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The Gradient Estimate of a Neumann Eigenfunction on a Compact Manifold with Boundary*

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Abstract Let $e_\lambda(x)$ be a Neumann eigenfunction with respect to the positive Laplacian Δ on a compact Riemannian manifold M with boundary such that $\Delta\, e_\lambda = \lambda^2 e_\lambda$ in the interior of M and the normal derivative of e_λ vanishes on the boundary of M. Let χ_λ be the unit band spectral projection operator associated with the Neumann Laplacian and f be a square integrable function on M. The authors show the following gradient estimate for $\chi_\lambda f$ as $\lambda \geq 1$: $\|\nabla\chi_\lambda\, f\|_\infty \leq C(\lambda\|\chi_\lambda\, f\|_\infty + \lambda^{-1}\|\Delta\chi_\lambda\, f\|_\infty)$, where C is a positive constant depending only on M. As a corollary, the authors obtain the gradient estimate of e_λ: For every $\lambda \geq 1$, it holds that $\|\nabla e_\lambda\|_\infty \leq C\,\lambda\,\|e_\lambda\|_\infty$.

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1 Introduction

Let (M,g) be an n-dimensional compact smooth Riemannian manifold with smooth boundary ∂M and Δ be the positive Laplacian on M. In the local coordinate chart $x = (x_1, \cdots, x_n)$, Δ can be expressed by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j} \partial_{x_i}(g^{ij}\sqrt{g}\ \partial_{x_j}),$$

where $(g^{ij}) = \big(g^{ij}(x)\big)$ is the inverse of the metric matrix $(g_{ij}) = \big(g_{ij}(x)\big) = g(\partial_{x_i}, \partial_{x_j})$, and $\sqrt{g} = \sqrt{g(x)} := \sqrt{\det\big(g_{ij}(x)\big)}$. In this paper, we always mean doing the summation from 1 to n when we omit the variation domain of indices. Let $L^2(M)$ be the space of square integrable functions on M with respect to the Riemannian density $\mathrm{d}V = \sqrt{\mathbf{g}(x)}\,\mathrm{d}x$. Let $e_1(x), e_2(x), \cdots$ be a complete orthonormal basis in $L^2(M)$ for Neumann eigenfunctions of Δ such that $0 = \lambda_1^2 < \lambda_2^2 \leq \lambda_3^2 \leq \cdots$ for the corresponding eigenvalues, where $e_j(x)$ $(j = 1, 2, \cdots)$ are real-valued smooth functions on M and λ_j are nonnegative numbers. Also, let $\mathbf{e}_j$ denote the projection of $L^2(M)$ onto the 1-dimensional space $\mathbf{C}e_j$. Thus, an L^2 function f can be written as $f = \sum\limits_{j=0}^{\infty} \mathbf{e}_j(f)$, where the partial sum converges in the L^2 norm. Let λ be a positive

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real number ≥ 1. We define the unit band spectral projection operator (UBSPO) χ_λ as

$$\chi_\lambda f := \sum_{\lambda_j \in (\lambda, \lambda+1]} \mathbf{e}_j(f).$$

We call that this χ_λ is associated with the Neumann Laplacian on M. The corresponding UBSPO χ_λ, where we use the same notion, can also be defined for both the Dirichlet Laplacian on M and the Laplacian on a closed Riemannian manifold.

Grieser [7] proved the following L^∞ estimate on χ_λ for both the Dirichlet Laplacian and the Neumann Laplacian,

$$\|\chi_\lambda\, f\|_\infty \leq C\lambda^{\frac{n-1}{2}} \|f\|_2, \tag{1.1}$$

where $\|f\|_r$ ($1 \leq r \leq \infty$) means the L^r norm of the function f on M. In the whole paper, C denotes a positive constant which depends only on (M, g) and may take different values at different places unless otherwise stated. The idea of Grieser is to use the standard wave kernel method outside a boundary layer of width $C\lambda^{-1}$ and a maximum principle argument inside that layer. On the other hand, Smith [12] proved a sharp $L^2 \to L^p$ estimate for χ_λ on a closed manifold with the Lipschitz metric. As a consequence, (1.1) holds for both the Dirichlet and the Neumann Laplacians provided that dim $M = 2$ or 3.

By using the maximum principle argument and the estimate (1.1), Xu [13–14] proved the following gradient estimate on χ_λ for both the Dirichlet and the Neumann Laplacians:

$$\|\nabla\chi_\lambda f\|_\infty \leq C\lambda^{\frac{n+1}{2}} \|f\|_2. \tag{1.2}$$

Here ∇ is the Levi-Civita connection on M. In particular, $\nabla f = \sum\limits_j g^{ij} \frac{\partial f}{\partial x_j}$ is the gradient vector field of a C^1 function f in a local coordinate chart $(x_1, \cdots, x_n)$, the square of whose length equals $\sum\limits_{i,j} g^{ij} \big(\frac{\partial f}{\partial x_i}\big)\big(\frac{\partial f}{\partial x_j}\big)$. One of his motivations is to prove the Hörmander multiplier theorem on compact manifolds with boundary. Seeger and Sogge [9] firstly proved the theorem on closed manifolds by using the parametrix of the wave kernel. Duong-Ouhabaz-Sikora [5] proved a general spectral multiplier theorem on closed manifolds by the L^2 norm estimate of the kernel of spectral multipliers and the Gaussian bounds for the corresponding heat kernel. As an application, they gave an alternative proof of the Hörmander multiplier theorem on closed manifolds by using the L^∞ estimate (1.1) of χ_λ and the heat kernel.

By rescaling $\chi_\lambda f$ at the scale of λ^{-1} both outside and inside the boundary layer of width $C\lambda^{-1}$, for χ_λ associated with the Dirichlet Laplacian, the last two authors [11] obtained by elliptic a priori $C^{1,\alpha}$ estimates the following estimate slightly better than (1.2),

$$\|\nabla\chi_\lambda\, f\|_\infty \leq C(\lambda\|\chi_\lambda\, f\|_\infty + \lambda^{-1}\|\Delta\chi_\lambda\, f\|_\infty), \quad f \in L^2(M). \tag{1.3}$$

See [11, Remark 1.2] for the argument that the above estimate with the help of (1.1) could imply the Dirichlet case of estimate (1.2) by Xu. On the other hand, an immediate consequence of our estimate (1.3) is as follows: There exists a constant C such that for each Dirichlet eigenfunction e_λ, i.e., $\Delta\, e_\lambda = \lambda^2\, e_\lambda$ in the interior of M and $e_\lambda = 0$ on the boundary of M, we have $\|\nabla e_\lambda\|_\infty \leq C\,\lambda\,\|e_\lambda\|_\infty$. Furthermore, following the idea of Brüning [2] and Zelditch [15, Theorem 4.1], the last two authors ([11, Lemma 2.2]) proved a basic geometry property of nodal sets for Dirichlet eigenfunctions, i.e., as λ is sufficiently large, every geodesic ball with radius $\frac{C}{\lambda}$ and lying in the interior $\mathrm{Int}(M)$ of M must contain at least one zero point of a Dirichlet eigenfunction with eigenvalue λ^2. We call this the equidistribution property of a non-trivial Dirichlet eigenfunction, using which we obtained a two-sided gradient estimate for a non-trivial Dirichlet eigenfunction e_λ,

$$C^{-1}\,\lambda\,\|e_\lambda\|_\infty \le \|\nabla e_\lambda\|_\infty \le C\,\lambda\,\|e_\lambda\|_\infty \quad \text{for all } \lambda \ge 1. \tag{1.4}$$

Soon after the last two authors wrote up the manuscript of [11], Professor C. D. Sogge asked the last author whether the two sided gradient estimate (1.4) holds for Neumann eigenfunctions. In the paper, we answer his question partially.

Theorem 1.1. *Let f be a square integrable function on the compact Riemannian manifold (M,g) with boundary ∂M. Let χ_λ be the UBSPO associated with the Neumann Laplacian. Then, for all $\lambda \ge 1$ and for all $f \in L^2(M)$, it holds that*

$$\|\nabla\chi_\lambda\, f\|_\infty \le C(\lambda\|\chi_\lambda\, f\|_\infty + \lambda^{-1}\|\Delta\chi_\lambda\, f\|_\infty). \tag{1.5}$$

In particular, letting $f = e_\lambda(x)$ be an eigenfunction with respect to the positive Neumann Laplacian on N, i.e., $\Delta\, e_\lambda = \lambda^2\, e_\lambda$ in the interior of N and the normal derivative of e_λ vanishes on the boundary of N, we obtain

$$\|\nabla e_\lambda\|_\infty \le C\,\lambda\,\|e_\lambda\|_\infty.$$

Remark 1.1 We shall prove Theorem 1.1 directly via the maximum principle argument in Section 3. It is quite different from that of the Dirichlet case (1.3) in [11, Section 3], where the last two authors used the $C^{1,\alpha}$ a priori estimate. Moreover, our maximum principle argument in this paper would not go through for the Dirichlet case. Heuristically speaking, we should owe the success of the maximum principle in the Neumann case to the following.

Fact 1 If a C^2 function g on the half real line $[0,\infty)$ satisfies $g'(0) = 0$, then the even extension of g is also C^2 on the real line $(-\infty,\infty)$.

Our failure of using the maximum principle argument in the Dirichlet case is partially due to the following.

Fact 2 If a C^2 function h on the half real line $[0,\infty)$ satisfies $h(0) = 0$, then the odd extension of h to the real line $(-\infty,\infty)$ is not C^2 on $(-\infty,\infty)$ in general.

Precisely speaking, by Fact 1, we can reduce the gradient estimate (1.5) near the boundary to the interior case, which will be proved by the standard maximum argument combined with the frequency dependent rescaling technique. However, Fact 2 prevents us from doing the similar thing for the Dirichlet case.

Remark 1.2 Theorem 1.1 strengthens the Neumann case of estimate (1.2) proved by Xu [14] in the sense that it shows how the gradient estimate on a Neumann eigenfunction depends on its supremum. In particular, the similar argument as [11, Remark 1.2] shows that estimate (1.5) together with (1.1) imply the Neumann case of estimate (1.2) by Xu [14]. However, (1.2) is strong enough for Xu to prove his Hörmander multiplier theorem associated with the Neumann

Laplacian on M. The authors' motivation is to prove the Neumann version of the result of Shi and Xu [11].

Remark 1.3 We conjecture that each Neumann eigenfunction has the equidistribution property, i.e., every geodesic ball with radius $\frac{C}{\lambda}$ lying in the interior $\mathrm{Int}(M)$ of M must contain at least one zero point of a Neumann eigenfunction with eigenvalue λ^2. If it were true, then we could prove the following lower bound estimate:

$$\|\nabla e_\lambda\|_\infty \geq C\,\|e_\lambda\|_\infty,$$

by a little modification of the argument in [11, Section 2]. However, the idea of the proof for the equidistribution property of a non-trivial Dirichlet eigenfunction in [11, Section 2] did not go through for a Neumann eigenfunction, because the restriction of a Neumann eigenfunction to one of its nodal domains only satisfies the mixed Dirichlet-Neumann boundary condition in general:

We conclude the introduction by explaining the organization of the left part of this paper. We use the even extension and the maximum principle to show (1.5), which implies the upper bound of ∇e_λ. We also provide an alternative proof of Theorem 1.1 by the same even extension and the $C^{1,\alpha}$ a priori estimate.

2 Estimate for the Gradient of Eigenfunction

2.1 Outside the boundary layer

Recall the principle: On a small scale comparable to the wavelength $\frac{1}{\lambda}$, the eigenfunction e_λ behaves like a harmonic function. It was developed in [3–4] and was used extensively there. In this section, for a square integrable function f on M, letting χ_λ be the UBSPO associated with the Neumann Laplacian, we shall give a modification of this principle, which can be applied to the Poisson equation

$$\Delta\,\chi_\lambda\, f = \sum_{\lambda_j\in(\lambda,\lambda+1]} \lambda_j^2\;\mathbf{e}_j(f) \quad \text{in } \mathrm{Int}(N)$$

with the Neumann boundary condition satisfied by $\chi_\lambda f$ on ∂M. Moreover, in this subsection, we only do analysis outside the boundary layer $L_{\frac{1}{\lambda}} = \{z\in N : d(z,\partial N)\leq \frac{1}{\lambda}\}$ of width $\frac{1}{\lambda}$.

Take a point p with $d(p,\partial M)\geq\frac{1}{\lambda}$. We may assume that $\frac{1}{\lambda}$ is sufficiently small such that there exists a geodesic normal coordinate chart $(x_1,\cdots,x_n)$ on the geodesic ball $B(p,\frac{1}{2\lambda})$ in M. In this chart, we may identify the ball $B(p,\frac{1}{2\lambda})$ with the n-dimensional Euclidean ball $\mathbb{B}(\frac{1}{2\lambda})$ centered at the origin 0, and think of the function $\chi_\lambda f$ in $B(p,\frac{1}{2\lambda})$ as a function in $\mathbb{B}(\frac{1}{2\lambda})$. Our aim in this subsection is to show the inequality

$$|(\nabla\chi_\lambda\; f)(p)| \leq C\Big(\lambda\|\chi_\lambda\; f\|_{L^\infty\left(\mathbb{B}(\frac{1}{2\lambda})\right)} + \lambda^{-1}\|\Delta\chi_\lambda\; f\|_{L^\infty\left(\mathbb{B}(\frac{1}{2\lambda})\right)}\Big). \tag{2.1}$$

For simplicity of notation, we rewrite $u=\chi_\lambda\, f$ and $v=\Delta\,\chi_\lambda\, f$ in what follows. The Poisson equation satisfied by u in $\mathbb{B}(\frac{1}{2\lambda})$ can be written as

$$-\frac{1}{\sqrt{g}}\,\sum_{i,j}\partial_{x_i}(g^{ij}\sqrt{g}\;\partial_{x_j}u) = v.$$

Consider the following rescaled functions:

$$u_\lambda(y) = u\Big(\frac{y}{\lambda}\Big) \quad \text{and} \quad v_\lambda(y) = v\Big(\frac{y}{\lambda}\Big) \quad \text{in the ball } \mathbb{B}\Big(\frac{1}{2}\Big).$$

The above estimate which we are after is equivalent to its rescaled version

$$|(\nabla u_\lambda)(0)| \le C\Big(\|u_\lambda\|_{L^\infty\left(\mathbb{B}(\frac{1}{2})\right)} + \lambda^{-2}\|v_\lambda\|_{L^\infty\left(\mathbb{B}(\frac{1}{2})\right)}\Big). \tag{2.2}$$

On the other hand, the rescaled version of the Poisson equation has the expression

$$\sum_{i,j} \partial_{y_i}\,(g_\lambda^{ij}\ \sqrt{g_\lambda}\ \partial_{y_j} u_\lambda) = -\lambda^{-2}\ \sqrt{g_\lambda}\ v_\lambda, \tag{2.3}$$

where

$$g_{ij,\lambda}(y) = g_{ij}\Big(\frac{y}{\lambda}\Big), \quad g_\lambda^{ij}(y) = g^{ij}\Big(\frac{y}{\lambda}\Big) \quad \text{and} \quad \sqrt{g_\lambda}(y) = (\sqrt{g})\Big(\frac{y}{\lambda}\Big).$$

The last two authors [11, Section 3.1] proved (2.2) by the interior $C^{1,\alpha}$ estimate (see [6, Theorem 8.32, p. 210]) for a second-order elliptic equation of the divergence type, where the C^α norm of coefficients $g_\lambda^{ij}\sqrt{g_\lambda}$ in the equation are involved. In the following paragraph, we shall give a different and more elementary proof of (2.2), where we use the maximum principle, however, the $C^{0,1}$ norm of coefficients $g_\lambda^{ij}\sqrt{g_\lambda}$ are involved. Note that the $C^{0,1}$ norms of $g_\lambda^{ij}\sqrt{g_\lambda}$ are uniformly bounded for all $\lambda \ge 1$.

For simplicity of notation, we set

$$u_\lambda = \phi, \quad h = -\lambda^{-2}\ \sqrt{g_\lambda}\ v_\lambda, \quad a_{ij} = g_\lambda^{ij}\ \sqrt{g_\lambda} \quad \text{and} \quad b_i = \sum_{j=1}^n \frac{a_{ij}}{\partial\, y_j}.$$

Then the rescaled Poission equation (2.3) can be written as

$$L\phi := \sum_{i,j} a_{ij}\ \frac{\partial^2\,\phi}{\partial y_i \partial y_j} + \sum_i b_i\ \frac{\partial\phi}{\partial y_i} = h, \quad y \in \mathbb{B}\Big(\frac{1}{2}\Big).$$

We learned this maximum principle argument for proving (2.2) from Brandt [1]. Moreover, we find that the constant-coefficient-assumption there could be removed. The idea is to construct a new function ϕ_1 from ϕ of $n+1$ variables and apply the maximum principle to ϕ_1. The details are as follows. Define

$$\phi_1(y_1, \cdots, y_n;\, z_1) := \frac{1}{2}(\phi(y_1 + z_1, y_2, \cdots, y_n) - \phi(y_1 - z_1, y_2, \cdots, y_n))$$

in the $(n+1)$-dimensional domain

$$\mathcal{R} = \Big\{(y_1, \cdots, y_n;\, z_1) : \ |y| < \frac{1}{4},\ 0 < z_1 < \frac{1}{4}\Big\}.$$

Writing

$$L_1 = L - \mu\,\frac{\partial^2}{\partial y_1^2} + \mu\,\frac{\partial^2}{\partial z_1^2} \quad (\mu > 0),$$

we observe that, for sufficiently small μ, this new operator is elliptic in the $n+1$ variables, and satisfies

$$|L_1\phi_1| = |L\phi_1| \le \|h\| \quad \text{in } \mathcal{R},$$

where we denote by $\|\cdot\|$ the L^∞ norm in $\mathbb{B}(\frac{1}{2})$. Choose a constant C sufficiently large and depending on the L^∞ norm of coefficients a_{ij} and b_i so that

$$L(|y|^2) \le 2\mu\, C$$

and introduce the comparison function

$$\overline{\phi_1} := \frac{1}{2\mu}\,\|h\|\,\Big(\frac{1}{4}\,z_1 - z_1^2\Big) + 16\,\|\phi\|\,\Big\{|y|^2 + z_1^2 + C\Big(\frac{1}{4}z_1 - z_1^2\Big)\Big\}.$$

Then we have

$$\begin{aligned} L_1\,\overline{\phi_1} &= -\|h\| + 16\,\|\phi\|\,\big(L(|y|^2) - 2\,\mu\,C\big) \\ &\le -\|h\| \le -|L_1\,\phi_1| \quad \text{in } \mathcal{R} \end{aligned}$$

and

$$\overline{\phi_1} \ge |\phi_1| \quad \text{on the boundary } \partial\mathcal{R}.$$

Thus, by the weak maximum principle (see [6, Theorem 3.1]), we obtain $|\phi_1| \le \overline{\phi_1}$. This implies that

$$\begin{aligned} \frac{1}{2}\,|\phi(z_1, 0, \cdots, 0) - \phi(-z_1, 0, \cdots, 0)| &\le \overline{\phi_1}(0, \cdots, 0, z_1) \\ &\le \frac{1}{2\mu}\,\frac{z_1}{4}\,\|h\| + 16\,\|\phi\|\Big(\frac{Cz_1}{4} + z_1^2\Big). \end{aligned}$$

Dividing through by z_1 and letting $z_1 \to 0$ yields the desired estimate

$$\Big|\frac{\partial\phi}{\partial y_1}(0)\Big| \le \frac{1}{8\mu}\|h\| + 4C\,\|\phi\|. \tag{2.4}$$

Therefore, we complete the proof of (2.1).

We remark that (2.1) can also be proved directly by the above maximum principle argument without doing the re-scaling. Here we prefer to do the rescaling before proceeding to the maximum principle argument because of the following two reasons:

(1) Re-scaling makes the dependence relation of the desired estimate on the eigenvalue λ^2 clear and reduce the question to the case of a fixed scale.

(2) It is convenient for the reader to compare the maximum principle argument here with the proof via the elliptic a priori estimate in [10–11].

2.2 Inside the boundary layer

Using the notations in subsection 3.1, we are going to prove the following estimate:

$$|\nabla u(p_0)| \le C(\lambda\|u\|_\infty + \lambda^{-1}\|v\|_\infty) \quad \text{for all } p_0 \in L_{\frac{1}{\lambda}}, \tag{2.5}$$

which combining with (2.1) completes the proof of Theorem 1.1.

Since the boundary ∂M is a compact sub-manifold in M of codimension 1, we can take λ sufficiently large such that there exists the boundary normal coordinate chart on the boundary layer $L_{\frac{3}{\lambda}} = \{p \in M : d(p, \partial M) \le \frac{3}{\lambda}\}$ with respect to the boundary ∂M (see [8, p. 51]). In particular, we have the map

$$\mathcal{B} : \partial M \times \Big[0, \frac{3}{\lambda}\Big] \to L_{\frac{3}{\lambda}}, \quad (p', \delta) \mapsto \mathcal{B}(p', \delta)$$

such that $\delta \mapsto \mathcal{B}(p', \delta)$, $\delta \in [0, \frac{3}{\lambda}]$, is the geodesic with an arc-length parameter normal to ∂N at p'. Moreover, for each point $(p', \delta) \in L_{\frac{3}{\lambda}}$, we have $0 \leq \delta \leq \frac{3}{\lambda}$ and

$$d((p', \delta), \partial M) = \delta.$$

Denote by $\mathcal{R}(r)$ the following n-dimensional rectangle in $\mathbf{R}^n$ sitting at the origin and having size r,

$$\mathcal{R}(r) = \{x = (x', x_n) = ((x_1, \cdots, x_{n-1}), x_n) \in \mathbf{R}^n : |(x', 0)| < r, 0 \leq x_n \leq r\}.$$

For a point q on ∂M, denote by Exp_q at q the exponential map on the sub-manifold ∂M. Since ∂M is compact and λ is sufficiently large, we may assume the existence of the geodesic normal chart for each metric ball of radius $\frac{3}{\lambda}$ on ∂M.

We choose and fix a point p_0 in $L_{\frac{1}{\lambda}}$, and write $p_0 = \mathcal{B}(q_0, \delta_0)$, where $q_0 \in \partial M$ and $\delta_0 \in [0, \frac{1}{\lambda}]$. We denote by $R(q_0, \frac{3}{\lambda})$ the rectangle in M sitting at q_0 and having size $\frac{3}{\lambda}$,

$$R\Big(q_0, \frac{3}{\lambda}\Big) = \Big\{(\mathrm{Exp}_{q_0}(x'), x_n)) : (x', x_n) \in \mathcal{R}\Big(\frac{3}{\lambda}\Big)\Big\}.$$

In this way, we identify the rectangle $R(q_0, \frac{3}{\lambda})$ in M sitting at q_0 and containing p_0 with the rectangle $\mathcal{R}(\frac{3}{\lambda})$ in $\mathbf{R}^n$. Thus we could look at u and v as functions in $\mathcal{R}(\frac{3}{\lambda})$.

We recall that u_λ and v_λ are the corresponding rescaled functions of u and v, respectively, i.e.,

$$u_\lambda(y) = u\Big(\frac{y}{\lambda}\Big), \quad v_\lambda(y) = v\Big(\frac{y}{\lambda}\Big) \quad \text{for all } y \text{ in } \mathcal{R}(3).$$

To prove (2.5), we only need to show the following estimate:

$$|(\nabla u)(p_0)| = |(\nabla u)(0, \delta_0)| \leq C\Big(\lambda \|u\|_{L^\infty\left(\mathcal{R}(\frac{3}{\lambda})\right)} + \lambda^{-1}\|v\|_{L^\infty\left(\mathcal{R}(\frac{3}{\lambda})\right)}\Big), \tag{2.6}$$

which can be reduced to the equivalent rescaled version

$$|(\nabla u_\lambda)(0, \lambda\,\delta_0)| \leq C\Big(\|u_\lambda\|_{L^\infty\left(\mathcal{R}(3)\right)} + \lambda^{-2}\|v_\lambda\|_{L^\infty\left(\mathcal{R}(3)\right)}\Big), \quad 0 \leq \lambda\,\delta_0 \leq 1, \tag{2.7}$$

where u_λ and v_λ are the rescaling functions of u and v, respectively. Observe that u_λ is the solution of the Poisson equation

$$\sum_{i,j} \partial_{y_i}\,(g_\lambda^{ij}\sqrt{g_\lambda}\;\partial_{y_j} u_\lambda) = -\lambda^{-2}\;\sqrt{g_\lambda}\;v_\lambda \quad \text{ in the interior of rectangle } \mathcal{R}(3) \tag{2.8}$$

and satisfies the Neumann boundary condition, i.e.,

$$\frac{\partial u_\lambda}{\partial y_n} = 0 \quad \text{ on the portion } \{x \in \mathcal{R}(3) : y_n = 0\} \text{ of the boundary } \partial\mathcal{R}(3).$$

We shall give two different proofs for (2.7).

The 1st proof The idea is to reduce, by Fact 1 in the introduction and the even extension, the question to the interior gradient estimate (2.2), which has been proved by the maximum principle in the former subsection. By the geometric property of the geodesic normal coordinate chart with respect to the boundary ∂M, we have

$$g^{nn}(x', x_n) = 1 \quad \text{and} \quad g^{jn}(x', x_n) = 0 \quad \text{for } j \neq n \text{ in } R\Big(q_0, \frac{3}{\lambda}\Big),$$

which implies that

$$g_\lambda^{nn}(y', y_n) = 1 \quad \text{and} \quad g_\lambda^{jn}(y', y_n) = 0 \quad \text{for } j \neq n \text{ in } \mathcal{R}(3).$$

Setting

$$a_{ij} := g_\lambda^{ij} \sqrt{g_\lambda} \quad \text{for } 1 \leq i, j \leq n-1, \qquad a_{nn} := \sqrt{g_\lambda}$$

and

$$b_i := \sum_{j=1}^{n-1} \frac{\partial a_{ij}}{\partial y_j} \quad \text{for } 1 \leq i \leq n-1, \quad b_n = \frac{\sqrt{g_\lambda}}{\partial y_n},$$

we can express the Poisson equation (2.8) as

$$\sum_{1 \leq i,j \leq n-1} a_{ij} \frac{\partial^2 \phi}{\partial y_i \partial y_j} + a_{nn} \frac{\partial^2 \phi}{\partial y_n^2} + \sum_{1 \leq k \leq n} b_k \frac{\partial \phi}{\partial y_k} = h \quad \text{in } \mathrm{Int}\big(\mathcal{R}(3)\big), \tag{2.9}$$

where $\phi = u_\lambda$ and $h = -\lambda^{-2} \sqrt{g_\lambda}\, v_\lambda$.

Set

$$\mathcal{S}(r) := \{y = (y', y_n) = ((y_1, \cdots, y_{n-1}), y_n) \in \mathbf{R}^n : |(y', 0)| < r, |y_n| \leq r\},$$

which is the union of rectangle $\mathcal{R}(r)$ and its reflection with respect to the hyperplane $\{y_n = 0\}$. We denote by $\widetilde{\phi}$ the even extension onto $\mathcal{S}(3)$ of the function ϕ defined on $\mathcal{R}(3)$, i.e.,

$$\widetilde{\phi}(y', y_n) = \begin{cases} \phi(y', y_n), & \text{if } y_n \geq 0, \\ \phi(y', -y_n), & \text{if } y_n < 0. \end{cases}$$

We do the even extension to h and the coefficients a_{ij}, a_{nn}, b_i for $1 \leq i, j \leq n-1$, and denote the corresponding extension functions on $\mathcal{S}(3)$ by

$$\widetilde{h}, \quad \widetilde{a_{ij}}, \quad \widetilde{a_{nn}}, \quad \widetilde{b_i}.$$

However, we do the odd extension to b_n, i.e.,

$$\widetilde{b_n}(y', y_n) = \begin{cases} b_n(y', y_n), & \text{if } y_n \geq 0, \\ -b_n(y', -y_n), & \text{if } y_n < 0. \end{cases}$$

We shall see soon that the possible discontinuity of $\widetilde{b_n}$ on the portion $\mathcal{S}(3) \cap \{y_n = 0\}$ would not cause any trouble.

Thus, we obtain the following Poisson equation about $\widetilde{\phi}$ with continuous coefficients:

$$\sum_{1 \leq i,j \leq n-1} \widetilde{a_{ij}} \frac{\partial^2 \widetilde{\phi}}{\partial y_i \partial y_j} + \widetilde{a_{nn}} \frac{\partial^2 \widetilde{\phi}}{\partial y_n^2} + \sum_{1 \leq k \leq n} \widetilde{b_k} \frac{\partial \widetilde{\phi}}{\partial y_k} = \widetilde{h} \quad \text{in } \mathrm{Int}\big(\mathcal{S}(3)\big)$$

except that $\widetilde{b_n}$ is bounded and possibly discontinuous on the portion $\mathcal{S}(3) \cap \{y_n = 0\}$. By Fact 1 in the introduction, which can be proved by simple calculus computation, we know that $\widetilde{\phi}$ is C^2 in $\mathcal{S}(3)$. The only point which we should take care of is whether $\widetilde{b_n} \frac{\partial \widetilde{\phi}}{\partial y_n}$ is an even continuous function in $\mathcal{S}(3)$ with respect to y_n. However, by the extension $\widetilde{\phi}$ of ϕ and the Neumann boundary condition, i.e., $\frac{\partial \phi}{\partial y_n} = 0$ on $\{y_n = 0\} \cap \mathcal{S}(3)$, $\frac{\partial \widetilde{\phi}}{\partial y_n}$ is an odd C^1 function vanishing on the portion $\mathcal{S}(3) \cap \{y_n = 0\}$. Since $\widetilde{b_n}$ is a bounded function in $\mathcal{S}(3)$ and is odd with respect to y_n, $\widetilde{b_n}\,\frac{\partial\widetilde{\phi}}{\partial y_n}$ is a continuous function being even with respect to y_n in $\mathcal{S}(3)$. Moreover, $\widetilde{b_n}\,\frac{\partial\widetilde{\phi}}{\partial y_n}$ vanishes on the portion $\{y_n = 0\} \cap \mathcal{S}(3)$. Therefore, we have reduced the proof of (2.7) to the estimate for $\widetilde{\phi}$ at the interior point $(0, \delta_0\,\lambda)$ of $\mathcal{S}(3)$ similar to (2.4) in the former subsection. The fact that the bounded coefficient b_n is possibly discontinuous on the portion $\{y_n = 0\}\cap\mathcal{S}(3)$ does not bring us any trouble in applying the weak maximum principle (see [6, (3.3), p. 31] and the related comments).

The 2nd proof The idea is to use the same even extension as above and the interior $C^{1,\alpha}$ estimate in [6, Theorem 8.32, p. 210]. Denote by $\widetilde{g}$ the even extension of the Riemannian metric g on $\mathcal{R}\big(\frac{3}{\lambda}\big)$ onto $\mathcal{S}\big(\frac{3}{\lambda}\big)$. Then $\widetilde{g}$ is a Lipschitz metric on $\mathcal{S}\big(\frac{3}{\lambda}\big)$ with the $C^{0,1}$ norm bounded by the C^1 norm of g. Denote the even extension of u and v on $\mathcal{S}\big(\frac{3}{\lambda}\big)$ by $\widetilde{u}$ and $\widetilde{v}$, respectively. We claim that $\widetilde{u}$ is a weak solution of the following Poisson equation:

$$-\frac{1}{\sqrt{\widetilde{g}}}\sum_{i,j}\partial_{x_i}(\widetilde{g}^{ij}\sqrt{\widetilde{g}}\,\partial_{x_j}\widetilde{u}) = \widetilde{v} \quad \text{in } \mathrm{Int}\Big(\mathcal{S}\Big(\frac{3}{\lambda}\Big)\Big),$$

that is, for each smooth function ψ compactly supported in $\mathrm{Int}\big(\mathcal{S}\big(\frac{3}{\lambda}\big)\big)$, the following integral equality holds:

$$\int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))}\sum_{i,j}\widetilde{g}^{ij}\;\partial_{x_i}\widetilde{u}\;\partial_{x_j}\psi\;\mathrm{d}x = \int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))}(-\widetilde{v})\;\psi\;\mathrm{d}x. \tag{2.10}$$

Actually, since $\frac{\partial\widetilde{u}}{\partial x_n} = 0$ on $\mathcal{S}\big(\frac{3}{\lambda}\big)\cap\{x_n = 0\}$, we find by the Green formula on Riemannian manifolds and the even extension of u and v,

$$\begin{aligned}
&\int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n>0\}}\sum_{i,j}\widetilde{g}^{ij}\;\partial_{x_i}\widetilde{u}\;\partial_{x_j}\psi\;\mathrm{d}x\\
&= \int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n>0\}}\Delta\,\widetilde{u}\;\psi\;\mathrm{d}x\\
&= \int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n>0\}}\Delta\,u\;\psi\;\mathrm{d}x\\
&= \int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n>0\}}(-v)\;\psi\;\mathrm{d}x\\
&= \int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n>0\}}(-\widetilde{v})\;\psi\;\mathrm{d}x.
\end{aligned}$$

Using the change of variable $x_n \mapsto -x_n$ and the above equality, we obtain

$$\int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n<0\}}\sum_{i,j}\widetilde{g}^{ij}\;\partial_{x_i}\widetilde{u}\;\partial_{x_j}\psi\;\mathrm{d}x = \int_{\mathrm{Int}(\mathcal{S}(\frac{3}{\lambda}))\cap\{x_n<0\}}(-\widetilde{v})\;\psi\;\mathrm{d}x,$$

where we also use $g^{in} = 0$ for all $i \neq n$. Summing these two equalities yields (2.10). Recall that the coefficient $\widetilde{g}^{ij}$ is Lipschitz and $\widetilde{v}$ is continuous on $\mathcal{S}\big(\frac{3}{\lambda}\big)$. On the other hand, we have the rescaled version of equation (2.10), i.e., for each smooth function ψ compactly supported in $\mathrm{Int}\big(\mathcal{S}(3)\big)$,

$$\int_{\mathrm{Int}(\mathcal{S}(3))}\sum_{i,j}\widetilde{g}_{\lambda}^{ij}\;\partial_{y_i}\widetilde{u}_{\lambda}\;\partial_{y_j}\psi\;\mathrm{d}x = \int_{\mathrm{Int}(\mathcal{S}(3))}(-\widetilde{v}_{\lambda})\;\psi\;\mathrm{d}x.$$

Thus, applying to it the interior $C^{1,\alpha}$ estimate in [6, Theorem 8.32, p. 210], we obtain that for every $0 < \alpha < 1$,

$$\|\widetilde{u}_\lambda\|_{C^{1,\alpha}(\mathcal{S}(2))} \leq C(\|\widetilde{u}_\lambda\|_{C^0(\mathcal{S}(3))} + \lambda^{-2}\ \|\widetilde{v}_\lambda\|_{C^0(\mathcal{S}(3))}),$$

which implies the desired estimate (2.7).

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