ (see [6, Remark 4.6] for more details). In order to prove Theorem 1.1, some elementary results on cyclic covers over integral schemes are needed. First of all, we recall an easy lemma in [2, Lemma 3.15(a)].

Lemma 3.1 *Let X be an integral scheme, and \mathcal{L} be an invertible sheaf on X . Let N be a positive integer prime to p , $0 \neq s \in H^0(X, \mathcal{L}^N)$, and $D = \text{div}_0(s)$ be the divisor of zeros of s . Let $\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}(\lfloor \frac{iD}{N} \rfloor)$, $Y = \mathbf{Spec} \mathcal{A}$, and $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of s . Then Y is reducible if and only if there is an integer $\mu > 1$ dividing N and a section $t \in H^0(X, \mathcal{L}^{N/\mu})$, such that $s = t^{\otimes \mu}$.*

Proof We can consider the problem over a dense open subset $\text{Spec } B \subset X \setminus D_{\text{red}}$. Since $H^0(X, \mathcal{L}^N) \cong B$, we may assume that $s \in H^0(X, \mathcal{L}^N)$ corresponds to an element $u \in B$. Since $\text{Spec } B[x]/(x^N - u)$ is a dense open subset of Y , Y is reducible if and only if $x^N - u$ is reducible in $B[x]$, which is equivalent to the existence of some $v \in B$ with $u = v^\mu$.

Definition 3.1 *Let X be a scheme, \mathcal{L} be an invertible sheaf on X , N be a positive integer, and $0 \neq s \in H^0(X, \mathcal{L}^N)$. The section s is said to be μ -divisible, if $\mu > 0$ divides N and there exists a section $t \in H^0(X, \mathcal{L}^{N/\mu})$, such that $s = t^{\otimes \mu}$. The section s is said to be maximally μ -divisible, if s is μ -divisible, and if s is also ν -divisible, then $\nu \leq \mu$.*

Lemma 3.2 *With the notation and assumptions as in Lemma 3.1, Y then has exactly μ irreducible components if and only if the section s is maximally μ -divisible.*

Proof First of all, we prove that if s is μ -divisible, then Y has at least μ irreducible components. Indeed, assume that s is μ -divisible, then there is a section $t \in H^0(X, \mathcal{L}^{N/\mu})$, such that $s = t^{\otimes \mu}$ and $D = \mu D_1$, where $D_1 = \text{div}_0(t)$. It follows from a direct calculation that $\pi : Y \rightarrow X$ factorizes into the composition of two cyclic covers: $Y \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$, where $\pi_1 : Y_1 \rightarrow X$ is the cyclic cover obtained by taking the μ -th root out of $1 \in H^0(X, (\mathcal{L}^{N/\mu}(-D_1))^\mu) = H^0(X, \mathcal{O}_X)$, and $\pi_2 : Y \rightarrow Y_1$ is the cyclic cover obtained by taking the N/μ -th root out of $\pi_1^* t \in H^0(Y_1, \pi_1^* \mathcal{L}^{N/\mu})$. Since π_1 is unramified, Y_1 has at least μ irreducible components, and so does Y .

If the section s is maximally μ -divisible, then Y has at least μ irreducible components by the above argument. If Y has exactly ν irreducible components with $\nu > \mu$, then by the proof of Lemma 3.1, s is also ν -divisible with $\nu > \mu$, which is absurd. Conversely, if Y has exactly μ irreducible components, then s is μ -divisible by the proof of Lemma 3.1, and furthermore, s is maximally μ -divisible by the above argument.

Definition 3.2 *With the notation and assumptions as in Definition 3.1, assume further*

that X has a lifting \tilde{X} over $W_2(k)$, and \mathcal{L} has a lifting $\tilde{\mathcal{L}}$ on \tilde{X} . A section $\tilde{s} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^N)$ is called a *divisible lifting* of $s \in H^0(X, \mathcal{L}^N)$, if the following conditions hold:

- (i) \tilde{s} is a lifting of s , i.e., $r(\tilde{s}) = s$.
- (ii) If there is an integer $\mu > 0$ dividing N and a section $t \in H^0(X, \mathcal{L}^{N/\mu})$, such that $s = t^{\otimes \mu}$, then there exists a section $\tilde{t} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^{N/\mu})$ lifting t , such that $\tilde{s} = \tilde{t}^{\otimes \mu}$.

It is easy to see that if s is maximally μ -divisible and \tilde{s} is a divisible lifting of s , then \tilde{s} is also maximally μ -divisible.

Lemma 3.3 *With the notation and assumptions as in Lemma 3.1, assume further that X has a lifting \tilde{X} over $W_2(k)$, \mathcal{L} has a lifting $\tilde{\mathcal{L}}$ on \tilde{X} , and s has a lifting $\tilde{s} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^N)$. Let $\tilde{D} = \text{div}_0(\tilde{s})$, $\tilde{\mathcal{A}} = \bigoplus_{i=0}^{N-1} \tilde{\mathcal{L}}^{-i}(\lfloor \frac{i\tilde{D}}{N} \rfloor)$ and $\tilde{Y} = \mathbf{Spec} \tilde{\mathcal{A}}$. If s is maximally μ -divisible and \tilde{s} is a divisible lifting of s , then \tilde{Y} has exactly μ irreducible components.*

Proof By factorizing $\tilde{\pi} : \tilde{Y} \rightarrow \tilde{X}$ into the composition of two cyclic covers, we can prove that \tilde{Y} has at least μ irreducible components, whose proof is almost identical to the argument given in the proof of Lemma 3.2 by changing the usual data into the lifted ones. Assume that \tilde{Y} has exactly ν irreducible components with $\nu > \mu$. Since $\tilde{Y} \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k = Y$ and irreducible components of \tilde{Y} have distinct underlying topological spaces, Y has at least ν irreducible components with $\nu > \mu$, which contradicts Lemma 3.2. Thus \tilde{Y} has exactly μ irreducible components.

Lemma 3.4 *With the notation and assumptions as in Lemma 3.1, let E be a prime divisor on X , which is not contained in $\text{Supp}(D)$, $\mathcal{B} = \bigoplus_{i=0}^{N-1} (\mathcal{L}|_E)^{-i}(\lfloor \frac{iD|_E}{N} \rfloor)$, and $\mathcal{A}|_E = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}(\lfloor \frac{iD}{N} \rfloor)|_E$ be the restriction of \mathcal{A} to E . Then there is a natural finite surjective morphism $\tau_E : \mathbf{Spec} \mathcal{B} \rightarrow \mathbf{Spec} \mathcal{A}|_E$.*

Proof It is easy to see that $\lfloor \frac{im}{N} \rfloor \geq m \lfloor \frac{i}{N} \rfloor$ holds for any $i \geq 0$ and $m \geq 1$. Thus there are injective homomorphisms $\mathcal{L}^{-i}(\lfloor \frac{iD}{N} \rfloor)|_E \rightarrow (\mathcal{L}|_E)^{-i}(\lfloor \frac{iD|_E}{N} \rfloor)$ for all $0 \leq i \leq N-1$, which induce a natural injective homomorphism of \mathcal{O}_E -algebras: $\bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}(\lfloor \frac{iD}{N} \rfloor)|_E \rightarrow \bigoplus_{i=0}^{N-1} (\mathcal{L}|_E)^{-i}(\lfloor \frac{iD|_E}{N} \rfloor)$. By [4, Subsection 6.D, Lemma 2], there is a natural dominant morphism $\tau_E : \mathbf{Spec} \mathcal{B} \rightarrow \mathbf{Spec} \mathcal{A}|_E$, which fits into a commutative diagram

$$\begin{array}{ccc} \mathbf{Spec} \mathcal{B} & \xrightarrow{\tau_E} & \mathbf{Spec} \mathcal{A}|_E \\ & \searrow \sigma & \swarrow \pi|_{\pi^{-1}(E)} \\ & E & \end{array}$$

where $\pi|_{\pi^{-1}(E)} : \mathbf{Spec} \mathcal{A}|_E = E \times_X Y = \pi^{-1}(E) \rightarrow E$ is the restriction of π to $\pi^{-1}(E)$ over E , and $\sigma : \mathbf{Spec} \mathcal{B} \rightarrow E$ is the cyclic cover obtained by taking the N -th root out of $s|_E$. Since \mathcal{B} is a finite \mathcal{O}_E -module, a finite $\mathcal{A}|_E$ -module, τ_E is finite. Since a finite morphism is closed (see [3, Exercise II.3.5]), τ_E is surjective.

Corollary 3.1 *With the notation and assumptions as in Lemma 3.4, assume further that E is smooth. Then $\tau_E : \mathbf{Spec} \mathcal{B} \rightarrow \mathbf{Spec} \mathcal{A}|_E$ is the normalization morphism of $\mathbf{Spec} \mathcal{A}|_E$.*

Proof Denote $\mathcal{A}'|_E = \bigoplus_{i=0}^{N-1} (\mathcal{L}|_E)^{-i}$. Then there are natural injective homomorphisms of \mathcal{O}_E -algebras: $\mathcal{A}'|_E \hookrightarrow \mathcal{A}|_E \hookrightarrow \mathcal{B}$, which induce morphisms: $\mathbf{Spec} \mathcal{B} \longrightarrow \mathbf{Spec} \mathcal{A}|_E \longrightarrow \mathbf{Spec} \mathcal{A}'|_E$. Since E is smooth and $\mathbf{Spec} \mathcal{B} \longrightarrow E$ is the cyclic cover obtained by taking the N -th root out of $s|_E$, by [2, Subsections 3.5 and 3.10], $\mathbf{Spec} \mathcal{B}$ is the normalization of $\mathbf{Spec} \mathcal{A}'|_E$, and hence of $\mathbf{Spec} \mathcal{A}|_E$.

Lemma 3.5 *With the notation and assumptions as in Lemma 3.3, let E be a prime divisor on X , which is not contained in $\mathrm{Supp}(D)$, $\tilde{E} \subset \tilde{X}$ be a lifting of $E \subset X$, $\tilde{\mathcal{B}} = \bigoplus_{i=0}^{N-1} (\tilde{\mathcal{L}}|_{\tilde{E}})^{-i}([\frac{i\tilde{D}}{N}])$, and $\tilde{\mathcal{A}}|_{\tilde{E}} = \bigoplus_{i=0}^{N-1} \tilde{\mathcal{L}}^{-i}([\frac{i\tilde{D}}{N}])|_{\tilde{E}}$ be the restriction of $\tilde{\mathcal{A}}$ to \tilde{E} . Then there is a natural finite surjective morphism $\tau_{\tilde{E}} : \mathbf{Spec} \tilde{\mathcal{B}} \longrightarrow \mathbf{Spec} \tilde{\mathcal{A}}|_{\tilde{E}}$, which is a lifting of $\tau_E : \mathbf{Spec} \mathcal{B} \longrightarrow \mathbf{Spec} \mathcal{A}|_E$ constructed as in Lemma 3.4.*

Proof It is similar to that of Lemma 3.4.

We give a simple example to show the difference between $\mathbf{Spec} \mathcal{B}$ and $\mathbf{Spec} \mathcal{A}|_E$ defined as in Lemma 3.4.

Example 3.1 Let $X = \mathbb{P}_k^2 = \mathrm{Proj} k[x, y, z]$, $\mathcal{L} = \mathcal{O}_X(1)$, $N = 2$, and $s = x^2 - yz \in H^0(X, \mathcal{L}^N)$ with $D = (x^2 - yz = 0)$, $\mathrm{char}(k) = p \geq 3$ and $E = (y = 0)$. Consider the cyclic cover $\pi : Y \longrightarrow X$ obtained by taking the square root out of s . Look at π over the affine piece $\mathbb{A}_k^2 = \mathrm{Spec} k[u, v]$, where $u = \frac{x}{z}$ and $v = \frac{y}{z}$. Then Y is defined by the equation $t^2 = u^2 - v$, and E is defined by the equation $v = 0$. It is easy to see that $\pi^{-1}(E)$ consists of two irreducible components, i.e., E_1 and E_2 , which are defined by the equations $t + u = 0$ and $t - u = 0$, respectively. Thus E_1 and E_2 are smooth, intersect transversally and map isomorphically onto E .

Since $s|_E$, the restriction of s to E , is defined by $x^2 = 0$ on $E = \mathrm{Proj} k[x, z]$, we have $D|_E = 2Q$, where Q is the point $[0 : 1]$ on E . Therefore, \mathcal{O}_E -algebras $\mathcal{B} = \bigoplus_{i=0}^{N-1} (\mathcal{L}|_E)^{-i}([\frac{iD|_E}{N}]) = \mathcal{O}_E \oplus \mathcal{O}_E$, $\mathcal{A}|_E = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}([\frac{iD}{N}])|_E = \mathcal{O}_E \oplus \mathcal{O}_E(-1)$. By assumption, $\mathbf{Spec} \mathcal{A}|_E = E \times_X Y = \pi^{-1}(E) = E_1 + E_2$, whereas by Corollary 3.1, $\mathbf{Spec} \mathcal{B} = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E) = F_1 \amalg F_2$ is a disjoint union of F_1 and F_2 , such that $\tau_E : F_1 \amalg F_2 \longrightarrow E_1 + E_2$ is the normalization morphism.

4 Proof of the Main Theorem

In this section, we prove the main theorem as follows.

Theorem 4.1 *With the notation and assumptions as in Theorem 3.1, fix such liftings $\tilde{X}, \tilde{\mathcal{L}}$ and \tilde{s} as in Theorem 3.1. Assume further that for any prime divisor E on X , which is not contained in $\mathrm{Supp}(D)$, there exists a lifting $\tilde{E} \subset \tilde{X}$ of $E \subset X$, such that $\tilde{s}|_{\tilde{E}} \in H^0(\tilde{E}, \tilde{\mathcal{L}}^N|_{\tilde{E}})$ is a divisible lifting of $s|_E \in H^0(E, \mathcal{L}^N|_E)$. Then Y is strongly liftable over $W_2(k)$.*

Before proving Theorem 4.1, we use Example 3.1 to illustrate the meaning of the further assumption made in Theorem 4.1.

Example 4.1 With the notation and assumptions as in Example 3.1, take liftings of X, \mathcal{L}, s, D and E as follows: $\tilde{X} = \mathbb{P}_{W_2(k)}^2 = \mathrm{Proj} W_2(k)[x, y, z]$, $\tilde{\mathcal{L}} = \mathcal{O}_{\tilde{X}}(1)$, $\tilde{s} = x^2 - yz \in$

$H^0(\tilde{X}, \tilde{\mathcal{L}}^N)$, $\tilde{D} = (x^2 - yz = 0)$ and $\tilde{E} = (y - pz = 0)$. Denote $\tilde{Y} = \mathbf{Spec} \bigoplus_{i=0}^{N-1} \tilde{\mathcal{L}}^{-i}([\frac{i\tilde{D}}{N}])$ and $\tilde{\pi} : \tilde{Y} \rightarrow \tilde{X}$ to be the induced morphism. Look at $\tilde{\pi}$ over the affine piece $\mathbb{A}_{W_2(k)}^2 = \mathbf{Spec} W_2(k)[u, v]$, where $u = \frac{x}{z}$ and $v = \frac{y}{z}$, \tilde{Y} is defined by $t^2 = u^2 - v$, and \tilde{E} is defined by $v = p$. It is easy to see that $\tilde{E}_{12} = \tilde{\pi}^{-1}(\tilde{E})$ is defined by $t^2 = u^2 - p$, which is irreducible. Hence by [6, Lemma 2.2], \tilde{E}_{12} is not a lifting of E_1 or E_2 or $E_1 + E_2$.

$$\begin{array}{ccc} E_1 + E_2 & \cdots \times \cdots \rightarrow & \tilde{E}_{12} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ E & \hookrightarrow & \tilde{E} \end{array}$$

The further assumption made in Theorem 4.1 guarantees that the choices of liftings \tilde{E} of E are so adequate that the above situation can be avoided. In our example, $s|_E$ is maximally 2-divisible, if we can choose a lifting \tilde{E} of E , such that $\tilde{s}|_{\tilde{E}}$ is a divisible lifting of $s|_E$ (so $\tilde{s}|_{\tilde{E}}$ is also maximally 2-divisible), and then we have a lifting $\mathbf{Spec} \tilde{\mathcal{B}} = \tilde{F}_1 \amalg \tilde{F}_2$ of $\mathbf{Spec} \mathcal{B} = F_1 \amalg F_2$, such that \tilde{F}_i is a lifting of F_i for $i = 1, 2$. Let $\tilde{E}_i = \tau_{\tilde{E}}(\tilde{F}_i)$. Then \tilde{E}_i is a lifting of E_i for $i = 1, 2$, since $\tau_{\tilde{E}}$ is a lifting of τ_E .

$$\begin{array}{ccc} F_1 \amalg F_2 & \hookrightarrow & \tilde{F}_1 \amalg \tilde{F}_2 \\ \tau_E \downarrow & & \downarrow \tau_{\tilde{E}} \\ E_1 + E_2 & \hookrightarrow & \tilde{E}_1 + \tilde{E}_2 \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ E & \hookrightarrow & \tilde{E} \end{array}$$

Proof of Theorem 4.1 Consider the following Cartesian square, where $\tilde{\pi} : \tilde{Y} \rightarrow \tilde{X}$ is the natural projection induced by the definition of \tilde{Y} in the proof of Theorem 3.1:

$$\begin{array}{ccc} Y & \xhookrightarrow{\iota} & \tilde{Y} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ X & \xhookrightarrow{\iota} & \tilde{X} \end{array}$$

Let E_Y be a prime divisor on Y , and $E = \pi_*(E_Y)$ be the induced prime divisor on X .

If $E \subset \text{Supp}(D)$, then let $\tilde{E} \subset \text{Supp}(\tilde{D})$ be the corresponding lifting of E . We can take an irreducible component \tilde{E}_Y of $\tilde{\pi}^{-1}(\tilde{E})$, such that $\tilde{E}_Y \times_{\tilde{E}} E = E_Y$, i.e., \tilde{E}_Y is a lifting of E_Y .

If $E \not\subset \text{Supp}(D)$, then $\pi^{-1}(E)$ may be reducible. Assume that $\pi^{-1}(E) = \sum_{i=1}^{\nu} E_i$ with $E_1 = E_Y$. Since $E \not\subset \text{Supp}(D)$, $0 \neq s|_E \in H^0(E, \mathcal{L}^N|_E)$ determines the effective divisor $D|_E$ on E . Let $\tau_E : \mathbf{Spec} \mathcal{B} = \sum_{j=1}^{\mu} F_j \rightarrow \mathbf{Spec} \mathcal{A}|_E = \pi^{-1}(E) = \sum_{i=1}^{\nu} E_i$ be the natural morphism defined as in Lemma 3.4, where F_j are distinct irreducible components. Since τ_E is finite and surjective, we may assume that $\tau_E(F_1) = E_1$. Since $\mathbf{Spec} \mathcal{B} = \sum_{j=1}^{\mu} F_j \rightarrow E$ is the cyclic cover obtained by taking the N -th root out of $s|_E$, by Lemma 3.2, the section $s|_E$ is maximally μ -divisible. Thus there exists a section $t_E \in H^0(E, \mathcal{L}^{N/\mu}|_E)$, such that $s|_E = t_E^{\otimes \mu}$. By assumption,

there is a lifting $\tilde{E} \subset \tilde{X}$ of $E \subset X$, such that $\tilde{s}|_{\tilde{E}}$ is a divisible lifting of $s|_E$, i.e., there is a section $\tilde{t}_E \in H^0(\tilde{E}, \tilde{\mathcal{L}}^{N/\mu}|_{\tilde{E}})$ lifting t_E , such that $\tilde{s}|_{\tilde{E}} = \tilde{t}_E^{\otimes \mu}$.

Consider $\tau_{\tilde{E}} : \mathbf{Spec} \tilde{\mathcal{B}} \rightarrow \mathbf{Spec} \tilde{\mathcal{A}}|_{\tilde{E}}$ defined as in Lemma 3.5, where $\mathbf{Spec} \tilde{\mathcal{B}} \rightarrow \tilde{E}$ is the cyclic cover obtained by taking the N -th root out of $\tilde{s}|_{\tilde{E}}$, and $\mathbf{Spec} \tilde{\mathcal{A}}|_{\tilde{E}} = \tilde{E} \times_{\tilde{X}} \tilde{Y} = \tilde{\pi}^{-1}(\tilde{E})$. Since $s|_E$ is maximally μ -divisible and $\tilde{s}|_{\tilde{E}}$ is a divisible lifting of $s|_E$, by Lemma 3.3, we may assume that $\mathbf{Spec} \tilde{\mathcal{B}} = \sum_{j=1}^{\mu} \tilde{F}_j$, where \tilde{F}_j are distinct irreducible components, and hence \tilde{F}_j have distinct underlying topological spaces. Since $\mathbf{Spec} \tilde{\mathcal{B}} \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k = \mathbf{Spec} \mathcal{B} = \sum_{j=1}^{\mu} F_j$ and $\tilde{F}_j \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k$ are distinct, up to permutation of indices, we can assume that $\tilde{F}_j \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k = F_j$ for any $1 \leq j \leq \mu$.

By Lemma 3.5, $\tau_{\tilde{E}}$ is finite and surjective, so there is an irreducible component of $\tilde{\pi}^{-1}(\tilde{E})$, i.e., \tilde{E}_1 , such that $\tau_{\tilde{E}}|_{\tilde{E}_1} : \tilde{F}_1 \rightarrow \tilde{E}_1$ is surjective. Since $\tau_{\tilde{E}}$ is a lifting of τ_E , we have $\tilde{E}_1 \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k = E_1$. Finally, we show that \tilde{E}_1 is flat over $W_2(k)$, whence \tilde{E}_1 is a lifting of $E_1 = E_Y$, and thus Y is strongly liftable over $W_2(k)$.

$$\begin{array}{ccc} F_1 + \cdots + F_{\mu} & \hookrightarrow & \tilde{F}_1 + \cdots + \tilde{F}_{\mu} \\ \tau_E \downarrow & & \downarrow \tau_{\tilde{E}} \\ E_1 + \cdots + E_{\nu} & \hookrightarrow & \tilde{E}_1 + \cdots + \tilde{E}_{\nu} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ E & \hookrightarrow & \tilde{E} \end{array}$$

Since $W_2(k)$ is an Artin local ring, to prove that \tilde{E}_1 is flat over $W_2(k)$, by the local criteria of flatness in [4, Subsection 20.C, Theorem 49], it suffices to show $\mathrm{Tor}_1^{W_2(k)}(\mathcal{O}_{\tilde{E}_1}, k) = 0$. Let $Z = \pi^{-1}(E)$, $\tilde{Z} = \tilde{\pi}^{-1}(\tilde{E})$, \mathcal{I} be the ideal sheaf of E_1 in Z , and $\tilde{\mathcal{I}}$ be the ideal sheaf of \tilde{E}_1 in \tilde{Z} . Then the structure sheaf of \tilde{Z} is $\tilde{\mathcal{A}}|_{\tilde{E}}$, which is locally free over \tilde{E} and \tilde{E} is flat over $W_2(k)$, so \tilde{Z} is flat over $W_2(k)$ and $\tilde{Z} \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k = Z$. Locally, E_1 is defined by one of the factors of the equation $x^N = s|_E$, and \tilde{E}_1 is defined by one of the factors of the equation $\tilde{x}^N = \tilde{s}|_{\tilde{E}}$. Since $\tilde{s}|_{\tilde{E}}$ is a divisible lifting of $s|_E$, we have that the reductions of the defining equations of \tilde{E}_1 modulo p are just the defining equations of E_1 , so $\tilde{\mathcal{I}} \times_{\mathbf{Spec} W_2(k)} \mathbf{Spec} k = \mathcal{I}$ holds. Considering the following exact sequence:

$$0 \rightarrow \tilde{\mathcal{I}} \rightarrow \mathcal{O}_{\tilde{Z}} \rightarrow \mathcal{O}_{\tilde{E}_1} \rightarrow 0,$$

and taking its long exact sequence for $-\otimes_{W_2(k)} k$, we obtain an exact sequence

$$0 \rightarrow \mathrm{Tor}_1^{W_2(k)}(\mathcal{O}_{\tilde{E}_1}, k) \rightarrow \tilde{\mathcal{I}} \otimes_{W_2(k)} k \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{E_1} \rightarrow 0,$$

which implies $\mathrm{Tor}_1^{W_2(k)}(\mathcal{O}_{\tilde{E}_1}, k) = 0$, since $\tilde{\mathcal{I}} \otimes_{W_2(k)} k = \mathcal{I}$ and $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{E_1} \rightarrow 0$ is exact.

Definition 4.1 A Noetherian scheme X is said to satisfy the H^i -vanishing condition, if $H^i(X, \mathcal{L}) = 0$ holds for any invertible sheaf \mathcal{L} on X . For example, the projective space \mathbb{P}_k^n satisfies the H^i -vanishing condition for any $1 \leq i \leq n-1$.

Corollary 4.1 *Let X be a smooth projective variety satisfying the H^i -vanishing condition for $i = 1, 2$. Then X is strongly liftable over $W_2(k)$, and for any cyclic cover $\pi : Y \rightarrow X$ constructed as in Theorem 3.1, Y is also strongly liftable over $W_2(k)$.*

Proof From the exact sequences (2.4) and (2.6) and Proposition 2.1, it follows that X is strongly liftable. By Theorem 4.1, we only have to show that for any prime divisor E on X , there exists a lifting $\tilde{E} \subset \tilde{X}$ of $E \subset X$, such that $\tilde{s}|_{\tilde{E}}$ is a divisible lifting of $s|_E$.

Assume that $s|_E \in H^0(E, \mathcal{L}^N|_E)$ is μ -divisible. Thus there is a section $t_E \in H^0(E, \mathcal{L}^{N/\mu}|_E)$, such that $s|_E = t_E^{\otimes \mu}$. Take an arbitrary lifting $\tilde{E} \subset \tilde{X}$ of $E \subset X$ and consider the following commutative diagram:

$$\begin{array}{ccc} H^0(\tilde{X}, \tilde{\mathcal{L}}^{N/\mu}) & \xrightarrow{r} & H^0(X, \mathcal{L}^{N/\mu}) \\ q_{\tilde{E}} \downarrow & & \downarrow q_E \\ H^0(\tilde{E}, \tilde{\mathcal{L}}^{N/\mu}|_{\tilde{E}}) & \xrightarrow{r} & H^0(E, \mathcal{L}^{N/\mu}|_E) \end{array}$$

where the surjectivity of the upper horizontal map r and the right vertical map q_E follows from the H^1 -vanishing condition for X by observing the exact sequence (2.6) and the following exact sequence:

$$0 \rightarrow \mathcal{O}_X(-E) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0. \quad (4.1)$$

Thus for $t_E \in H^0(E, \mathcal{L}^{N/\mu}|_E)$, there exists a section $\tilde{t} \in H^0(\tilde{X}, \tilde{\mathcal{L}}^{N/\mu})$, such that $q_E \circ r(\tilde{t}) = t_E$. Let $\tilde{t}_E = q_{\tilde{E}}(\tilde{t})$. Then $\tilde{t}_E \in H^0(\tilde{E}, \tilde{\mathcal{L}}^{N/\mu}|_{\tilde{E}})$ is a lifting of t_E .

The exact sequence (4.1) gives rise to an exact sequence of cohomology groups

$$0 = H^1(X, \mathcal{O}_X) \rightarrow H^1(E, \mathcal{O}_E) \rightarrow H^2(X, \mathcal{O}_X(-E)) = 0,$$

so we have $H^1(E, \mathcal{O}_E) = 0$. Taking cohomology groups of the following exact sequence, which is the exact sequence (2.3) for \tilde{E}

$$0 \rightarrow \mathcal{O}_E \xrightarrow{q} \mathcal{O}_{\tilde{E}}^* \xrightarrow{r} \mathcal{O}_E^* \rightarrow 0,$$

we have an exact sequence of cohomology groups

$$0 = H^1(E, \mathcal{O}_E) \rightarrow H^1(\tilde{E}, \mathcal{O}_{\tilde{E}}^*) \xrightarrow{r} H^1(E, \mathcal{O}_E^*), \quad (4.2)$$

which implies that $\tilde{\mathcal{L}}_1 \cong \tilde{\mathcal{L}}_2$ if and only if $\mathcal{L}_1 \cong \mathcal{L}_2$, where $\tilde{\mathcal{L}}_i$ are invertible sheaves on \tilde{E} and $\mathcal{L}_i = \tilde{\mathcal{L}}_i|_E$ for $i = 1, 2$.

Since $r(\tilde{s}|_{\tilde{E}}) = s|_E = t_E^{\otimes \mu} = r(\tilde{t}_E^{\otimes \mu})$, there exists a unit $\tilde{u} \in \mathcal{O}_{\tilde{E}}^*$, such that $r(\tilde{u}) = 1$ and $\tilde{s}|_{\tilde{E}} = \tilde{u} \tilde{t}_E^{\otimes \mu}$. Since $p \nmid N$, we have $p \nmid \mu$, and hence there exists a unit $\tilde{v} \in \mathcal{O}_{\tilde{E}}^*$, such that $\tilde{v}^\mu = \tilde{u}$ and $r(\tilde{v}) = 1$. Redefine \tilde{t}_E by $\tilde{v} \tilde{t}_E$, so then \tilde{t}_E is a lifting of t_E and $\tilde{s}|_{\tilde{E}} = \tilde{t}_E^{\otimes \mu}$ is μ -divisible.

Corollary 4.2 *Let $X = \mathbb{P}_k^n$ with $n \geq 3$, and \mathcal{L} be an invertible sheaf on X . Let N be a positive integer prime to p , and D be an effective divisor on X with $\mathcal{L}^N = \mathcal{O}_X(D)$ and $\text{Sing}(D_{\text{red}}) = \emptyset$. Let $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of D . Then Y is a smooth projective scheme which is strongly liftable over $W_2(k)$.*

Proof Since the projective space \mathbb{P}_k^n ($n \geq 3$) satisfies the H^i -vanishing condition for $i = 1, 2$, the conclusion follows from Corollary 4.1.

By means of Corollary 4.2, we can construct many strongly liftable varieties of the general type.

Example 4.2 Let $X = \mathbb{P}_k^n$, $\mathcal{L} = \mathcal{O}_X(1)$, and N be a positive integer such that $n \geq 3$, $(N, p) = 1$ and $N > n + 2$. Let H be a general element in the linear system of $\mathcal{O}_X(N)$. Then H is a smooth irreducible hypersurface of degree N in X with $\mathcal{L}^N = \mathcal{O}_X(H)$. Let $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of H . Then by Corollary 4.2, Y is a strongly liftable smooth projective variety. By Hurwitz's formula, we have $K_Y = \pi^*(K_X + \frac{N-1}{N}H)$. Since the degree of $K_X + \frac{N-1}{N}H$ is $N - (n + 2) > 0$, K_Y is an ample divisor on Y , and hence Y is of the general type.

Obviously, the H^i -vanishing condition for $i = 1, 2$ is too strong to give more applications. Although there are no further evidences besides Corollary 4.2, we would like to put forward the following conjecture, i.e., cyclic covers over toric varieties should be strongly liftable over $W_2(k)$, whereas the liftability has already been proved in [6, Corollary 4.4].

Conjecture 4.1 *Let X be a smooth projective toric variety, and \mathcal{L} be an invertible sheaf on X . Let N be a positive integer prime to p , and D be an effective divisor on X with $\mathcal{L}^N = \mathcal{O}_X(D)$ and $\text{Sing}(D_{\text{red}}) = \emptyset$. Let $\pi : Y \rightarrow X$ be the cyclic cover obtained by taking the N -th root out of D . Then Y is a smooth projective scheme which is strongly liftable over $W_2(k)$.*

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