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Chow-Type Maximal Inequality for Conditional Demimartingales and Its Applications*

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Abstract In this paper, the Chow-type maximal inequality for conditional demimartingales is established. By using the Chow-type maximal inequality, the authors provide the maximal inequality for conditional demimartingales based on concave Young functions. At last, the moment inequalities for conditional demimartingales are established.

Keywords Conditional demimartingales, Chow-type maximal inequality, Concave Young functions
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1 Introduction

Let X and Y be random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ with $EX^2 < \infty$ and $EY^2 < \infty$. Let $\mathcal{F}$ be a sub-σ algebra of $\mathcal{A}$. The notion of the conditional covariance of X and Y given $\mathcal{F}$ ($\mathcal{F}$-covariance for short) is defined as

$$\mathrm{Cov}^{\mathcal{F}}(X,Y) = E^{\mathcal{F}}((X - E^{\mathcal{F}}X)(Y - E^{\mathcal{F}}Y)),$$

where $E^{\mathcal{F}}Z$ denotes the conditional expectation of a random variable Z given $\mathcal{F}$. In contrast to the ordinary concept of variance, conditional variance of X given $\mathcal{F}$ is defined as $\mathrm{Var}^{\mathcal{F}}X = \mathrm{Cov}^{\mathcal{F}}(X,X)$.

Firstly, let us recall some definitions.

Definition 1.1 *Let $S_1, S_2, \cdots$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \cdots$,*

$$E\{(S_{j+1} - S_j) f(S_1, \cdots, S_j)\} \geq 0 \tag{1.1}$$

for all coordinatewise nondecreasing functions f, such that the expectation is defined. Then $\{S_j, j \geq 1\}$ is called a demimartingale. If in addition, the function f is assumed to be nonnegative, then the sequence $\{S_j, j \geq 1\}$ is called a demisubmartingale.

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Definition 1.2 *Let $S_1, S_2, \cdots$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \cdots$,*

$$E\{(S_{j+1} - S_j)f(S_1, \cdots, S_j)\} \leq 0 \tag{1.2}$$

for all coordinatewise nondecreasing functions f, such that the expectation is defined. Then $\{S_j, j \geq 1\}$ is called an N-demimartingale. If in addition, the function f is assumed to be nonnegative, then the sequence $\{S_j, j \geq 1\}$ is called an N-demisupermartingale.

Definition 1.1 is due to Newman and Wright [1]. Many authors studied this concept and provided interesting results and applications (see [1–10]). Christofides [11] introduced the class of N-demimartingales. Many authors obtained some results for N-demimartingales (see [8, 10, 12–17]).

Hadjikyriakou [18] introduced the following concept of conditional demimartingales.

Definition 1.3 *A sequence of L^1 random variables $\{S_n, n \geq 1\}$ is called an $\mathcal{F}$-demimartingale, if for $1 \leq i < j < \infty$,*

$$E^{\mathcal{F}}\{(S_j - S_i)f(S_1, S_2, \cdots, S_i)\} \geq 0 \quad a.s. \tag{1.3}$$

for every componentwise nondecreasing function f and whenever the conditional expectation is defined. If in addition, f is assumed to be nonnegative, the sequence $\{S_n, n \geq 1\}$ is called an $\mathcal{F}$-demisubmartingale.

It is easy to check that for $i \geq 1$, (1.3) is equivalent to

$$E^{\mathcal{F}}\{(S_{i+1} - S_i)f(S_1, S_2, \cdots, S_i)\} \geq 0 \quad \text{a.s.}$$

Definition 1.4 *A finite collection of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be $\mathcal{F}$-associated if for any two componentwise nondecreasing functions f and g on $\mathbb{R}^n$,*

$$\mathrm{Cov}^{\mathcal{F}}(f(X_1, X_2, \cdots, X_n), g(X_1, X_2, \cdots, X_n)) \geq 0 \quad a.s.,$$

whenever the conditional covariance exists. An infinite collection $\{X_n, n \geq 1\}$ is said to be $\mathcal{F}$-associated if every finite subcollection is $\mathcal{F}$-associated.

Remark 1.1 It is easy to verify that the partial sum of $\mathcal{F}$-associated random variables with conditional mean zero is an $\mathcal{F}$-demimartingale by Property P2 in [19] and the definition of $\mathcal{F}$-association. The details on $\mathcal{F}$-association are due to [20–21] etc. Yuan and Yang [19] pointed out that the conditional association of random variables does not imply association and the opposite implication is not true, either.

Hence one does have to derive some properties under certain conditions if there is a need for such results even though the results and their proofs may be analogous to those under the non-conditioning setup. This is one of the reasons for developing results for sequences of conditional random variables in this paper.

The main purpose of this paper is to establish some maximal and moment inequalities for $\mathcal{F}$-demimartingales, which can be applied to obtain other inequalities for $\mathcal{F}$-demimartingales. The organization of this paper is as follows. Some useful lemmas are presented in Section 2.

The Chow-type maximal inequality for $\mathcal{F}$-demimartingales is established in Section 3, which will be used to prove other maximal inequalities for $\mathcal{F}$-demimartingales including the Doob's maximal inequality. The maximal inequalities for $\mathcal{F}$-demimartingales based on concave Young functions are provided in Section 4. Finally, the moment inequalities for $\mathcal{F}$-demimartingales are established in Section 5.

Throughout this paper, I_A denotes the indicator function of the set A, and let $P^{\mathcal{F}}(A) = E^{\mathcal{F}}(I_A)$, $x^+ = \max\{0, x\}$, $x^- = \max\{0, -x\}$ and $a \vee b = \max\{a, b\}$. Let $\{S_n, n \geq 1\}$ be an $\mathcal{F}$-demimartingale, and $g(\cdot)$ be a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$. Let $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables. Denote $S_n^* = \max\limits_{1\leq k\leq n} c_k g(S_k)$, $T_n = \sum\limits_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))$, $n \geq 1$, $S_0 = 0$ and $S_0^* = 0$.

2 Preliminaries

In this section, we give some lemmas which are very useful to prove the main results of this paper. The first one is a very important property of $\mathcal{F}$-demimartingales which was proved by Hadjikyriakou [18].

Lemma 2.1 *Let $\{S_n, n \geq 1\}$ be an $\mathcal{F}$-demimartingale (or $\mathcal{F}$-demisubmartingale), and g be a nondecreasing convex function. Then $\{g(S_n), n \geq 1\}$ is an $\mathcal{F}$-demisubmartingale.*

The next one is a useful (deterministic) inequality for nonnegative real numbers obtained by Christofides and Hadjikyriakou [14].

Lemma 2.2 *Let $x, y \geq 0$ and $p \geq 2$. Then*

$$y^p \geq x^p + px^{p-1}(y - x) + (y - x)^p.$$

The following conditional version of the Fubini theorem, which was proved by Roussas [20], will play an essential role in the proof of the main results of this paper.

Lemma 2.3 *Let $X(\cdot,\cdot,\cdot) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{A} \times \mathcal{B}^2$-measurable and either nonnegative or $P \times \mu \times \mu$-integrable, where μ is the Lebesgue measure, and let $\mathcal{F}$ be a sub-σ-field of $\mathcal{A}$. Then*

$$E^{\mathcal{F}} \int_{\mathbb{R}^2} X(\cdot, t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2 = \int_{\mathbb{R}^2} [E^{\mathcal{F}} X(\cdot, t_1, t_2)]\mathrm{d}t_1\mathrm{d}t_2 \quad a.s.$$

3 Chow-Type Maximal Inequality for $\mathcal{F}$-Demimartingales

In this section, we present the Chow-type maximal inequality for $\mathcal{F}$-demimartingales, which will play an important role in the proof of other inequalities for $\mathcal{F}$-demimartingales, such as Doob's maximal inequality. Denote $S_n^* = \max\limits_{1\leq k\leq n} c_k g(S_k)$, $n \geq 1$ and $S_0^* = 0$. The Chow-type maximal inequality for $\mathcal{F}$-demimartingales is as follows.

Theorem 3.1 *Let $\{S_n, n \geq 1\}$ be an $\mathcal{F}$-demimartingale, and $g(\cdot)$ be a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$. Let $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables. Then for any $\mathcal{F}$-measurable random variable $\varepsilon > 0$ a.s.,*

$$\varepsilon P^{\mathcal{F}}(S_n^* \geq \varepsilon) \leq \sum_{i=1}^{n} c_i E^{\mathcal{F}}[\big(g(S_i) - g(S_{i-1})\big)I(S_n^* \geq \varepsilon)] \quad a.s. \tag{3.1}$$

Proof Let

$$u(x)=\begin{cases} g(x), & x\ge 0,\\ 0, & x<0,\end{cases}\qquad v(x)=\begin{cases} 0, & x\ge 0,\\ g(x), & x<0.\end{cases}$$

By the definition of g, we have

$$g(x)=u(x)+v(x)=\max\{u(x),v(x)\},\quad x\in\mathbb{R}. \tag{3.2}$$

It is easy to see that $u(x)$ is a nonnegative nondecreasing convex function, and $v(x)$ is a nonnegative nonincreasing convex function. By the definitions of $u(x)$ and $v(x)$, we have

$$\begin{aligned}\varepsilon P^{\mathcal{F}}(S_n^*\ge\varepsilon) &= \varepsilon P^{\mathcal{F}}\Big(\max_{1\le k\le n} c_k\max(u(S_k),v(S_k))\ge\varepsilon\Big)\\ &= \varepsilon P^{\mathcal{F}}(\max(c_1\max(u(S_1),v(S_1)),\dots,c_n\max(u(S_n),v(S_n)))\ge\varepsilon)\\ &\le \varepsilon P^{\mathcal{F}}\Big(\max_{1\le k\le n} c_k u(S_k)\ge\varepsilon\Big)+\varepsilon P^{\mathcal{F}}\Big(\max_{1\le k\le n} c_k v(S_k)\ge\varepsilon\Big).\end{aligned} \tag{3.3}$$

Therefore, in order to prove (3.1), we only need to show that

$$\varepsilon P^{\mathcal{F}}\Big(\max_{1\le k\le n} c_k u(S_k)\ge\varepsilon\Big)\le\sum_{i=1}^n c_i E^{\mathcal{F}}[(u(S_i)-u(S_{i-1}))I(S_n^*\ge\varepsilon)]\quad\text{a.s.}, \tag{3.4}$$

$$\varepsilon P^{\mathcal{F}}\Big(\max_{1\le k\le n} c_k v(S_k)\ge\varepsilon\Big)\le\sum_{i=1}^n c_i E^{\mathcal{F}}[(v(S_i)-v(S_{i-1}))I(S_n^*\ge\varepsilon)]\quad\text{a.s.} \tag{3.5}$$

Now, we prove (3.4). Let m be a nonnegative nondecreasing function on $\mathbb{R}$ with $m(0)=0$ and define $H_n=\max\limits_{1\le k\le n} c_k u(S_k)$ with $H_0=0$. First, we prove that

$$E^{\mathcal{F}}\Big[\int_0^{H_n} t\mathrm{d}m(t)\Big]\le\sum_{i=1}^n c_i E^{\mathcal{F}}[(u(S_i)-u(S_{i-1}))m(H_n)]\quad\text{a.s.} \tag{3.6}$$

It is easy to see that

$$\begin{aligned}E^{\mathcal{F}}\Big[\int_0^{H_n} t\mathrm{d}m(t)\Big] &= \sum_{i=1}^n E^{\mathcal{F}}\Big[\int_{H_{i-1}}^{H_i} t\mathrm{d}m(t)\Big]\\ &\le \sum_{i=1}^n E^{\mathcal{F}}[H_i(m(H_i)-m(H_{i-1}))]\quad\text{a.s.}\end{aligned} \tag{3.7}$$

By the definitions of H_n and m, we can see that for $H_i\ge H_{i-1}$, either $H_i=c_iu(S_i)$ or $m(H_i)=m(H_{i-1})$. Hence, we have by (3.7) that

$$\begin{aligned}&E^{\mathcal{F}}\Big[\int_0^{H_n} t\mathrm{d}m(t)\Big]\\ &\le\sum_{i=1}^n c_iE^{\mathcal{F}}[u(S_i)(m(H_i)-m(H_{i-1}))]\\ &=\sum_{i=1}^n c_iE^{\mathcal{F}}[(u(S_i)-u(S_{i-1}))m(H_n)]\\ &\quad-\Big\{\sum_{i=1}^{n-1}E^{\mathcal{F}}[(c_{i+1}u(S_{i+1})-c_iu(S_i))m(H_i)]+\sum_{i=1}^{n-1}(c_i-c_{i+1})E^{\mathcal{F}}[u(S_i)m(H_n)]\Big\}\\ &\doteq\sum_{i=1}^n c_iE^{\mathcal{F}}[(u(S_i)-u(S_{i-1}))m(H_n)]-A\quad\text{a.s.}\end{aligned}$$

To prove (3.6), it suffices to show that

$$\begin{aligned} A \doteq \sum_{i=1}^{n-1} & E^{\mathcal{F}}[(c_{i+1}u(S_{i+1}) - c_i u(S_i))m(H_i)] \\ & + \sum_{i=1}^{n-1}(c_i - c_{i+1})E^{\mathcal{F}}[u(S_i)m(H_n)] \geq 0 \quad \text{a.s.} \end{aligned} \tag{3.8}$$

Denote the left-hand derivative of function u by

$$h(x) = \lim_{\triangle x \to 0^-} \frac{u(x+\triangle x) - u(x)}{\triangle x}.$$

By the convexity of u, h is a nondecreasing function, and furthermore, we have

$$u(y) - u(x) \geq (y-x)h(x). \tag{3.9}$$

Since $(c_i - c_{i+1})u(S_i) \geq 0$, $i = 1, 2, \cdots, n-1$, it follows by (3.9) that

$$\begin{aligned} A &\geq \sum_{i=1}^{n-1} E^{\mathcal{F}}[(c_{i+1}u(S_{i+1}) - c_i u(S_i))m(H_i)] + \sum_{i=1}^{n-1}(c_i - c_{i+1})E^{\mathcal{F}}[u(S_i)m(H_i)] \\ &= \sum_{i=1}^{n-1} c_{i+1}E^{\mathcal{F}}[(u(S_{i+1}) - u(S_i))m(H_i)] \\ &\geq \sum_{i=1}^{n-1} c_{i+1}E^{\mathcal{F}}[(S_{i+1} - S_i)h(S_i)m(H_i)] \quad \text{a.s.} \end{aligned} \tag{3.10}$$

It is a simple fact that $h(S_i)m(H_i)$ is a nondecreasing function of $S_1, S_2, \cdots, S_i$, so the right-hand side of (3.10) is nonnegative by the definition of $\mathcal{F}$-demimartingale which yields (3.6).

Taking $m(t) = I(t \geq \varepsilon)$ in (3.6), by the definition of H_n, we have

$$\begin{aligned} & \varepsilon P^{\mathcal{F}}\Big(\max_{1\leq k\leq n} c_k u(S_k) \geq \varepsilon\Big) \\ \leq & \sum_{i=1}^{n} c_i E^{\mathcal{F}}[\big(u(S_i) - u(S_{i-1})\big)I(H_n \geq \varepsilon)] \\ = & \sum_{i=1}^{n-1}(c_i - c_{i+1})E^{\mathcal{F}}[u(S_i)I(H_n \geq \varepsilon)] + c_n E[u(S_n)I(H_n \geq \varepsilon)] \\ \leq & \sum_{i=1}^{n-1}(c_i - c_{i+1})E^{\mathcal{F}}[u(S_i)I(S_n^* \geq \varepsilon)] + c_n E[u(S_n)I(S_n^* \geq \varepsilon)] \\ = & \sum_{i=1}^{n-1} c_i E^{\mathcal{F}}[\big(u(S_i) - u(S_{i-1})\big)I(S_n^* \geq \varepsilon)] \quad \text{a.s.,} \end{aligned}$$

where the second inequality follows from the simple fact that $H_n \leq S_n^*$. Hence, (3.4) has been proved. Similarly, we can get (3.5). Finally, (3.1) follows from (3.3)–(3.5) immediately. This completes the proof of the theorem.

Remark 3.1 If we take $c_k \equiv 1$ for each $k \geq 1$ in Theorem 3.1, then we have by Theorem 3.1 that

$$\begin{aligned}\varepsilon P^{\mathcal{F}}\Big(\max_{1\leq k\leq n} g(S_k) \geq \varepsilon\Big) &\leq \sum_{i=1}^{n} E^{\mathcal{F}}\Big[(g(S_i)-g(S_{i-1}))I\Big(\max_{1\leq k\leq n} g(S_k)\geq\varepsilon\Big)\Big] \\ &= E^{\mathcal{F}}\Big[g(S_n)I\Big(\max_{1\leq k\leq n} g(S_k)\geq\varepsilon\Big)\Big] \quad \text{a.s.}\end{aligned}$$

Furthermore, if $g(x) = |x|^r$ for some $r \geq 1$, then we can obtain by the inequality above that

$$\varepsilon P^{\mathcal{F}}\Big(\max_{1\leq k\leq n} |S_k|^r \geq \varepsilon\Big) \leq E^{\mathcal{F}}\Big[|S_n|^r I\Big(\max_{1\leq k\leq n} |S_k|^r \geq\varepsilon\Big)\Big] \quad \text{a.s.}$$

4 Maximal Inequality for $\mathcal{F}$-Demimartingales Based on Concave Young Functions

Let ϕ be a right continuous nonincreasing function on $(0,\infty)$, which satisfies the condition

$$\phi(\infty) \doteq \lim_{t\to\infty} \phi(t) = 0.$$

Assume further that ϕ is also integrable with respect to the Lebesgue measure on any finite interval $(0, x)$. Let

$$\Phi(x) = \int_0^x \phi(t)\mathrm{d}t, \quad x \geq 0.$$

Then the function $\Phi(x)$ is a nonnegative nondecreasing concave function such that $\Phi(0) = 0$. Further assume that $\Phi(\infty) = \infty$. Then $\Phi(x)$ is called a concave Young function.

For more details and properties of concave Young functions, one can refer to [22]. An example of such a function is $\Phi(x) = x^p$, $0 < p < 1$. Agbeko [22] obtained the maximal inequality for nonnegative submartingales based on the class of concave Young functions. Inspired by [22], Christofides [11] obtained some maximal inequalities for concave Young functions for N-demimartingales. Our goal in this paper is to extend these results to $\mathcal{F}$-demimartingales based on the classes of concave Young functions. Denote $S_n^* = \max_{1\leq k\leq n} c_k g(S_k)$, $T_n = \sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))$, $n \geq 1$ and $S_0^* = 0$. Our results are as follows.

Theorem 4.1 *Suppose that the conditions of Theorem 3.1 are satisfied. Let $\Phi(x)$ be a concave Young function. Denote $\xi(x) = \Phi(x) - x\phi(x)$. Then we have*

(i)

$$E^{\mathcal{F}}\xi(S_n^*) \leq \inf_{x_0>0}[\xi(x_0) + \phi(x_0)E^{\mathcal{F}}T_n] \quad a.s. \tag{4.1}$$

(ii) *If*

$$\limsup_{x\to\infty} \frac{x\phi(x)}{\Phi(x)} < 1, \tag{4.2}$$

then the inequality

$$(1-b)E^{\mathcal{F}}\Phi(S_n^*) - a \leq E^{\mathcal{F}}\xi(S_n^*) \quad a.s. \tag{4.3}$$

is valid for some constants $a \geq 0$ and $0 < b < 1$.

(iii) *If* (4.2) *holds true, then*

$$E^{\mathcal{F}}\Phi(S_n^*) \leq C_\Phi\Big(1 + \inf_{x_0>0}[\xi(x_0) + \phi(x_0)E^{\mathcal{F}}T_n]\Big) \quad a.s. \tag{4.4}$$

for some positive constant C_Φ depending only on Φ.

Proof (i) Theorem 3.1 implies that for all real numbers $x > 0$,

$$xP^{\mathcal{F}}(S_n^* \geq x) \leq E^{\mathcal{F}}[T_n I(S_n^* \geq x)] \quad \text{a.s.} \tag{4.5}$$

By integrating on $[x_0, \infty)$, $x_0 > 0$, with respect to the measure $d(-\phi(x))$, we can get by (4.5) and Lemma 2.3 that

$$\begin{aligned} E^{\mathcal{F}}\Big[\int_{x_0}^{S_n^*\vee x_0} x\mathrm{d}(-\phi(x))\Big] &\leq E^{\mathcal{F}}\Big[T_n\int_{x_0}^{S_n^*\vee x_0}\mathrm{d}(-\phi(\mathrm{x}))\Big] \\ &= -E^{\mathcal{F}}[T_n\phi(S_n^*\vee x_0)] + \phi(x_0)E^{\mathcal{F}}T_n \\ &\leq \phi(x_0)E^{\mathcal{F}}T_n \quad \text{a.s.} \end{aligned} \tag{4.6}$$

The last inequality follows from the fact that $T_n\phi(S_n^*\vee x_0) \geq 0$. Integrating by parts, we obtain by the notation of ξ that

$$\begin{aligned} &E^{\mathcal{F}}\Big[\int_{x_0}^{S_n^*\vee x_0} x\mathrm{d}(-\phi(x))\Big] \\ &= x_0\phi(x_0) - E^{\mathcal{F}}[(S_n^*\vee x_0)\phi(S_n^*\vee x_0)] + E^{\mathcal{F}}\Big[\int_{x_0}^{S_n^*\vee x_0}\phi(x)\mathrm{d}x\Big] \\ &= x_0\phi(x_0) - E^{\mathcal{F}}[(S_n^*\vee x_0)\phi(S_n^*\vee x_0)] + E^{\mathcal{F}}\Phi(S_n^*\vee x_0) - \Phi(x_0) \\ &= E^{\mathcal{F}}\xi(S_n^*\vee x_0) - \xi(x_0) \quad \text{a.s.} \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we can get that

$$E^{\mathcal{F}}\xi(S_n^*\vee x_0) \leq \xi(x_0) + \phi(x_0)E^{\mathcal{F}}T_n \quad \text{a.s.} \tag{4.8}$$

It is easy to check that the function $\xi(x) = \Phi(x) - x\phi(x)$ is nondecreasing for $x > 0$. Thus, by (4.8), we have

$$E^{\mathcal{F}}\xi(S_n^*) \leq E^{\mathcal{F}}\xi(S_n^*\vee x_0) \leq \xi(x_0) + \phi(x_0)E^{\mathcal{F}}T_n \quad \text{a.s.}$$

The desired result (4.1) follows from the inequality above immediately.

(ii) By (4.2), we can see that there exist constants $a \geq 0$ and $0 < b < 1$, such that for all $x > 0$,

$$x\phi(x) \leq a + b\Phi(x). \tag{4.9}$$

It follows by (4.9) and the definition of ξ that

$$E^{\mathcal{F}}\xi(S_n^*) = E^{\mathcal{F}}\Phi(S_n^*) - E^{\mathcal{F}}[S_n^*\phi(S_n^*)] \geq E^{\mathcal{F}}\Phi(S_n^*) - bE^{\mathcal{F}}\Phi(S_n^*) - a \quad \text{a.s.},$$

which yields the desired result (4.3) by reorganizing the inequality above.

(iii) The validity of inequality (4.4) follows from (i)–(ii) immediately. This completes the proof of the theorem.

Corollary 4.1 *Suppose that the conditions of Theorem* 4.1 *are satisfied. Then for any* $0 < p < 1$,

$$E^{\mathcal{F}}(S_n^*)^p \le \frac{1}{1-p}(E^{\mathcal{F}}T_n)^p \quad a.s. \tag{4.10}$$

Proof Taking $\Phi(x) = x^p$, $0 < p < 1$, we have

$$\phi(x) = px^{p-1}, \quad \xi(x) = \Phi(x) - x\phi(x) = (1-p)x^p.$$

Therefore, by (4.1), we have

$$E^{\mathcal{F}}(S_n^*)^p \le \inf_{x_0>0}\Big(x_0^p + \frac{px_0^{p-1}}{1-p}E^{\mathcal{F}}T_n\Big) \quad \text{a.s.} \tag{4.11}$$

The right-hand side of (4.11) is minimized at $x_0 = E^{\mathcal{F}}T_n$. Hence, the desired result (4.10) can be easily obtained by taking $x_0 = E^{\mathcal{F}}T_n$ in the right-hand side of (4.11).

If $c_k \equiv 1$ for each $k \ge 1$ in Corollary 4.1, we can get the following result.

Corollary 4.2 *Suppose that the conditions of Corollary* 4.1 *are satisfied with* $c_k \equiv 1$ *for each* $k \ge 1$. *Then*

$$E^{\mathcal{F}}\Big[\max_{1\le k\le n} g(S_k)\Big]^p \le \frac{1}{1-p}[E^{\mathcal{F}}g(S_n)]^p \quad a.s. \tag{4.12}$$

If we take $g(x) = |x|$ in Corollary 4.2, then we have the following corollary.

Corollary 4.3 *Let* $\{S_n, n \ge 1\}$ *be an* $\mathcal{F}$*-demisubmartingale. Then for any* $0 < p < 1$,

$$E^{\mathcal{F}}\Big(\max_{1\le k\le n}|S_k|\Big)^p \le \frac{1}{1-p}(E^{\mathcal{F}}|S_n|)^p \quad a.s. \tag{4.13}$$

If we take $\Phi(x) = \ln(1+x)$ in Theorem 4.1, then we can get the following result.

Corollary 4.4 *Suppose that the conditions of Theorem* 4.1 *are satisfied. Then*

$$E^{\mathcal{F}}\ln(1+S_n^*) \le 1 + \ln(1+E^{\mathcal{F}}T_n) \quad a.s. \tag{4.14}$$

Particularly, if $c_k \equiv 1$ *for each* $k \ge 1$, *then*

$$E^{\mathcal{F}}\ln[1+g(S_n)] \le E^{\mathcal{F}}\ln\Big[1+\max_{1\le k\le n} g(S_k)\Big] \le 1+\ln[1+E^{\mathcal{F}}g(S_n)] \quad a.s. \tag{4.15}$$

Proof Taking $\Phi(x) = \ln(1+x)$, $x \ge 0$ in Theorem 4.1, we have

$$\phi(x) = \frac{1}{1+x}, \quad \xi(x) = \Phi(x) - x\phi(x) = \ln(1+x) - \frac{x}{1+x}.$$

Therefore, we have by (4.1) that

$$\begin{aligned} E^{\mathcal{F}}\xi(S_n^*) &= E^{\mathcal{F}}\ln(1+S_n^*) - E^{\mathcal{F}}\Big(\frac{S_n^*}{1+S_n^*}\Big) \\ &\le \inf_{x_0>0}\Big(\ln(1+x_0) - \frac{x_0}{1+x_0} + \frac{1}{1+x_0}E^{\mathcal{F}}T_n\Big) \quad \text{a.s.} \end{aligned} \tag{4.16}$$

The right-hand side of (4.16) is minimized at $x_0 = E^{\mathcal{F}} T_n$. Hence, the desired result (4.14) can be easily obtained by taking $x_0 = E^{\mathcal{F}} T_n$ in the right-hand side of (4.16). (4.15) follows from (4.14) by taking $c_k \equiv 1$ for each $k \geq 1$ immediately. This completes the proof of the corollary.

In the following, we will continue to study the estimate for $E^{\mathcal{F}}\Phi(S_n^*)$ under a different assumption from (4.2).

Theorem 4.2 *Suppose that the conditions of Theorem* 3.1 *are satisfied. Let* $\Phi(x)$ *be a concave Young function, and suppose that*

$$\int_1^\infty \frac{\phi(t)}{t}\mathrm{d}t = C_\phi < \infty, \tag{4.17}$$

where C_ϕ *is a positive constant depending only on* ϕ. *Then*

$$E^{\mathcal{F}}\Phi(S_n^*) \leq \Phi(1) + C_\phi \sum_{j=1}^n c_j E^{\mathcal{F}}(g(S_j) - g(S_{j-1})) = \Phi(1) + C_\phi E^{\mathcal{F}} T_n \quad a.s. \tag{4.18}$$

Proof Theorem 3.1 implies that for all real numbers $x > 0$,

$$xP^{\mathcal{F}}(S_n^* \geq x) \leq \sum_{j=1}^n c_j E^{\mathcal{F}}[(g(S_j) - g(S_{j-1}))I(S_n^* \geq x)], \quad x > 0, \tag{4.19}$$

which we shall integrate on $[1, \infty)$, with respect to the measure generated by the nondecreasing function $\int_1^x \frac{\phi(t)}{t}\mathrm{d}t$, $x \geq 1$. It follows by (4.19) and Lemma 2.3 that

$$\begin{aligned}
\int_1^\infty P^{\mathcal{F}}(S_n^* \geq x)\phi(x)\mathrm{d}x &\leq \int_1^\infty \sum_{j=1}^n c_j E^{\mathcal{F}}[(g(S_j) - g(S_{j-1}))I(S_n^* \geq x)]\frac{\phi(x)}{x}\mathrm{d}x \\
&= \sum_{j=1}^n c_j E^{\mathcal{F}}\Big[(g(S_j) - g(S_{j-1}))\int_1^{S_n^* \vee 1} \frac{\phi(x)}{x}\mathrm{d}x\Big] \\
&= \sum_{j=1}^{n-1}(c_j - c_{j+1})E^{\mathcal{F}}\Big[g(S_j)\int_1^{S_n^* \vee 1} \frac{\phi(x)}{x}\mathrm{d}x\Big] \\
&\quad + c_n E^{\mathcal{F}}\Big[g(S_n)\int_1^{S_n^* \vee 1} \frac{\phi(x)}{x}\mathrm{d}x\Big] \\
&\leq C_\phi \sum_{j=1}^{n-1}(c_j - c_{j+1})E^{\mathcal{F}} g(S_j) + C_\phi c_n E^{\mathcal{F}} g(S_n) \\
&= C_\phi \sum_{j=1}^n c_j E^{\mathcal{F}}(g(S_j) - g(S_{j-1})) \quad \text{a.s.}
\end{aligned} \tag{4.20}$$

On the other hand, it follows by Lemma 2.3 that

$$\begin{aligned}
\int_1^\infty P^{\mathcal{F}}(S_n^* \geq x)\phi(x)\mathrm{d}x &= E^{\mathcal{F}}\Big[\int_1^{S_n^* \vee 1} \phi(x)\mathrm{d}x\Big] \\
&= E^{\mathcal{F}}\Phi(S_n^* \vee 1) - \Phi(1) \\
&\geq E^{\mathcal{F}}\Phi(S_n^*) - \Phi(1).
\end{aligned} \tag{4.21}$$

Together with (4.20)–(4.21), we can get the desired result (4.18) immediately. This completes the proof of the theorem.

Remark 4.1 If we take $\Phi(x)=x^p$, $0<p<1$ in Theorem 4.2, then we have $\phi(x)=px^{p-1}$ and $C_\phi=\int_1^\infty \frac{\phi(t)}{t}\mathrm{d}t=\frac{p}{1-p}$. Therefore, (4.18) implies that

$$E^{\mathcal{F}}(S_n^*)^p \le 1+\frac{p}{1-p}\sum_{j=1}^n c_j E^{\mathcal{F}}(g(S_j)-g(S_{j-1})).$$

Furthermore, if we set $c_k\equiv 1$ for each $k\ge 1$, then we have

$$E^{\mathcal{F}}\Big(\max_{1\le k\le n} g(S_k)\Big)^p \le 1+\frac{p}{1-p}E^{\mathcal{F}}g(S_n).$$

5 Moment Inequalities for $\mathcal{F}$-Demimartingales

In this section, we provide some moment inequalities for $\mathcal{F}$-demimartingales. The main idea is inspired by Christofides and Hadjikyriakou [14]. Our main results are as follows.

Theorem 5.1 *Let $\{S_n, n\ge 1\}$ be a nonnegative $\mathcal{F}$-demisubmartingale. Then for $p\ge 2$,*

$$E^{\mathcal{F}}S_n^p \ge \sum_{j=1}^n E^{\mathcal{F}}d_j^p \quad a.s., \tag{5.1}$$

where $d_j=S_j-S_{j-1},\ j=1,2,\cdots,n$.

Proof By Lemma 2.2, we can see that

$$\begin{aligned} E^{\mathcal{F}}S_{j+1}^p &\ge E^{\mathcal{F}}S_j^p + pE^{\mathcal{F}}[S_j^{p-1}(S_{j+1}-S_j)] + E^{\mathcal{F}}d_{j+1}^p \\ &\ge E^{\mathcal{F}}S_j^p + E^{\mathcal{F}}d_{j+1}^p \quad \text{a.s.,} \end{aligned}$$

where the last inequality follows from the definition of $\mathcal{F}$-demisubmartingales. The desired result (5.1) can be obtained by using induction. The proof is complete.

For the special case of p being a positive even number, the previous result can be extended to $\mathcal{F}$-demimartingales.

Theorem 5.2 *Let $\{S_n, n\ge 1\}$ be an $\mathcal{F}$-demimartingale and p be a positive even integer. Then*

$$E^{\mathcal{F}}|S_n|^p \ge \frac{1}{2^{p-1}}\sum_{j=1}^n E^{\mathcal{F}}|d_j|^p \quad a.s., \tag{5.2}$$

where $d_j=S_j-S_{j-1}$, $j=1,2,\cdots,n$.

Proof Applying the C_r-inequality for a positive even integer p, we have

$$(x-y)^p \le 2^{p-1}[(x^+-y^+)^p+(x^--y^-)^p]. \tag{5.3}$$

By Lemma 2.1, we can see that $\{S_n^+, n\ge 1\}$ is a nonnegative $\mathcal{F}$-demisubmartingale. Let $Y_n=-S_n,\ n\ge 1$. It is easy to see that the sequence $\{Y_n, n\ge 1\}$ is also an $\mathcal{F}$-demimartingale and $Y_n^+ = S_n^-$. Hence it follows that the sequence $\{S_n^-, n \geq 1\}$ is also a nonnegative $\mathcal{F}$-demisubmartingale. Thus, by Theorem 5.1 and (5.3), we have

$$\begin{aligned} E^{\mathcal{F}}|S_n|^p &= E^{\mathcal{F}}(S_n^+)^p + E^{\mathcal{F}}(S_n^-)^p \\ &\geq \sum_{j=1}^n E^{\mathcal{F}}(S_j^+ - S_{j-1}^+)^p + \sum_{j=1}^n E^{\mathcal{F}}(S_j^- - S_{j-1}^-)^p \quad \text{(by Theorem 5.1)} \\ &\geq \frac{1}{2^{p-1}} \sum_{j=1}^n E^{\mathcal{F}}(S_j - S_{j-1})^p \quad \text{(by (5.3))} \\ &= \frac{1}{2^{p-1}} \sum_{j=1}^n E^{\mathcal{F}}|d_j|^p \quad \text{a.s.,} \end{aligned}$$

which implies (5.2). This completes the proof of theorem.

An immediate application of Theorem 5.2 for $\mathcal{F}$-associated random variables with conditional mean zero is as follows.

Corollary 5.1 *Let $\{X_n, n \geq 1\}$ be $\mathcal{F}$-associated random variables with conditional mean zero. Then for a positive even integer p,*

$$E^{\mathcal{F}}|S_n|^p \geq \frac{1}{2^{p-1}} \sum_{j=1}^n E^{\mathcal{F}}|X_j|^p, \tag{5.4}$$

where $S_n = \sum\limits_{j=1}^n X_j$.

Proof It follows by Remark 1.1 that the partial sum of $\mathcal{F}$-associated random variables with conditional mean zero is an $\mathcal{F}$-demimartingale. Hence, the desired result (5.4) follows from Theorem 5.2 immediately. The proof is complete.

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