Chin. Ann. Math.
36B(6), 2015, 947–956
DOI: 10.1007/s11401-015-0932-6

Chinese Annals of Mathematics, Series B

On the Serrin's Regularity Criterion for the β-Generalized Dissipative Surface Quasi-geostrophic Equation*

Jihong ZHAO[1] Qiao LIU[2]

Abstract The authors establish a Serrin's regularity criterion for the β-generalized dissipative surface quasi-geostrophic equation. More precisely, it is shown that if the smooth solution θ satisfies $\nabla\theta \in L^q(0,T;L^p(\mathbb{R}^2))$ with $\frac{\alpha}{q}+\frac{2}{p} \le \alpha+\beta-1$, then the solution θ can be smoothly extended after time T. In particular, when $\alpha+\beta \ge 2$, it is shown that if $\partial_y\theta \in L^q(0,T;L^p(\mathbb{R}^2))$ with $\frac{\alpha}{q}+\frac{2}{p} \le \alpha+\beta-1$, then the solution θ can also be smoothly extended after time T. This result extends the regularity result of Yamazaki in 2012.

Keywords β-generalized quasi-geostrophic equation, Weak solution, Serrin's regularity criterion
2010 MR Subject Classification 35B65, 35Q86, 35R11

1 Introduction

In this paper, we study the two dimensional β-generalized surface quasi-geostrophic equation as follows:

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = 0, & (x,y)\in\mathbb{R}^2,\ t>0,\\ \theta(0,x,y) = \theta_0(x,y), & (x,y)\in\mathbb{R}^2.\end{cases} \tag{1.1}$$

Here $\alpha\in(0,1]$, $\beta\in[1,2)$, $\kappa>0$ is the dissipative coefficient, and $\theta=\theta(t,x,y):(0,\infty)\times\mathbb{R}^2\mapsto\mathbb{R}$ is a real-valued function of a time variable t and two space variables (x,y), and represents the potential temperature of the fluid, while $u=(u^1,u^2):(0,\infty)\times\mathbb{R}^2\mapsto\mathbb{R}^2$ is the velocity field of the fluid which is defined by

$$u=(u^1,u^2)=\Lambda^{1-\beta}\mathcal{R}^\perp\theta=\Lambda^{1-\beta}(-\mathcal{R}_2\theta,\mathcal{R}_1\theta), \tag{1.2}$$

where the fractional power of the Laplacian $\Lambda^\alpha=(-\Delta)^{\frac{\alpha}{2}}$ is defined by the Fourier transform $\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\widehat{f}(\xi)$, and $\mathcal{R}_1,\ \mathcal{R}_2$ are Riesz transforms defined by $\widehat{\mathcal{R}_j f}(\xi)=-\frac{\mathrm{i}\xi_j}{|\xi|}\widehat{f}(\xi)$ for $j=1,2$.

Manuscript received December 7, 2013. Revised January 21, 2015.
[1]College of Science, Northwest A&F University, Yangling 712100, Shaanxi, China. E-mail: jihzhao@163.com
[2]Department of Mathematics, Hunan Normal University, Changsha 410081, China.
E-mail: liuqao2005@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 11501453, 11371294, 11326155, 11401202), the Fundamental Research Funds for the Central Universities (No. 2014YB031), the Fundamental Research Project of Natural Science in Shaanxi Province–Young Talent Project (No. 2015JQ1004), the Scientific Research Fund of Hunan Provincial Education Department (No. 14B117) and the China Postdoctoral Science Foundation (No. 2015M570053).

The β-generalized surface quasi-geostrophic equation (1.1) was introduced by Kiselev in [21]. For $\beta = 1$, (1.1) reduces to the following dissipative surface quasi-geostrophic equation:

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = 0, & (x,y)\in\mathbb{R}^2,\ t>0,\\ u = \mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), & (x,y)\in\mathbb{R}^2,\ t>0,\\ \theta(0,x,y) = \theta_0(x,y), & (x,y)\in\mathbb{R}^2. \end{cases} \tag{1.3}$$

(1.3) is an important model in geophysical fluid dynamics used in meteorology and oceanography, and they are special cases of the general quasi-geostrophic approximations for atmosphere and oceanic fluid flow with small Rossy and Ekman numbers (see [12, 27] for more details about its physical background). Due to its analogy with 3D incompressible Navier-Stokes/Euler equations, in the last two decades, (1.3) attracted enormous attention and many important results were obtained. For the global well-posedness of (1.3) in the subcritical case $\alpha > 1$, we refer the readers to [2, 13, 28]. For the global well-posedness with small initial data in various functional spaces (e.g., Sobolev spaces, Besov spaces, Hölder spaces, etc.) of (1.3) in the critical case $\alpha = 1$, we refer the readers to [1, 7, 9–10, 14, 24]. Recently, the global regularity of weak solutions in the critical case $\alpha = 1$ was addressed by the following two mathematical groups: Kiselev, Nazarov and Volberg [22] proved global well-posedness of (1.3) with periodic C^∞ data by using a certain non-local maximum principle for a suitable chosen modulus of continuity; Caffarelli and Vasseur [4] obtained a global regular weak solution to (1.3) with merely L^2 initial data by using the modified De Georgi interation. For the global regularity of the supercritical case $\alpha < 1$, we refer the readers to [3, 8, 29, 35]. Parts of the above global well-posedness results were subsequently extended to (1.1) with $\beta \in [1, 2)$ by [11, 26, 31–32].

Although the global existence of smooth solutions to (1.1) with suitable choices of α and β was established (see [32]), the regularity issue of weak solutions in the supercritical case is still an open problem, so the development of the regularity criterion of weak solutions is of major importance for both theoretical and practical purposes. For $\beta = 1$, Constantin, Majda and Tabak [12] proved that the maximum norm of $\nabla^\perp\theta$ controls the breakdown of the smooth solution to (1.3) in both viscous and invisid cases, i.e., they proved that if

$$\int_0^T \|\nabla^\perp\theta(t)\|_{L^\infty}\mathrm{d}t < \infty, \tag{1.4}$$

then the solution θ can be extended beyond time T. Chae [6] established that if

$$\int_0^T \|\nabla\theta(t)\|_{L^p}^r\mathrm{d}t < \infty \quad \text{for } \frac{2}{p} + \frac{\alpha}{r} \le \alpha,\ \frac{2}{\alpha} < p < \infty, \tag{1.5}$$

then there is no singularity up to time T. For some improvements of (1.5), we refer the readers to [15–17, 19, 30, 34]. For the β-generalized surface quasi-geostrophic equation (1.1), under the hypothesis that $\alpha + \beta = 2$, Yamazaki [33] established that if

$$\int_0^T \frac{\|\partial_y\theta(t)\|_{L^p}^q}{\ln(\mathrm{e} + \|\nabla\theta(t)\|_{L^2}^2)}\mathrm{d}t < \infty \quad \text{for } \frac{2}{p} + \frac{\alpha}{q} \le 1,\ 2 < p < \infty, \tag{1.6}$$

then there is no singularity up to time T.

Motivated by the above cited results, the first purpose of this paper is to establish a similar Serrin's regularity criterion (1.5) for the β-generalized surface dissipative quasi-geostrophic equation (1.1). In the sequel, if $\beta = 1$, we let $\frac{2}{\beta-1} = \infty$.

Theorem 1.1 *Let $\alpha \in (0,1]$ and $\beta \in [1,2)$, such that $\alpha + 2\beta < 4$. Assume that θ is a smooth solution to (1.1) with initial data $\theta_0 \in H^3(\mathbb{R}^2)$. Assume further that for some $T > 0$,*

$$\int_0^T \frac{\|\nabla\theta(t)\|_{L^p}^q}{\ln(\mathrm{e} + \|\nabla\theta(t)\|_{L^\infty}^2)}\mathrm{d}t < \infty \quad \textit{for } \frac{2}{p} + \frac{\alpha}{q} \le \alpha + \beta - 1,\ \frac{2}{\alpha+\beta-1} < p < \frac{2}{\beta-1}. \tag{1.7}$$

Then the solution θ can be smoothly extended after time T.

Remark 1.1 (i) Theorem 1.1 is clearly a generalization of (1.5).

(ii) The conditions $\alpha + 2\beta < 4$ and $p < \frac{2}{\beta-1}$ appear due to the Gagliardo-Nirenberg inequalities and the Hardy-Littlewood-Sobolev inequalities which we will use in the proof of Theorem 1.1.

The second purpose of this paper is based on the observation that the velocity field u is divergence free, i.e., $\partial_x u^1 + \partial_y u^2 = 0$, so we can establish the following regularity criterion in terms of partial derivatives of the solution θ.

Theorem 1.2 *Let $\alpha \in (0,1]$ and $\beta \in [1,2)$, such that $\alpha + \beta \ge 2$ and $\alpha + 2\beta < 4$. Assume that θ is a smooth solution to (1.1) with initial data $\theta_0 \in H^3(\mathbb{R}^2)$. Assume further that for some $T > 0$,*

$$\int_0^T \frac{\|\partial_y\theta(t)\|_{L^p}^q}{\ln(\mathrm{e} + \|\nabla\theta(t)\|_{L^2}^2)}\mathrm{d}t < \infty \quad \textit{for } \frac{2}{p} + \frac{\alpha}{q} \le \alpha + \beta - 1,\ \frac{2}{\alpha+\beta-1} < p < \frac{2}{\beta-1}. \tag{1.8}$$

Then the solution θ can be smoothly extended after time T.

Remark 1.2 (i) The role of $\partial_y\theta$ can be replaced by $\partial_x\theta$ in Theorem 1.2. This implies that one direction of the derivative of the solution θ controls the regularity of the solution θ.

(ii) Theorem 1.2 covers the supercritical case, and the distinction between Theorem 1.2 and the regularity result of Yamazaki [33] is that we improve the condition $\alpha + \beta = 2$ to $\alpha + \beta \ge 2$.

(iii) Using a single partial derivative of the solution to control the regularity of weak solutions was observed in many equations in fluid dynamics, e.g., for the Navier-Stokes equations (see [18, 23, 36]), for the MHD equations (see [5]), and for the nematic liquid crystal flows (see [25]).

The remaining part of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Throughout this paper, C stands for a generic positive constant which may vary from line to line, and $\|\cdot\|_X$ denotes the norm of the Banach space X.

2 The Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. Multiplying (1.1) by θ, integrating over $\mathbb{R}^2$ and using the fact $\nabla\cdot u=0$, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^2}^2+\kappa\|\Lambda^{\frac{\alpha}{2}}\theta(t)\|_{L^2}^2=0,$$

and it follows that

$$\|\theta(t)\|_{L^2}^2+2\kappa\int_0^t\|\Lambda^{\frac{\alpha}{2}}\theta(\tau)\|_{L^2}^2\mathrm{d}\tau\le\|\theta_0\|_{L^2}^2\quad\text{for all } t\ge 0. \tag{2.1}$$

Applying $\Lambda^3\theta$ to (1.1), multiplying the resulting identity by $\Lambda^3\theta$, and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^3\theta\|_{L^2}^2+\kappa\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2=-\int_{\mathbb{R}^2}\Lambda^3(u\cdot\nabla\theta)\Lambda^3\theta\mathrm{d}x\mathrm{d}y. \tag{2.2}$$

Thanks to the fact that $\nabla\cdot u=0$, we have

$$\int_{\mathbb{R}^2}u\cdot\nabla\Lambda^3\theta\Lambda^3\theta\mathrm{d}x\mathrm{d}y=0. \tag{2.3}$$

Thus we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^3\theta\|_{L^2}^2+\kappa\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2=-\int_{\mathbb{R}^2}(\Lambda^3(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^3\theta)\Lambda^3\theta\mathrm{d}x\mathrm{d}y:=\mathrm{I}. \tag{2.4}$$

To estimate the right-hand side of (2.4), we need to use the following well-known commutator estimate (see [20]): For $s>1$, we have

$$\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\le C(\|\Lambda f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}}+\|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}) \tag{2.5}$$

with $1<p,\ q_1,\ p_2<\infty$, such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$. Moreover, we split the proof of Theorem 1.1 into the following two cases.

For the case of $\frac{2}{\alpha+\beta-1}<p<\frac{4}{\alpha+2\beta-2}$, by using (2.5), we see that

$$\begin{aligned}
\mathrm{I}&\le\|\Lambda^3(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^3\theta\|_{L^{\frac{4p}{4+4p-\alpha p-2\beta p}}}\|\Lambda^3\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le C(\|\nabla\theta\|_{L^p}\|\Lambda^3u\|_{L^{\frac{4}{4-\alpha-2\beta}}}+\|\nabla u\|_{L^{\frac{2p}{2+p-\beta p}}}\|\Lambda^3\theta\|_{L^{\frac{4}{2-\alpha}}})\|\Lambda^3\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le C(\|\nabla\theta\|_{L^p}\|\Lambda^{3+\frac{\alpha}{2}}\mathcal{R}^\perp\theta\|_{L^2}+\|\nabla\mathcal{R}^\perp\theta\|_{L^p}\|\Lambda^3\theta\|_{L^{\frac{4}{2-\alpha}}})\|\Lambda^3\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le C\|\nabla\theta\|_{L^p}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}\|\Lambda^3\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le C\|\nabla\theta\|_{L^p}\|\Lambda^3\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}\\
&\le\frac{\kappa}{2}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\Lambda^3\theta\|_{L^2}^2,
\end{aligned} \tag{2.6}$$

where we used the Hardy-Littlewood-Sobolev inequalities ($\alpha+2\beta<4$)

$$\|\Lambda^{1-\beta}u\|_{L^{\frac{4}{4-\alpha-2\beta}}}\le C\|\Lambda^{\frac{\alpha}{2}}u\|_{L^2},\quad\|\Lambda^{1-\beta}u\|_{L^{\frac{2p}{2+p-\beta p}}}\le C\|u\|_{L^p},$$

the boundedness of Riesz operators in $L^p(\mathbb{R}^2)$ with $1 < p < \infty$ and the following Gagliardo-Nirenberg inequality ($\frac{2}{\alpha+\beta-1} < p < \frac{4}{\alpha+2\beta-2}$):

$$\|\Lambda^3\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}} \le C\|\Lambda^3\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}} \|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{4+2p-\alpha p-2\beta p}{\alpha p}}.$$

For the case of $\frac{4}{\alpha+2\beta-2} \le p < \frac{2}{\beta-1}$, by using (2.5) again, we obtain

$$\begin{aligned}
\mathrm{I} &\le \|\Lambda^3(u\cdot\nabla\theta) - u\cdot\nabla\Lambda^3\theta\|_{L^{\frac{2p}{2+2p-\beta p}}} \|\Lambda^3\theta\|_{L^{\frac{2p}{\beta p-2}}} \\
&\le C(\|\nabla\theta\|_{L^p}\|\Lambda^3 u\|_{L^{\frac{2}{2-\beta}}} + \|\nabla u\|_{L^{\frac{2p}{2+p-\beta p}}}\|\Lambda^3\theta\|_{L^2})\|\Lambda^3\theta\|_{L^{\frac{2p}{\beta p-2}}} \\
&\le C(\|\nabla\theta\|_{L^p}\|\Lambda^3\mathcal{R}^\perp\theta\|_{L^2} + \|\nabla\mathcal{R}^\perp\theta\|_{L^p}\|\Lambda^3\theta\|_{L^2})\|\Lambda^3\theta\|_{L^{\frac{2p}{\beta p-2}}} \\
&\le C\|\nabla\theta\|_{L^p}\|\Lambda^3\theta\|_{L^2}\|\Lambda^3\theta\|_{L^{\frac{2p}{\beta p-2}}} \\
&\le C\|\nabla\theta\|_{L^p}\|\Lambda^3\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}} \\
&\le \frac{\kappa}{2}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-2-p}}\|\Lambda^3\theta\|_{L^2}^2,
\end{aligned} \tag{2.7}$$

where we used the Hardy-Littlewood-Sobolev's inequalities ($p < \frac{2}{\beta-1}$)

$$\|\Lambda^{1-\beta}u\|_{L^{\frac{2}{2-\beta}}} \le C\|u\|_{L^2} \quad \text{and} \quad \|\Lambda^{1-\beta}u\|_{L^{\frac{2p}{2+p-\beta p}}} \le C\|u\|_{L^p},$$

and the Gagliardo-Nirenberg's inequality ($\frac{4}{\alpha+2\beta-2} \le p < \infty$)

$$\|\Lambda^3\theta\|_{L^{\frac{2p}{\beta p-2}}} \le C\|\Lambda^3\theta\|_{L^2}^{\frac{\alpha p+2\beta p-2p-4}{\alpha p}} \|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}.$$

Let $q = \frac{\alpha p}{\alpha p+\beta p-2-p}$. It is easy to verify that $\frac{\alpha}{q} + \frac{2}{p} = \alpha+\beta-1$. Then, by (2.6)–(2.7), one obtains that for $\frac{2}{\alpha+\beta-1} < p < \frac{2}{\beta-1}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^3\theta\|_{L^2}^2 + \kappa\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2 &\le C\|\nabla\theta\|_{L^p}^q\|\Lambda^3\theta\|_{L^2}^2 \\
&\le C\frac{\|\nabla\theta\|_{L^p}^q}{\ln(1+\|\nabla\theta\|_{L^\infty}^2)}\ln(1+\|\nabla\theta\|_{L^\infty}^2)\|\Lambda^3\theta\|_{L^2}^2 \\
&\le C\frac{\|\nabla\theta\|_{L^p}^q}{\ln(1+\|\nabla\theta\|_{L^\infty}^2)}\ln(1+\|\Lambda^3\theta\|_{L^2}^2)\|\Lambda^3\theta\|_{L^2}^2,
\end{aligned} \tag{2.8}$$

where we used the following Sobolev interpolation inequality:

$$\ln(\mathrm{e}+\|\nabla\theta\|_{L^\infty}^2) \le C\ln(\mathrm{e}+\|\theta\|_{L^2}^{\frac{2}{3}}\|\Lambda^3\theta\|_{L^2}^{\frac{4}{3}}) \le C\ln(\mathrm{e}+\|\Lambda^3\theta\|_{L^2}^2).$$

Hence, we obtain from (2.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(\mathrm{e}+\|\Lambda^3\theta(t)\|_{L^2}^2) \le C\frac{\|\nabla\theta(t)\|_{L^p}^q}{\ln(1+\|\nabla\theta(t)\|_{L^\infty}^2)}\ln(\mathrm{e}+\|\Lambda^3\theta(t)\|_{L^2}^2). \tag{2.9}$$

Applying Gronwall's inequality to (2.9) on the time interval $[0,T]$ and using the condition (1.7), we can easily see that

$$\ln(\mathrm{e}+\|\Lambda^3\theta(T)\|_{L^2}^2) \le \ln(\mathrm{e}+\|\Lambda^3\theta_0\|_{L^2}^2)\exp\Big(C\int_0^T \frac{\|\nabla\theta(t)\|_{L^p}^q}{\ln(1+\|\nabla\theta(t)\|_{L^\infty}^2)}\mathrm{d}t\Big) < \infty. \tag{2.10}$$

Combining (2.10) with the energy inequality (2.1), we get the boundedness of $\|\theta(t)\|_{H^3}$ on the time interval $[0,T]$. The proof of Theorem 1.1 is complete.

3 The Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2. Applying ∂_x to (1.1), multiplying the resultant by $\partial_x\theta$, and integrating over $\mathbb{R}^2$, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x\theta\|_{L^2}^2+\kappa\|\Lambda^{\frac{\alpha}{2}}\partial_x\theta\|_{L^2}^2=-\int_{\mathbb{R}^2}\partial_x(u\cdot\nabla\theta)\partial_x\theta\mathrm{d}x\mathrm{d}y. \tag{3.1}$$

Since $\nabla\cdot u=0$, it follows that

$$\int_{\mathbb{R}^2}u\cdot\nabla\partial_x\theta\partial_x\theta\mathrm{d}x\mathrm{d}y=0. \tag{3.2}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x\theta\|_{L^2}^2+\kappa\|\Lambda^{\frac{\alpha}{2}}\partial_x\theta\|_{L^2}^2&=-\int_{\mathbb{R}^2}\partial_xu\cdot\nabla\theta\partial_x\theta\mathrm{d}x\mathrm{d}y\\&=-\int_{\mathbb{R}^2}(\partial_xu^1\partial_x\theta\partial_x\theta+\partial_xu^2\partial_y\theta\partial_x\theta)\mathrm{d}x\mathrm{d}y\\&=\int_{\mathbb{R}^2}\partial_yu^2\partial_x\theta\partial_x\theta\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\partial_xu^2\partial_y\theta\partial_x\theta\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y\theta\|_{L^2}^2+\kappa\|\Lambda^{\frac{\alpha}{2}}\partial_y\theta\|_{L^2}^2&=-\int_{\mathbb{R}^2}\partial_yu\cdot\nabla\theta\partial_y\theta\mathrm{d}x\mathrm{d}y\\&=-\int_{\mathbb{R}^2}\partial_yu^1\partial_x\theta\partial_y\theta\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\partial_yu^2\partial_y\theta\partial_y\theta\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.4}$$

Hence, by (3.3)–(3.4), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^2}^2+\kappa\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2&=\int_{\mathbb{R}^2}\partial_yu^2\partial_x\theta\partial_x\theta\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\partial_xu^2\partial_y\theta\partial_x\theta\mathrm{d}x\mathrm{d}y\\&\quad-\int_{\mathbb{R}^2}\partial_yu^1\partial_x\theta\partial_y\theta\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}\partial_yu^2\partial_y\theta\partial_y\theta\mathrm{d}x\mathrm{d}y\\&=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4.\end{aligned} \tag{3.5}$$

For the case of $\frac{2}{\alpha+\beta-1}<p<\frac{4}{\alpha+2\beta-2}$, we proceed in the same way as the proof of (2.6) to estimate the terms I_i $(i=1,2,3,4)$ as follows:

$$\begin{aligned}\mathrm{I}_1&\le\|\partial_yu^2\|_{L^{\frac{2p}{2+p-\beta p}}}\|\partial_x\theta\|_{L^{\frac{4}{2-\alpha}}}\|\partial_x\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\&\le C\|\partial_y\theta\|_{L^p}\|\nabla\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}\\&\le\frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2+C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\mathrm{I}_2&\le\|\partial_y\theta\|_{L^p}\|\partial_xu^2\|_{L^{\frac{4}{4-\alpha-2\beta}}}\|\partial_x\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\&\le C\|\partial_y\theta\|_{L^p}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\&\le\frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2+C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{3.7}$$

$$\mathrm{I}_3\le\|\partial_y\theta\|_{L^p}\|\partial_yu^1\|_{L^{\frac{4}{4-\alpha-2\beta}}}\|\partial_x\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}$$

$$
\begin{aligned}
&\le C\|\partial_y\theta\|_{L^p}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le \frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 + C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2, \qquad (3.8)\\
\mathrm{I}_4 &\le \|\partial_y\theta\|_{L^p}\|\partial_y u^2\|_{L^{\frac{4}{4-\alpha-2\beta}}}\|\partial_y\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le C\|\partial_y\theta\|_{L^p}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^{\frac{4p}{\alpha p+2\beta p-4}}}\\
&\le \frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 + C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2. \qquad (3.9)
\end{aligned}
$$

For the case of $\frac{4}{\alpha+2\beta-2} \le p < \frac{2}{\beta-1}$, in a way similar to the proof of (2.7), we estimate the terms I_i $(i=1,2,3,4)$ as follows:

$$
\begin{aligned}
\mathrm{I}_1 &\le \|\partial_y u^2\|_{L^{\frac{2p}{2+p-\beta p}}}\|\partial_x\theta\|_{L^2}\|\partial_x\theta\|_{L^{\frac{2p}{\beta p-2}}}\\
&\le C\|\partial_y\theta\|_{L^p}\|\nabla\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}\\
&\le \frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 + C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2, \qquad (3.10)\\
\mathrm{I}_2 &\le \|\partial_y\theta\|_{L^p}\|\partial_x u^2\|_{L^{\frac{2}{2-\beta}}}\|\partial_x\theta\|_{L^{\frac{2p}{\beta p-2}}}\\
&\le C\|\partial_y\theta\|_{L^p}\|\nabla\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}\\
&\le \frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 + C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2, \qquad (3.11)\\
\mathrm{I}_3 &\le \|\partial_y\theta\|_{L^p}\|\partial_y u^1\|_{L^{\frac{2}{2-\beta}}}\|\partial_x\theta\|_{L^{\frac{2p}{\beta p-2}}}\\
&\le C\|\partial_y\theta\|_{L^p}\|\nabla\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}\\
&\le \frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 + C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2, \qquad (3.12)\\
\mathrm{I}_4 &\le \|\partial_y\theta\|_{L^p}\|\partial_y u^2\|_{L^{\frac{2}{2-\beta}}}\|\partial_y\theta\|_{L^{\frac{2p}{\beta p-2}}}\\
&\le C\|\partial_y\theta\|_{L^p}\|\nabla\theta\|_{L^2}^{\frac{2\alpha p+2\beta p-2p-4}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^{\frac{4+2p-2\beta p}{\alpha p}}\\
&\le \frac{\kappa}{8}\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 + C\|\partial_y\theta\|_{L^p}^{\frac{\alpha p}{\alpha p+\beta p-p-2}}\|\nabla\theta\|_{L^2}^2. \qquad (3.13)
\end{aligned}
$$

Note that if we set $q = \frac{\alpha p}{\alpha p+\beta p-p-2}$, which satisfies $\frac{\alpha}{q}+\frac{2}{p} = \alpha+\beta-1$, then by putting the above estimates (3.6)–(3.13) together, we get for all $\frac{2}{\alpha+\beta-1} < p < \frac{2}{\beta-1}$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^2}^2 + \kappa\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^2 &\le C\|\partial_y\theta\|_{L^p}^q\|\nabla\theta\|_{L^2}^2\\
&\le C\frac{\|\partial_y\theta\|_{L^p}^q}{\ln(1+\|\nabla\theta\|_{L^2}^2)}\ln(1+\|\nabla\theta\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2. \qquad (3.14)
\end{aligned}
$$

Dividing both sides of (3.14) by $(\mathrm{e}+\|\nabla\theta\|_{L^2}^2)$, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\ln(\mathrm{e}+\|\nabla\theta\|_{L^2}^2) \le C\frac{\|\partial_y\theta\|_{L^p}^q}{\ln(1+\|\nabla\theta\|_{L^2}^2)}\ln(\mathrm{e}+\|\nabla\theta\|_{L^2}^2). \qquad (3.15)
$$

Applying Gronwall's inequality to (3.15), it follows from the condition (1.8) that

$$
\|\nabla\theta(T)\|_{L^2}^2 \le (\mathrm{e}+\|\nabla\theta_0\|_{L^2}^2)\exp\Big(C\int_0^T\frac{\|\partial_y\theta\|_{L^p}^q}{\ln(1+\|\nabla\theta\|_{L^2}^2)}\mathrm{d}t\Big) < \infty. \qquad (3.16)
$$

Going back to (3.14), and integrating on the time interval $[0,T]$, we obtain

$$\begin{aligned}&\|\nabla\theta(T)\|_{L^2}^2+\kappa\int_0^T\|\Lambda^{\frac{\alpha}{2}}\nabla\theta(t)\|_{L^2}^2\mathrm{d}t\\ &\le\|\nabla\theta_0\|_{L^2}^2+\sup_{0\le t\le T}(\ln(\mathrm{e}+\|\nabla\theta(t)\|_{L^2}^2)\|\nabla\theta(t)\|_{L^2}^2)\int_0^T\frac{\|\partial_y\theta\|_{L^p}^q}{\ln(1+\|\nabla\theta\|_{L^2}^2)}\mathrm{d}t<\infty.\end{aligned}\tag{3.17}$$

In particular, we notice that $\int_0^T\|\Lambda^{\frac{\alpha}{2}}\nabla\theta(t)\|_{L^2}^2\mathrm{d}t<\infty$.

Now we are in a position to derive the desired estimate of $\Lambda^3\theta$. In a way similar to the proof of Theorem 1.1, by using (2.5), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^3\theta\|_{L^2}^2+\kappa\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2\\ &=-\int_{\mathbb{R}^2}(\Lambda^3(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^3\theta)\cdot\Lambda^3\theta\mathrm{d}x\mathrm{d}y\\ &\le\|\Lambda^3(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^3\theta\|_{L^2}\|\Lambda^3\theta\|_{L^2}\\ &\le C(\|\nabla u\|_{L^{\frac{4}{4-\alpha-2\beta}}}\|\Lambda^3\theta\|_{L^{\frac{4}{\alpha+2\beta-2}}}+\|\nabla\theta\|_{L^{\frac{4}{2-\alpha}}}\|\Lambda^3u\|_{L^{\frac{4}{\alpha}}})\|\Lambda^3\theta\|_{L^2}\\ &\le C(\|\Lambda^{\frac{\alpha}{2}}\nabla\mathcal{R}^\perp\theta\|_{L^2}\|\Lambda^3\theta\|_{L^{\frac{4}{\alpha+2\beta-2}}}\\ &\quad+\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}\|\Lambda^{4-\beta}\mathcal{R}^\perp\theta\|_{L^{\frac{4}{\alpha}}})\|\Lambda^3\theta\|_{L^2}\\ &\le C\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}\|\Lambda^3\theta\|_{L^2}^{\frac{3\alpha+2\beta-4}{\alpha}}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{4-\alpha-2\beta}{\alpha}}\\ &\le\frac{\kappa}{2}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^2+C\|\Lambda^{\frac{\alpha}{2}}\nabla\theta\|_{L^2}^{\frac{2\alpha}{3\alpha+2\beta-4}}\|\Lambda^3\theta\|_{L^2}^2,\end{aligned}\tag{3.18}$$

where we used, under the assumptions $\alpha+\beta\ge 2$ and $\alpha+2\beta<4$, the following Gagliardo-Nirenberg inequalities:

$$\begin{aligned}\|\Lambda^3\theta\|_{L^{\frac{4}{\alpha+2\beta-2}}}&\le C\|\Lambda^3\theta\|_{L^2}^{\frac{2\alpha+2\beta-4}{\alpha}}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{\alpha+2\beta-4}{\alpha}},\\ \|\Lambda^{4-\beta}\theta\|_{L^{\frac{4}{\alpha}}}&\le C\|\Lambda^3\theta\|_{L^2}^{\frac{2\alpha+2\beta-4}{\alpha}}\|\Lambda^{3+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{\alpha+2\beta-4}{\alpha}}.\end{aligned}$$

Since $\int_0^T\|\Lambda^{\frac{\alpha}{2}}\nabla\theta(t)\|_{L^2}^2\mathrm{d}t<\infty$ and $\frac{2\alpha}{3\alpha+2\beta-4}\le 2$, it follows from Gronwall's inequality that

$$\|\Lambda^3\theta(t)\|_{L^2}<\infty\quad\text{for all } t\in[0,T].$$

Combining this with (2.1) yields the boundedness of $\|\theta(t)\|_{H^3}$ on the time interval $[0,T]$. We complete the proof of Theorem 1.2.

References

[1] Abidi, H. and Hmidi, T., On the global well-posedness of the critical quasi-geostrophic equation, *SIAM J. Math. Anal.,* **40**, 2008, 167–185.

[2] Benameura, J. and Benhamedb, M., Global existence of the two-dimensional QGE with sub-critical dissipation, *J. Math. Anal. Appl.,* **423**, 2015, 1330–1347.

[3] Biswas, A., Gevrey regularity for the supercritical quasi-geostrophic equation, *J. Diff. Eq.,* **257**, 2014, 1753–1772.

[4] Caffarelli, L. and Vasseur, A., Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. of Math.,* **171**, 2010, 1903–1930.

[5] Cao, C. and Wu, J., Two regularity criteria for the 3D MHD equations, *J. Diff. Eq.,* **248**, 2010, 2263–2274.

[6] Chae, D., On the regularity conditions for the dissipative quasi-geostrophic equations, *SIAM J. Math. Anal.,* **37**, 2006, 1649–1656.

[7] Chae, D. and Lee, J., Global well-posedness in the super-critical dissipative quasigeostrophic equations, *Commun. Math. Phys.,* **233**, 2003, 297–311.

[8] Chen. Q., Miao, C. and Zhang, Z., A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation, *Commun. Math. Phys.,* **271**, 2007, 821–838.

[9] Chen, Q. and Zhang, Z., Global well-posedness of the 2D critical dissipative quasigeostrophic equation in the Triebel-Lizorkin spaces, *Nonlinear Anal.,* **67**, 2007, 1715–1725.

[10] Constantin, P., Córdoba, D. and Wu, J., On the critical dissipative quasi-geostrophic equation, *Indiana Univ. Math. J.,* **50**, 2001, 97–107.

[11] Constantin, P., Iyer, G. and Wu, J., Global regularity for a modified critical dissipative quasi-geostrophic equation, *Indiana Univ. Math. J.,* **57**, 2011, 97–107.

[12] Constantin, P., Majda, A. J. and Tabak, E., Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar, *Nonlinearity,* **7**, 1994, 1495–1533.

[13] Constantin, P. and Wu, J., Behavior of solutions of 2D quasi-geostrophic equations, *SIAM J. Math. Anal.,* **30**, 1999, 937–948.

[14] Córdoba, A. and Córdoba, D., A maximum principle applied to quasi-geostrophic equations, *Commun. Math. Phys.,* **249**, 2004, 511–528.

[15] Dong, B. and Chen, Z., A remark on regularity criterion for the dissipative quasi-geostrophic equations, *J. Math. Anal. Appl.,* **329**, 2007, 1212–1217.

[16] Dong, H. and Pavlovic, N., A regularity criterion for the dissipation quasi-geostrophic equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **26**, 2009, 1607–1619.

[17] Fan, J., Gao, H. and Nakamura, G., Regularity criteria for the generalized magnetohydrodynamic equations and the quasi-geostrophic equations, *Taiwanese J. Math.,* **15**(3), 2011, 1059–1073.

[18] He, C., Regularity for solutions to the Navier-Stokes equations with one velocity component regular, *Electronic J. Diff. Eq.,* **29**, 2002, 1–13.

[19] Jia, Y. and Dong, B., Remarks on the logarithmical regularity criterion of the supercritical surface quasi-geostrophic equation in Morrey spaces, *Appl. Math. Lett.,* **43**, 2015, 80–84.

[20] Kato, T. and Ponce, G., Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.,* **41**, 1988, 891–907.

[21] Kiselev, A., Regularity and blow-up for active scalars, *Math. Model. Nat. Phenom.,* **5**(4), 2010, 225–255.

[22] Kiselev, A., Nazarov, F. and Volberg, A., Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, *Invent. Math.,* **167**, 2007, 445–453.

[23] Kukavica, I. and Ziane, M., Navier-Stokes equations with regularity in one direction, *J. Math. Phys.,* **48**, 2007, 065203.

[24] Lazer, O., Global existence for the critical dissipative surface quasi-geostrophic equation, *Commun. Math. Phys.,* **322**, 2013, 73–93.

[25] Liu, Q., Zhao, J. and Cui, S., A regularity criterion for the three-dimensional nematic liquid crystal flow in terms of one directional derivative of the velocity, *J. Math. Phys.,* **52**, 2011, 033102.

[26] Miao, C. and Xue, L., Global wellposedness for a modified critical dissipative quasi-geostrophic equation, *J. Diff. Eq.,* **252**(1), 2012, 792–818.

[27] Pedlosky, J., Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.

[28] Resnick, S., Dynamical Problems in Nonlinear Advective Partial Differential Equations, Ph.D. Thesis, University of Chicago, Chicago, 1995.

[29] Silvestre, L., Vicol, V. and Zlatoš, A., On the loss of continuity for super-critical drift-diffusion equations, *Arch. Rational Mech. Anal.,* **207**, 2013, 845–877.

[30] Xiang, Z., A regularity criterion for the critical and supercritical dissipative quasi-geostrophic equations, *Appl. Math. Lett.,* **23**, 2010, 1286–1290.

[31] Xue, L. and Zheng, X., Note on the well-posedness of a slightly supercritical surface quasi-geostrophic equation, *J. Diff. Eq.,* **253**, 2012, 795–813.

[32] Yamazaki, K., A remark on the global well-posedness of a modified critical quasi-geostrophic equation. arXiv: 1006.0253v2

[33] Yamazaki, K., On the regularity criteria of a surface quasi-geostrophic equation, *Nonlinear Analysis,* **75**, 2012, 4950–4956.

[34] Yuan, J., On regularity criterion for the dissipative quasi-geostrophic equations, *J. Math. Anal. Appl.,* **340**, 2008, 334–339.

[35] Zelati, M. C. and Vicol, V., On the global regularity for the supercritical SQG equation. arXiv: 1410.3186v1

[36] Zhou, Y., A new regularity criterion for weak solutions to the Navier-Stokes equations, *J. Math. Pures Appl.,* **84**, 2005, 1496–1514.