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OD-Characterization of Alternating and Symmetric Groups of Degree $p + 5$*

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Abstract In this paper, it is proved that all the alternating groups A_{p+5} are OD-characterizable and the symmetric groups S_{p+5} are 3-fold OD-characterizable, where $p+4$ is a composite number and $p+6$ is a prime and $5 \neq p \in \pi(1000!)$.

Keywords Prime graph, Degree pattern, Degree of a vertex, Finite simple groups, Alternating and symmetric groups
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1 Introduction

Throughout this paper, groups under consideration are finite and simple groups that are non-Abelian. For any group G, we use $\pi_e(G)$ to denote the set of orders of its elements and denote by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a graph of G, denoted by $\Gamma(G)$ (see [1]). The vertex set of this graph is $\pi(G)$, and two distinct vertices p, q are adjacent by an edge if and only if $pq \in \pi_e(G)$. In this case, we write $p \sim q$. We also denote by $\pi(n)$ the set of all primes dividing n, where n is a positive integer.

In this article, we use the following symbols. For a finite group G, the socle of G is defined as the subgroup generated by minimal normal subgroups of G, denoted as $\mathrm{Soc}(G)$. $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, and P_r denotes a Sylow r-subgroup of G for $r \in \pi(G)$. Furthermore, the symmetric and alternating groups of degree n are denoted by S_n and A_n, respectively. Let q be a prime, and we use $\mathrm{Exp}(m, q)$ to denote the exponent of the largest power of a prime q in the factorization of a positive integer m (> 1). All further unexplained symbols and notations are standard and can be found in [2], for instance.

Definition 1.1 (see [3]) *Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_is are primes and α_is are integers. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) \mid p \sim q\}|$, which we call the degree of p. We also define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the degree pattern of G.*

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Definition 1.2 (see [3]) *A group M is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups G such that* (1) $|G| = |M|$ *and* (2) $D(G) = D(M)$. *Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.*

It is an interesting and difficult topic to determine the structure of finite groups by their orders and degree patterns. This topic is related to the following open problem.

Open problem (see [3]) Let G and M be finite groups satisfying the conditions (1) $|G| = |M|$ and (2) $D(G) = D(M)$. Then

(i) How far do these conditions affect the structure of G?

(ii) Does the number of non-isomorphic groups satisfy (1) and (2) finite?

At present, we mention that the problem is still unsolved completely and till now we may not be able to provide a suitable answer for the above questions. This topic was studied in several articles. For example, in a series of articles (see [3–20]), it was shown that many finite almost-simple groups are m-fold OD-characterizable, where m is a positive integer and $m \geq 1$. For convenience, we summarize some results of these articles in the following Propositions 1.1–1.4.

Proposition 1.1 (see [3–12]) *Let p be a prime. A finite group G is OD-characterizable if G is isomorphic to one of the following groups:*

(1) *The alternating groups A_p, A_{p+1} and A_{p+2};*

(2) *the alternating groups A_{p+3}, where $7 \neq p \in \pi(100!)$;*

(3) *all finite almost simple K_3-groups except* $\mathrm{Aut}(A_6)$ *and* $\mathrm{Aut}(U_4(2))$;

(4) *the symmetric groups S_p and S_{p+1};*

(5) *all finite simple K_4-groups except A_{10};*

(6) *all finite simple $C_{2,2}$-groups;*

(7) *the simple groups of Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ for certain prime power q;*

(8) *all sporadic simple groups and their automorphism groups except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(M^cL)$;

(9) *the almost simple groups:* $\mathrm{Aut}(F_4(2))$, $\mathrm{Aut}(O_{10}^+(2))$ *and* $\mathrm{Aut}(O_{10}^-(2))$;

(10) *almost simple group $L_2(49) \cdot 2^i$ $(i = 1, 2, 3)$;*

(11) *projective general linear group $PGL(2, q)$ for certain odd prime power q;*

(12) *all finite simple groups whose orders are less than 10^8 except for A_{10} and $U_4(2)$.*

Proposition 1.2 (see [13–16]) *A finite group G is 2-fold OD-characterizable if G is one of the following groups: A_{10}, $U_4(2)$, $S_6(3)$, $O_7(3)$, $B_2(q)$, $C_p(3)$, almost simple groups $2 \cdot F_4(2)$,* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(M^cL)$.

Proposition 1.3 (see [17–20]) *A finite group G is 3-fold OD-characterizable if G is one of the following groups:*

(1) *The almost simple groups of $U_3(5) \cdot 3$ and $U_6(2) \cdot 3$;*

(2) *the symmetric groups S_n, where $n \leq 100$ and $n \neq 10$, p, $p + 1$.*

Proposition 1.4 (see [13]) *Let G be a finite group with $|G| = |S_{10}|$ and $D(G) = D(S_{10})$, then G is 8-fold OD-characterizable.*

2 Main Results

According to Proposition 1.1(1), the alternating groups A_p, A_{p+1} and A_{p+2} are OD-characterizable, and A_{p+3} with $7 \neq p \in \pi(100!)$ is OD-characterizable. Proposition 1.2 says that the alternating group A_{10} is 2-fold OD-characterizable. Now, omitting all the above alternating groups except A_{10}, there remain the following alternating groups:

$$A_{10}, A_{106}, A_{112}, A_{116}, \cdots, A_{126}, A_{134}, A_{135}, A_{136}, A_{142}, \cdots . \tag{2.1}$$

By [1], it is easy to check that all the groups in (2.1) have the connected prime graph. By Proposition 1.1, we see that it is difficult to investigate how many-fold OD-characterization of alternating groups are. In this paper, we continue to investigate this topic and get the following theorem.

Theorem 2.1 *All alternating groups A_{p+5}, where $p+4$ is a composite number and $p+6$ is a prime and $5 \neq p \in \pi(1000!)$, are OD-characterizable.*

Proposition 1.2 says that A_{10} is 2-fold OD-characterizable. It is worth mentioning that A_{10} is the first alternating group which has not been considered as OD-characterizable. Up to now, we have not found an alternating group A_n $(n \neq 10)$ which is not OD-characterizable. Hence, we put forward the following question.

Question 2.1 Are the alternating groups A_n $(n \neq 10)$ OD-characterizable?

In fact, Theorem 2.1 and Proposition 1.1(1)–(3) imply the following corollary.

Corollary 2.1 *Let A_n be an alternating group of degree n. Assume that one of the following conditions is fulfilled:*

(1) *$n = p$, $p+1$ or $p+2$, where p is a prime;*

(2) *$n = p+3$, where $7 \neq p \in \pi(100!)$;*

(3) *$n = p+5$, where $p+4$ is a composite number and $p+6$ is a prime and $5 \neq p \in \pi(1000!)$. Then A_n is OD-characterizable.*

In this article, we will also show the following theorem.

Theorem 2.2 *All symmetric groups S_{p+5}, where $p+4$ is a composite number and $p+6$ is a prime and $5 \neq p \in \pi(1000!)$, are 3-fold OD-characterizable.*

3 Preliminaries

In this section, we give some results which will be applied for our further investigations. We shall utilize the following Lemma 3.1 concerning the set of elements of the alternating and symmetric groups (see [21]).

Lemma 3.1 (see [21]) *The group S_n (or A_n) has an element of order $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_1, p_2, \cdots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \cdots \alpha_s$ are natural numbers, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ (or $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ for m odd, and $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n-2$ for m even).*

As an immediate corollary of Lemma 3.1, we have the following lemma.

Lemma 3.2 *Let A_n (or S_n) be an alternating group (or a symmetric group) of degree n. Then the following assertions hold:*

(1) *Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p+q \leq n$.*

(2) *Let $p \in \pi(A_n)$ be an odd prime. Then $2 \sim p$ if and only if $p+4 \leq n$.*

(3) *Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p+q \leq n$.*

Lemma 3.3 (see [22]) *Let G be a finite solvable group, all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 3.4 *Let A_{p+5} be an alternating group of degree $p+5$, where p is a prime, and $p+2$ and $p+4$ are composite numbers. Suppose that $|\pi(A_{p+5})| = d$. Then the following assertions hold:*

(1) $\deg(2) = d-1$. *Particularly,* $2 \sim r$ *for each* $r \in \pi(A_{p+5})$.

(2) $\deg(3) = \deg(5) = d-1$, *i.e.,* $3 \sim r$ *and* $5 \sim r$ *for each* $r \in \pi(A_{p+5})$.

(3) $\deg(p) = 3$. *In other words,* $p \sim r$, *where* $r \in \pi(A_{p+5})$, *if and only if* $r = 2$, $r = 3$ *or* $r = 5$.

(4) $\mathrm{Exp}(|A_{p+5}|, 2) = \sum\limits_{i=1}^{+\infty} [\frac{p+5}{2^i}] - 1$. *In particular,* $\mathrm{Exp}(|A_{p+5}|, 2) < p+5$.

(5) $\mathrm{Exp}(|A_{p+5}|, r) = \sum\limits_{i=1}^{+\infty} [\frac{p+5}{r^i}]$ *for each* $r \in \pi(A_{p+5}) \backslash \{2\}$. *Furthermore,* $\mathrm{Exp}(|A_{p+5}|, r) < \frac{p-1}{3}$, *where* $7 \leq r \in \pi(A_{p+5})$. *Particularly, if* $r > [\frac{p+5}{2}]$, *then* $\mathrm{Exp}(|A_{p+5}|, r) = 1$.

Proof By Lemma 3.2, one has that $2p \in \pi_e(A_{p+5})$. Clearly, since $r+4 \leq p+5$ for any $r \in \pi(A_{p+5}) \backslash \{p\}$, it follows that $2 \sim r$, so $\deg(2) = d-1$. Similarly, we have $\deg(3) = \deg(5) = d-1$. For $r \in \pi(A_{p+5}) \backslash \{2, p\}$, by Lemma 3.2, it is easy to check that $p \sim r$ if and only if $r \leq 5$. Hence $r = 2$, 3 or 5. Thus $\deg(p) = 3$. Till now we have proved that (1)–(3) hold.

By the definition of Gauss's integer function, we have that

$$\begin{aligned}\mathrm{Exp}(|A_{p+5}|, 2) &= \sum_{i=1}^{+\infty} \left[\frac{p+5}{2^i}\right] - 1 = \left(\left[\frac{p+5}{2}\right] + \left[\frac{p+5}{2^2}\right] + \left[\frac{p+5}{2^3}\right] + \cdots\right) - 1 \\ &\leq \left(\frac{p+5}{2} + \frac{p+5}{2^2} + \frac{p+5}{2^3} + \cdots\right) - 1 \\ &= (p+5)\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) - 1 = p+4.\end{aligned}$$

Hence $\mathrm{Exp}(|A_{p+5}|, 2) < p+5$. So (4) follows.

Again, using Gauss's integer function, we have that $\mathrm{Exp}(|A_{p+5}|, r) = \sum\limits_{i=1}^{+\infty} [\frac{p+5}{r^i}] - 1 = ([\frac{p+5}{r}] + [\frac{p+5}{r^2}] + [\frac{p+5}{r^3}] + \cdots) - 1 \leq (\frac{p+5}{r} + \frac{p+5}{r^2} + \frac{p+5}{r^3} + \cdots) - 1 = (p+5)(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots) - 1 = \frac{p+5}{r-1} \leq \frac{p+5}{6} < \frac{2p-2}{6} = \frac{p-1}{3}$. Therefore $\mathrm{Exp}(|A_{p+5}|, r) < \frac{p-1}{3}$, where $7 \leq r \in \pi(A_{p+5})$. Obviously, if $r > [\frac{p+5}{2}]$, then $\mathrm{Exp}(|A_{p+5}|, r) = 1$. Hence (5) follows. This completes the proof of Lemma 3.4.

Lemma 3.5 (see [23]) *Let a be an arbitrary integer and m a positive integer. If $(a, m) = 1$, then the equation $a^x \equiv 1 \pmod{m}$ has solutions. Moreover, if the order of a modulo m is $h(a)$, then $h(a) \mid \varphi(m)$, where $\varphi(m)$ is an Euler's function of m.*

Lemma 3.6 *Let A_{p+5} be an alternating group of degree $p+5$, where $p+4$ is a composite number, $p+6$ is a prime and $100 < p \in \pi(1000!)$. Set $P \in \mathrm{Syl}_p(A_{p+5})$ and $Q \in \mathrm{Syl}_q(A_{p+5})$, where $7 \leq q < p$. Then the following assertions hold:*

(i) $q^{s(q)} \nmid |N_G(P)|$, *where* $s(q) = \mathrm{Exp}(|A_{p+5}|, q)$.

(ii) *If* $p \in \{131, 167, 173, 233, 251, 257, 263, 373, 383, 433, 503, 541, 557, 563, 587, 607, 647, 677, 727, 733, 941, 977\}$, *then* $p \nmid |N_G(Q)|$.

(iii) *If* $p \in \{157, 271, 331, 353, 367, 443, 571, 593, 601, 653, 751, 947, 971\}$, *then there exists at least a prime number, say* r, *such that the order of* r *modulo* p *is less than* $p-1$, *where* $7 \leq r < p$ *and* $r \in \pi(A_{p+5})$.

Proof Obviously, the equation $q^x \equiv 1 \pmod p$ has solutions by Lemma 3.5. Suppose that the order of q modulo p is $h(q)$. If $h(q) = p-1$, then we call q a primitive root of modulo p. By hypothesis, it is easy to check that there are only 35 such groups satisfying the conditions that $p+4$ is a composite number, $p+6$ is a prime number and $100 < p \in \pi(1000!)$. Using Magma, for each $q \in \pi(A_{p+5})$ and $7 \le q < p$, we can obtain $h(q)$. For convenience, we have tabulated p and $h(q)$ in Table 1 of this article.

Table 1 (p and $h(q)$)

p	$h(q)$	Condition	p	$h(q)$	Condition	p	$h(q)$	Condition
131	$2\cdot5\cdot13$	none	157	$2^2\cdot3\cdot13$	$q\neq13$	157	6	$q=13$
167	$2\cdot83$	none	173	$2^2\cdot43$	none	233	$2^3\cdot29$	none
251	$2\cdot5^3$	none	257	2^8	none	263	$2\cdot131$	none
271	$2\cdot3^3\cdot5$	$q\neq29$	271	6	$q=29$	331	$2\cdot3\cdot5\cdot11$	$q\neq31$
331	3	$q=31$	353	$2^5\cdot11$	$q\neq7$	353	32	$q=7$
367	$2\cdot3\cdot61$	$q\neq83$	367	3	$q=83$	373	$2^2\cdot3\cdot31$	none
383	$2\cdot191$	none	433	$2^4\cdot3^3$	none	443	$2\cdot13\cdot17$	$q\neq13$
443	17	$q=13$	503	$2\cdot251$	none	541	$2^2\cdot3^3\cdot5$	none
557	$2^2\cdot139$	none	563	$2\cdot281$	none	571	$2\cdot3\cdot5\cdot19$	$q\neq109$
571	3	$q=109$	587	$2\cdot293$	none	593	$2^4\cdot37$	$q\neq59$
593	8	$q=59$	601	$2^3\cdot3\cdot5^2$	$q\neq13,59$	601	$2^2\cdot5$	$q=13$
601	8	$q=59$	607	$2\cdot3\cdot101$	none	647	$2\cdot17\cdot19$	none
653	$2^2\cdot163$	$q\neq149$	653	4	$q=149$	677	$2^2\cdot13^2$	none
727	$2\cdot3\cdot11^2$	none	733	$2^2\cdot3\cdot61$	none	751	$2\cdot3\cdot5^3$	$q\neq73$
751	$2\cdot3$	$q=73$	941	$2^2\cdot5\cdot47$	none	947	$2\cdot11\cdot43$	$q\neq7,17$
947	$2\cdot43$	$q=7$	947	$2\cdot11$	$q=17$	971	$2\cdot5\cdot97$	$q\neq7$
971	97	$q=7$	977	$2^4\cdot61$	none			

Now, using the N-C theorem, the factor group $N_G(P)/C_G(P) \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$. Hence, $|N_G(P)/C_G(P)| \mid (p-1)$. By Lemma 3.4(3), one has that $\pi(N_G(P)) \subseteq \{2,3,5\} \cup \pi(p-1)$. By Table 1, if there exists a prime, say q, where $7 \le q < p$ and $q \in \pi(A_{p+5})$, such that $q^{s(q)} \mid |N_G(P)|$, then $q \mid |C_G(P)|$. Thus $\deg(p) \ge 4$, a contradiction to Lemma 3.4(3), and (i) is proved.

Assume that $p \in \{131, 167, 173, 233, 251, 257, 263, 373, 383, 433, 503, 541, 557, 563, 587, 607, 647, 677, 727, 733, 941, 977\}$. If $p \mid |N_G(Q)|$, by Table 1 and $\mathrm{Exp}(|A_{p+5}|, q) < \frac{p-1}{3} < p-1$, $p \mid |C_G(Q)|$, a contradiction. Thus (ii) follows. The remaining parts of (iii) follow at once from Table 1. This completes the proof of Lemma 3.6.

Lemma 3.7 (see [2, 24]) *Let M be a finite non-Abelian simple group with order having prime divisors at most* 997. *Then M is isomorphic to one of the simple groups listed in Tables* 1–3 *in* [24]. *Particularly, if $|\pi(\mathrm{Out}(M))| \neq 1$, then $\pi(\mathrm{Out}(M)) \subseteq \{2,3,5,7\}$.*

Lemma 3.8 (see [25]) *Let $S = P_1 \times P_2 \times \cdots \times P_r$, where P_i $(i = 1, 2, \cdots, r)$ are isomorphic non-Abelian simple groups. Then* $\mathrm{Aut}(S) = (\mathrm{Aut}(P_1) \times \mathrm{Aut}(P_2) \times \cdots \times \mathrm{Aut}(P_r)) \rtimes S_r$, *where S_r is a symmetric group of degree r.*

4 *OD*-Characterization of the Alternating Group A_{p+5}

In this section, we will prove the following Theorem 2.1. It is worth mentioning that this result not only generalizes the results in [4] but also gives an affirmative answer to the Question 1.1 of this article for the alternating group A_{p+5}.

Proof of Theorem 2.1 Let G be a finite group satisfying the conditions that (1) $|G| = |A_{p+5}|$ and (2) $D(G) = D(A_{p+5})$, where $p+4$ is a composite number, $p+6$ is a prime and

$5 \neq p \in \pi(1000!)$. By [17], we only need to discuss the alternating groups A_{p+5}, where $p+4$ is a composite number, $p+6$ is a prime and $100 < p \in \pi(1000!)$. By these hypotheses, we obtain that $\{2, r, 2r\} \cup \{rs \mid r+s \leq p+5\} \subseteq \pi_e(G)$ and $\{rs \mid r+s > p+5\} \cap \pi_e(G) = \emptyset$, where $2 \neq r, s \in \pi(G)$. By Lemma 3.4, the prime graph of G is connected since $\deg(2) = |\pi(G)| - 1$. Moreover, by the structure of the degree pattern $D(G)$, it is easy to check by Magma of computation group software that $\Gamma(G) = \Gamma(A_{p+5})$. In the following, we shall prove that $G \cong A_{p+5}$. For convenience, we divide the proof of Theorem 2.1 into three separate cases.

Case 1 Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is a nonsolvable group.

We first show that K is a p'-group. Assume to the contrary, and let p divide the order of K. Set $P \in \mathrm{Syl}_p(G)$. By Lemma 3.6 (i), we have $q^{s(q)} \nmid |N_G(P)|$ for any $q \in \pi(G)$ and $7 \leq q < p$. If $q \mid |N_G(P)|$, then either $q \mid |C_G(P)|$ or $q \in \pi(K)$. For the former, by Lemma 3.4(3), this leads to an obvious contradiction since $q \sim p$. In the latter case, i.e., if $q \in \pi(K)$, it is easy to check by Table 1 that there necessarily exists such a prime r such that $r \not\sim q$, where $7 \leq r < p$ and $r \in \pi(K)$. In fact, by Lemma 3.2(1), it is sufficient to find such a prime r such that $r + q > p$, so then $r \not\sim q$. Since K is solvable, it possesses a Hall $\{p, q, r\}$-subgroup T. It follows that T is solvable. Since there exists no edge between any two distinct vertices of p, q and r in $\Gamma(G)$, all elements in T are of prime power order. Hence $|\pi(T)| \leq 2$ by Lemma 3.3, a contradiction. Hence K is a p'-group.

We shall argue next that K is a q'-group for any $q \in \pi(G) \backslash \{2, 3, 5, p\}$. Set $Q \in \mathrm{Syl}_q(K)$, where $q \in \pi(K)$. Suppose that the order of q modulo p is $h(q)$. By the Frattini argument, $G = KN_G(Q)$, and hence p divides the order of $N_G(Q)$. By Lemma 3.6(ii)–(iii), it follows that p can only be equal to one of the primes: 157, 271, 331, 353, 367, 443, 571, 593, 601, 653, 751, 947 and 971. In this case, there necessarily exists at least a prime, say q, such that $h(q) < p - 1$. We prove the lemma up to the choice of p one by one.

Subcase 1.1 To prove the case follows if $p = 157$.

By Table 1, If there exists a prime q such that $p \mid |N_G(Q)|$, where $Q \in \mathrm{Syl}_q(G)$, then $q = 13$. By the N-C theorem, $N_G(Q)/C_G(Q) \lesssim \mathrm{Aut}(Q)$. By Lemma 3.4(5), we have $\mathrm{Exp}(|G|, 13) = 12$, and thus $|N_G(Q)/C_G(Q)| \mid \prod_{i=1}^{12} 13^{66} \cdot (13^i - 1)$. By Magma, it is easy to check that $151 \nmid \prod_{i=1}^{12} 13^{66} \cdot (13^i - 1)$. If $151 \mid |N_G(Q)|$, then $151 \in \pi(C_G(Q))$. Thus $13 \sim 151$, a contradiction. Hence $151 \in \pi(K)$. Since K is solvable, it possesses a Hall $\{13, 151\}$-subgroup H of order $13^{12} \cdot 151$. Clearly, H is nilpotent, so $13 \sim 151$, a contradiction.

Subcase 1.2 To prove the case follows if $p = 271$.

In this case, we know that there exists a prime, say q, such that $p \mid |N_G(Q)|$, where $Q \in \mathrm{Syl}_q(G)$. Then $q = 29$ by Table 1. On the other hand, the factor group $N_G(Q)/C_G(Q)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q)$ by N-C theorem and $\mathrm{Exp}(|G|, 29) = 9$ by Lemma 3.4, so $|N_G(Q)/C_G(Q)| \mid \prod_{i=1}^{9} 29^{36} \cdot (29^i - 1)$. It is easy to check that $269 \nmid \prod_{i=1}^{9} 29^{36} \cdot (29^i - 1)$. If $269 \mid |N_G(Q)|$, then $269 \in \pi(C_G(Q))$. Thus $269 \sim 29$, a contradiction. Hence $269 \in \pi(K)$. Since K is solvable, it possesses a Hall $\{29, 269\}$-subgroup H of order $29^9 \cdot 269$. Clearly, H is Abelian, so $29 \sim 269$, a contradiction.

Subcase 1.3 To prove the case follows if $p = 331$.

In the case, there exists a prime, say q, such that $p \mid |N_G(Q)|$, where $Q \in \mathrm{Syl}_q(G)$. Then $q = 31$ by Table 1. On the other hand, we have that $N_G(Q)/C_G(Q) \lesssim \mathrm{Aut}(Q)$ by the N-C theorem and $\text{Exp}(|G|, 31) = 10$ by Lemma 3.4, so $|N_G(Q)/C_G(Q)| \mid \prod_{i=1}^{10} 31^{45} \cdot (31^i - 1)$. It is easy to compute that $47 \nmid \prod_{i=1}^{10} 31^{45} \cdot (31^i - 1)$. If $47 \mid |N_G(Q)|$, then $47 \in \pi(C_G(Q))$. Set $N = N_G(Q)$, $C = C_G(Q)$ and $K_{47} \in \text{Syl}_{47}(C_G(Q))$. By Lemma 3.4, we have $\text{Exp}(|G|, 47) = 7$. Again, by the Frattini argument, we have that $N = CN_N(K_{47})$ and hence $p \nmid |N_N(K_{47})|$. Thus $p \mid |C_G(Q)|$, so $\deg(p) \geq 4$, a contradiction. Therefore $47 \nmid |N_G(Q)|$ and $47 \in \pi(K)$. Set $P_{47} \in \text{Syl}_{47}(K)$. Since $G = KN_G(P_{47})$, then $p \mid |N_G(P_{47})|$. It is easy to check by Table 1 that there exists no such a prime p such that $p \mid |N_G(P_{47})|$, a contradiction.

Till now we have proved that K is a q'-group while $p = 157, 271$ or 331. Assume that $p \in \{353, 367, 443, 571, 593, 601, 653, 751, 947, 971\}$. Now, we have to discuss ten cases. If K is a q-group for every $q \in \pi(G)\backslash\{2, 3, 5, p\}$, it is easy to know that this is impossible by checking each choice of p one by one. Since the methods used below are the same as in Subcase 1.3, we omit the detailed processes. Therefore K is a $\{2, 3, 5\}$-group. Since $K \neq G$, it follows at once that G is nonsolvable. This completes the proof of Case 1.

Case 2 The quotient group G/K is an almost simple group. In other words, there exists a non-Abelian simple group S such that $S \lesssim G/K \lesssim \text{Aut}(S)$.

Proof Let $\overline{G} := G/K$ and $S := \text{Soc}(\overline{G})$. Then $S = B_1 \times B_2 \times B_3 \times \cdots \times B_n$, where B_j $(j = 1, 2, \cdots, n)$ are non-Abelian simple groups and $S \lesssim \overline{G} \lesssim \text{Aut}(S)$. We assert that $n = 1$ and $S = B_1$.

Suppose that $n \geq 2$. We first show that p does not divide the order of S. If not, there exists a prime r such that $r \in \pi(S)$ and $r \notin \pi(K)$. Hence $\deg(p) \geq 4$, a contradiction. Therefore, for each j, one has that $B_j \in \mathcal{F}_p$. On the other hand, by Lemma 3.7, we see that $p \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$. Thus p divides the order of $\text{Out}(S)$. But $\text{Out}(S) = \text{Out}(S_1) \times \text{Out}(S_2) \times \cdots \times \text{Out}(S_r)$, where the groups S_j are direct products of all isomorphic $B_k^{'}s$ such that $S = S_1 \times S_2 \times \cdots \times S_r$. Therefore for some i, p divides the order of an outer automorphism group of a direct product S_i of m isomorphic simple groups B_j for some $1 \leq j \leq n$. Since $B_j \in \mathcal{F}_p$, it follows that $|\text{Out}(B_j)|$ is not divided by p by Lemma 3.7. Now, by Lemma 3.8, we obtain $|\text{Aut}(S_i)| = |\text{Aut}(B_j)|^m \cdot m!$. Therefore $m \geq p$ and so 2^{2p} must divide the order of G. However, $\text{Exp}(|A_{p+5}|, 2) < p + 5 < 2p$ by Lemma 3.4 (4), a contradiction. Thus $n = 1$ and $S = B_1$. This completes the proof of Case 2.

Case 3 $G \cong A_{p+5}$. In other words, A_{p+5} is OD-characterizable.

Proof By Lemma 3.7 and Case 1, assume that $|S| = \frac{|G|}{2^{u_1} \cdot 3^{u_2} \cdot 5^{u_3}} \cdot 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3}$, where $2 \leq \alpha_1 \leq |G|_2 = \text{Exp}(|A_{p+5}|, 2) = u_1$, $1 \leq \alpha_2 \leq |G|_3 = \text{Exp}(|A_{p+5}|, 3) = u_2$, $1 \leq \alpha_3 \leq |G|_5 = \text{Exp}(|A_{p+5}|, 5) = u_3$. Let $p_1, p_2, p_3, \cdots, p_s$ be distinct consecutive prime numbers and $2 = p_1 < 3 = p_2 < 5 = p_3 < \cdots < p_s = p$, so then $|G|_{p_j} = \text{Exp}(|A_{p+5}|, p_j)$ for every $j \geq 3$. By Tables 1–3 listed in [24], S can only be isomorphic to one of A_p, A_{p+1}, A_{p+2}, A_{p+3}, A_{p+4} and A_{p+5}.

If $S \cong A_p$, then $A_p \leq \overline{G} \leq S_p$. Hence, $3 \cdot p \in \pi_e(\overline{G})\backslash\pi_e(S_p)$, a contradiction.

For the same reason, $S \ncong A_{p+1}$ or A_{p+2}. Therefore, S is isomorphic to one of the simple groups: A_{p+3}, A_{p+4} and A_{p+5}.

Let q be an odd prime and $5 < q < p$. Set $E(q, p) = \text{Exp}(|G|, q)$, where $p + 4$ is a composite number, $p + 6$ is a prime number and $100 < p \in \pi(1000!)$. By Lemma 3.6, we know that there are only 35 such groups satisfying the conditions above. Using Magma, we can obtain every value of $E(q, p)$. In order to prove $G \cong A_{p+5}$, we need to discuss the difference between the values of $E(q,p)$ for each prime p. For convenience, for each p, we have tabulated some values of $E(q,p)$ in Table 2 of this article.

Table 2 $E(q,p)$

G	$E(17,131)$	$E(7,157)$	$E(43,167)$	$E(89,173)$	$E(7,233)$	$E(17,251)$	$E(131,257)$
A_{p+3}	7	25	3	1	37	14	1
A_{p+4}	7	25	3	1	37	15	1
A_{p+5}	8	26	4	2	38	15	2
G	$E(67,263)$	$E(23,271)$	$E(7,331)$	$E(179,353)$	$E(31,367)$	$E(7,373)$	$E(43,383)$
A_{p+3}	3	11	53	1	11	61	8
A_{p+4}	3	11	53	1	11	61	9
A_{p+5}	4	12	54	2	12	62	9
G	$E(73,433)$	$E(7,443)$	$E(127,503)$	$E(7,541)$	$E(281,557)$	$E(71,563)$	$E(23,571)$
A_{p+3}	5	73	3	89	1	7	25
A_{p+4}	5	73	3	89	1	7	26
A_{p+5}	6	74	4	90	2	8	26
G	$E(37,587)$	$E(199,593)$	$E(11,601)$	$E(17,607)$	$E(7,647)$	$E(7,653)$	$E(7,677)$
A_{p+3}	15	2	58	37	106	107	66
A_{p+4}	15	3	60	37	106	107	66
A_{p+5}	16	3	60	38	107	108	67
G	$E(17,727)$	$E(41,733)$	$E(151,751)$	$E(11,941)$	$E(7,947)$	$E(13,971)$	$E(491,977)$
A_{p+3}	44	17	4	92	156	79	1
A_{p+4}	45	17	5	92	156	80	1
A_{p+5}	45	18	5	93	157	80	2

If $p \in \{131, 173, 167, 233, 257, 263, 271, 331, 353, 367, 373, 433, 443, 503, 541, 557, 563, 571, 653, 587, 607, 677, 733, 941, 947, 977\}$, by Table 2, S can not be isomorphic to the simple group A_{p+3} or A_{p+4}. Otherwise, there exists at least a prime q with $5 < q < p$ such that $q^{\mathrm{Exp}(|G|,q)} \nmid |G|$, a contradiction.

If $p \in \{157, 251, 383, 593, 601, 647, 727, 751, 971\}$, by Table 2 and Case 1, K is a $\{2,3\}$-group. In this case, $S \cong A_m$, where $m = p+3$ or $p+4$. By Case 2, we have that $A_m \le G/K \le \mathrm{Aut}(A_m) \cong S_m$. But $5 \cdot p \in \pi_e(G/K) \setminus \pi_e(S_n)$, a contradiction.

Hence, $S \cong A_{p+5}$. By Case 2, one has that $A_{p+5} \lesssim G/K \lesssim \mathrm{Aut}(A_{p+5}) \cong S_{p+5}$. Since $|G| = |A_{p+5}|$, $G/K \ncong S_{p+5}$. If $G/K \cong A_{p+5}$, then by comparing the orders we deduce that $G \cong A_{p+5}$, which completes the proof of Case 3 and also the proof of Theorem 2.1.

In 1989, Shi W. J. put forward the following conjecture.

Corollary 4.1 (see [26]) *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if* (1) $|G| = |M|$ *and* (2) $\pi_e(G) = \pi_e(M)$.

The above conjecture was proved by joint works of many mathematicians and the last part of the proof was given by Mozurov V. D. etc. in [27]. That is, the following theorem holds.

Theorem 4.1 (see [27]) *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if* (1) $|G| = |M|$ *and* (2) $\pi_e(G) = \pi_e(M)$.

About the relation of Theorem 4.1 and OD-characterizable groups, we have the following facts: For two finite groups G and M, if $\pi_e(G) = \pi_e(M)$, then G and M must have the same prime graph. Hence they have the same degree pattern. Therefore, we can get the following corollary by Theorem 2.1.

Corollary 4.2 *If G is a finite group such that* (1) $|G| = |A_{p+5}|$ *and* (2) $\pi_e(G) = \pi_e(A_{p+5})$, *where* $5 \neq p \in \pi(1000!)$, *then* $G \cong A_{p+5}$.

5 OD-Characterization of the Symmetric Group S_{p+5}

As we already mentioned, the symmetric groups S_p and S_{p+1}, where p is a prime, are OD-characterizable. Proposition 1.5 says that the symmetric groups S_n with $10 \neq n \leq 100$ and $n \neq p, p+1$ are 3-fold OD-characterizable. On the other hand, according to Proposition 1.6, S_{10} is 8-fold OD-characterizable, and S_{10} is the first symmetric group which is not OD-characterizable. Till now, we have not found a symmetric group S_n $(n \neq p, p+1)$, except S_{10}, which is not 3-fold OD-characterizable. Hence, it is an interesting and difficult topic to investigate how many-fold OD-characterization of symmetric groups are. Therefore, the first author of this article put forward the following conjecture.

Conjecture 5.1 All the symmetric groups S_n $(n \neq p, p+1)$, except S_{10}, are 3-fold OD-characterizable.

In this section, we are going to give an affirmative answer to this conjecture for the symmetric group S_{p+5}. In other words, we will prove Theorem 2.2.

Proof of Theorem 2.2 Let G be a finite group satisfying $|G| = |S_{p+5}|$ and $D(G) = D(S_{p+5})$, where $p+4$ is a composite number, $p+6$ is a prime and $5 \neq p \in \pi(1000!)$. By [17], we only need to discuss the primes p such that $p+4$ is a composite number, $p+6$ is a prime and $100 < p \in \pi(1000!)$. By these hypotheses and Lemma 3.2, one has that $\{r\} \cup \{rs \mid r+s \leq p+5\} \subseteq \pi_e(G)$ and $\{rs \mid r+s > p+5\} \cap \pi_e(G) = \emptyset$, where $r, s \in \pi(G)$. By Lemma 3.4(2), $\deg(2) = |\pi(G)| - 1$, so the prime graph of G is connected. By the structure of $D(G)$, it is easy to check by the Magma software that $\Gamma(G) = \Gamma(S_{p+5})$.

Let K denote the maximal normal solvable subgroup of G. For the similar reason as the proof of Theorem 2.1, K is a $\{2, 3, 5\}$-group and $A_{p+5} \lesssim G/K \lesssim \mathrm{Aut}(A_{p+5}) \cong S_{p+5}$. Hence $G/K \cong A_{p+5}$ or S_{p+5}. If $G/K \cong S_{p+5}$, then by comparing the order we get that $G \cong S_{p+5}$. If $G/K \cong A_{p+5}$, then $|K| = 2$ and $K \leq G' \cap Z(G)$. Therefore G is a central extension of Z_2 by A_{p+5}. If G is a non-split extension of Z_2 by A_{p+5}, then $G \cong Z_2 \cdot A_{p+5}$. If G is a split extension of Z_2 by A_{p+5}, then $G \cong Z_2 \times A_{p+5}$.

We omit the details for S_{p+5} because the arguments are quite similar to those for A_{p+5}. We only mention that the non-isomorphic groups $Z_2 \cdot A_{p+5}$ and $Z_2 \times A_{p+5}$ have the same order and the degree pattern as S_{p+5}. Hence S_{p+5} is 3-fold OD-characterizable, which completes the proof of Theorem 2.2.

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