

Gradient Estimates for a Nonlinear Diffusion Equation on Complete Manifolds*

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Abstract This paper deals with the gradient estimates of the Hamilton type for the positive solutions to the following nonlinear diffusion equation:

$$u_t = \Delta u + \nabla \phi \cdot \nabla u + a(x)u \ln u + b(x)u$$

on a complete noncompact Riemannian manifold with a Bakry-Emery Ricci curvature bounded below by $-K$ ($K \geq 0$), where ϕ is a C^2 function, $a(x)$ and $b(x)$ are C^1 functions with certain conditions.

Keywords Gradient estimate, Bakry-Emery Ricci curvature, Nonlinear diffusion equation

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1 Introduction

The notion of Bakry-Emery Ricci tensor associated with a diffusion operator was introduced by Bakry [1], which we recall as follows.

Definition 1.1 *Given an n -dimensional Riemannian manifold (M, g) and a C^2 function ϕ on M , one has a diffusion operator $L := \Delta + \nabla \phi \cdot \nabla$, where the Δ and ∇ are the Laplace operator and the gradient operator on M respectively. Then the Bakry-Emery Ricci tensor associated with the diffusion operator L is defined as the following symmetric 2-tensor:*

$$\widetilde{\text{Ric}} := \text{Ric} - \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n},$$

where the constant $m \geq n$; if $m = n$, we assume $\phi = 0$. Denote by Ric_∞ the limit $\lim_{m \rightarrow \infty} \widetilde{\text{Ric}} = \text{Ric} - \nabla^2 \phi$.

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In this note, we want to study the gradient estimates of the Hamilton type for the positive solution to the following nonlinear diffusion equation:

$$u_t = Lu + a(x)u \ln u + b(x)u \quad (1.1)$$

on a complete noncompact Riemannian manifold with the above Bakry-Emery Ricci curvature bounded below by $-K$ ($K \geq 0$), where $a(x)$ and $b(x)$ are C^1 functions with certain conditions (for details, see Theorem 1.3).

The elliptic case of the equation (1.1) with $\phi = 0$, namely

$$\Delta u + au \ln u + bu = 0, \quad (1.2)$$

was first considered by Ma [6] in the case that a and b are constants and $a < 0$ when he studied the gradient Ricci Soliton. He also pointed out that it is interesting to consider the gradient estimates for the positive solutions to the corresponding parabolic equation

$$u_t = \Delta u + au \ln u + bu. \quad (1.3)$$

Later, Yang [12] studied the above parabolic equation and obtained the gradient estimate of Li-Yau type (see [3]) for the solutions to (1.3). Here we should also mention that Li [5] studied earlier the following equation:

$$u_t = \Delta u + bu^\alpha \quad (1.4)$$

for some $\alpha > 0$, and got the gradient estimates and the Harnack inequality which generalize the corresponding estimates of Li-Yau [3].

When $a = b = 0$, the equation was studied by Li [5]. He obtained a gradient estimate of the Li-Yau type.

There is another kind of gradient estimates developed by Hamilton [2]. He considered the heat equation on compact manifolds and obtained the following estimate, which we call the gradient estimate of Hamilton type.

Theorem 1.1 *Let M be a compact manifold without boundary and with Ricci curvature bounded below by $-K$, $K \geq 0$. Suppose that u is any positive solution to the heat equation $u_t = \Delta u$ with $u \leq C$ for all $(x, t) \in M \times (0, +\infty)$. Then*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \left(\ln \frac{C}{u}\right).$$

In [10], Souplet and Zhang extend the above gradient estimate to noncompact manifolds.

Theorem 1.2 (Souplet-Zhang) *Let M be an n -dimensional complete noncompact manifold with the Ricci curvature bounded below by $-K$, $K \geq 0$. Suppose that u is any positive solution to the heat equation $u_t = \Delta u$ in $Q_{2R, 2T} \equiv B(x_0, 2R) \times [t_0 - 2T, t_0]$, and $u \leq C$ in $Q_{2R, 2T}$. Then one has in $Q_{R, T}$,*

$$\frac{|\nabla u|}{u} \leq C_1 \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right) \left(1 + \ln \frac{C}{u} \right),$$

where C_1 is some positive constant depending only on the dimension n of M .

In this note, we will study the equation (1.1), and hope to get the gradient estimates of Hamilton type for the solutions under the condition of Bakry-Emery Ricci curvature. Our method is motivated by [9], where the second author obtained the gradient estimate of Hamilton type for the solution to the equation (1.1) when $a(x) \equiv 0$. We should point out that the equation (1.1) is a nonlinear PDE when $a(x) \neq 0$. Our main result can be stated as follows.

Theorem 1.3 *Let (M, g) be an n -dimensional complete non-compact Riemannian manifold with Bakry-Emery curvature $\widetilde{\text{Ric}} \geq -K$ for some constant $K \geq 0$, $p \in M$, $B(p, R)$ the geodesic ball with radius R and the center at p . Suppose that $u(x, t)$ is a positive smooth solution to the diffusion equation (1.1) in $Q_{2R, 2T_0} \equiv B(p, 2R) \times [0, 2T_0] \subset M \times [0, \infty)$. Let $\alpha = \sup_{(x, t) \in Q_{2R, 2T_0}} \ln u + 1$, $\gamma = \max\{1, |\alpha - 1|\}$ and assume that $\frac{|\nabla a|^2}{|a|} \leq C_0$, $\frac{|\nabla b|^2}{|b|} \leq C_0$, $|a| \leq C_0$, $|b| \leq C_0$ for some constant $C_0 > 0$. Then one has the following estimate in Q_{R, T_0} :*

$$\frac{|\nabla u|}{u} \leq C_1 \left(\sqrt{K} + \sqrt{\alpha} + \sqrt{\gamma} + \frac{\gamma + \sqrt{m}}{R} + \frac{\sqrt{(m-1)K}}{\sqrt{R}} + \frac{1}{\sqrt{t}} \right) (\alpha - \ln u),$$

where C_1 is a general constant independent of the dimension n of M and depending only on C_0 .

Setting $R \rightarrow +\infty$, we can get a global gradient estimate for the equation (1.1), which is independent of the dimension n of M .

Corollary 1.1 *Under the conditions of Theorem 1.3, then one has in $M \times [0, T_0]$,*

$$\frac{|\nabla u|}{u} \leq C_1 \left(\sqrt{K} + \sqrt{\alpha} + \sqrt{\gamma} + \frac{1}{\sqrt{t}} \right) (\alpha - \ln u).$$

2 Gradient Estimates of Hamilton Type

In this section, we use a certain cut-off function and the maximum principle to show Theorem 1.3.

We first state a general Laplacian comparison theorem (see [5, 8]), which will be used in the following proof. Given an n -dimensional Riemannian manifold (M, g) and a C^2 function ϕ on M , one has a diffusion operator $L := \Delta + \nabla \phi \cdot \nabla$, where the Δ and ∇ are the Laplace operator and the gradient operator on M respectively. One has the so-called Bakry-Emery Ricci tensor associated with the diffusion operator L . The comparison theorem associated with L can be stated as follows.

Lemma 2.1 *Let M be an n -dimensional complete Riemannian manifold with the Bakry-Emery curvature $\widetilde{\text{Ric}}$ associated with L greater than $-K$ ($K > 0$). One then has*

$$L\rho(x) \leq (m-1)\sqrt{K} \coth(\sqrt{K}\rho), \quad \forall x \in M, \quad C_p \quad (2.1)$$

where the C_p denotes the cut locus of p .

We also need a C^2 cut-off function $\eta = \eta(t)$ on $[0, +\infty)$ (also see [3]), which is defined as follows:

$$\eta(t) = \begin{cases} 1, & t \in [0, 1], \\ > 0, & t \in (1, 2), \\ 0, & t \in [2, +\infty), \end{cases} \quad (2.2)$$

and satisfies that $\forall t > 0$, $0 \geq \frac{\eta'(t)}{\eta^{1/2}(t)} \geq -C$, $\eta''(t) \geq -C$, where C is a positive constant. Here and henceforth, unless otherwise stated, by C , C_0 , C_1 , etc, we always mean some general constants independent of the dimension n of M .

Let $\rho(x)$ be the distance function at p , and define

$$\psi(x) = \eta\left(\frac{\rho(x)}{R}\right).$$

Then we have

$$\frac{|\nabla\psi|^2}{\psi} = \frac{|\eta'|^2}{R^2\eta}\left(\frac{\rho(x)}{R}\right) \leq \frac{C^2}{R^2}. \quad (2.3)$$

By (2.1), one then has

$$L\psi(x) = \frac{\eta''|\nabla\rho|^2}{R^2} + \frac{\eta' L\rho}{R} \geq \frac{-mC}{R^2} - \frac{\sqrt{(m-1)KC}}{R}. \quad (2.4)$$

For convenience, we introduce two bilinear operators Γ and Γ_2 (see [1]) as follows. For $u, v \in C^2(M)$, set

$$\Gamma(u, v) = \frac{1}{2}\{L(uv) - uLv - vLu\}$$

and

$$\Gamma_2(u, v) = \frac{1}{2}\{L\Gamma(u, v) - \Gamma(Lu, v) - \Gamma(u, Lv)\}.$$

A simple computation shows that $\forall u \in C^2(M)$,

$$\Gamma(u, u) = |\nabla u|^2$$

and

$$\Gamma_2(u, u) = \frac{1}{2}\{\square\Gamma(u, u) - 2\Gamma(\square u, u)\}, \quad (2.5)$$

where $\square := L - \partial_t$.

In the following, we denote $\Gamma(u, u)$ and $\Gamma_2(u, u)$ by $\Gamma(u)$ and $\Gamma_2(u)$ respectively. Thus, by the assumption of the Bakry-Emory curvature $\text{Ric} \geq -K$ and the Bochner's formula, one has, for any $u \in C^2(M)$,

$$\Gamma_2(u) \geq -K\Gamma(u). \quad (2.6)$$

Proof of Theorem 1.3 We now begin to show our main theorem. Set $f = \ln u$ and

$$\omega = |\nabla \ln(\alpha - f)|^2.$$

Rewriting the equation (1.1) as

$$\frac{u_t}{u} = \frac{\Delta u}{u} + \frac{\nabla\phi \cdot \nabla u}{u} + a(x) \ln u + b(x),$$

one then has

$$\begin{aligned}
 \square \ln(\alpha - f) &= -\frac{Lf}{\alpha - f} + \frac{f_t}{\alpha - f} - \omega \\
 &= -\frac{Lu}{u(\alpha - f)} + \frac{u_t}{u(\alpha - f)} + (\alpha - f - 1)\omega \\
 &= \frac{af}{\alpha - f} + \frac{b}{\alpha - f} + (\alpha - f - 1)\omega.
 \end{aligned} \tag{2.7}$$

Thus, using (2.5)–(2.7), one has

$$\begin{aligned}
 \square \omega &= \square \Gamma(\ln(\alpha - f)) \\
 &= 2\Gamma_2(\ln(\alpha - f)) + 2\Gamma(\square(\ln(\alpha - f)), \ln(\alpha - f)) \\
 &\geq -2K\omega + 2\Gamma\left(\frac{af}{\alpha - f}, \ln(\alpha - f)\right) + 2\Gamma\left(\frac{b}{\alpha - f}, \ln(\alpha - f)\right) \\
 &\quad + 2\Gamma((\alpha - f - 1)\omega, \ln(\alpha - f)).
 \end{aligned} \tag{2.8}$$

We now estimate the last three terms in (2.8).

$$\begin{aligned}
 2\Gamma\left(\frac{af}{\alpha - f}, \ln(\alpha - f)\right) &= -\frac{2f\nabla a \nabla f}{(\alpha - f)^2} - 2a\omega - \frac{2af}{\alpha - f}\omega \\
 &\geq -\frac{2|f||\nabla a|}{\alpha - f}\omega^{1/2} - \frac{2\alpha|a|\omega}{\alpha - f} \\
 &\geq -\frac{|\nabla a|^2}{|a|} \cdot \frac{f^2}{\alpha(\alpha - f)} - \frac{3\alpha|a|\omega}{\alpha - f} \\
 &\geq -\frac{C_0 f^2}{\alpha(\alpha - f)} - \frac{3\alpha|a|\omega}{\alpha - f},
 \end{aligned} \tag{2.9}$$

$$\begin{aligned}
 2\Gamma\left(\frac{b}{\alpha - f}, \ln(\alpha - f)\right) &= 2\frac{\nabla b \nabla \ln(\alpha - f)}{\alpha - f} - \frac{2b\omega}{\alpha - f} \\
 &\geq -\frac{2|\nabla b|\omega^{1/2}}{\alpha - f} - \frac{2|b|\omega}{\alpha - f} \\
 &\geq -\frac{|\nabla b|^2/|b|}{\alpha - f} - \frac{3|b|\omega}{\alpha - f} \\
 &\geq -\frac{C_0}{\alpha - f} - \frac{3|b|\omega}{\alpha - f}
 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
 &2\Gamma((\alpha - f - 1)\omega, \ln(\alpha - f)) \\
 &= 2(\alpha - f)\omega^2 - 2(\alpha - 1)\frac{\nabla \omega \nabla f}{\alpha - f} + 2f\frac{\nabla \omega \nabla f}{\alpha - f}.
 \end{aligned} \tag{2.11}$$

Substituting (2.9)–(2.11) into (2.8), one obtains

$$\begin{aligned}
 \square \omega &\geq -2K\omega - \frac{3\alpha|a|\omega}{\alpha - f} - \frac{C_0 f^2}{\alpha(\alpha - f)} - \frac{3|b|\omega}{\alpha - f} - \frac{C_0}{(\alpha - f)} \\
 &\quad + 2(\alpha - f)\omega^2 - \frac{2(\alpha - 1)\nabla \omega \nabla f}{\alpha - f} + \frac{2f\nabla \omega \nabla f}{\alpha - f}.
 \end{aligned} \tag{2.12}$$

Let $\varphi = t\psi$ and suppose that the function $\varphi\omega$ takes the maximum at the point (x_1, t_1) in $B(p, 2R) \times [0, T] \subset M \times [0, T_0]$. It is well-known (see [3]) that one may assume that x_1 is not in the cut locus of p . Then the maximum principle implies

$$\begin{cases} \nabla(\varphi\omega)(x_1, t_1) = 0, \\ \Delta(\varphi\omega)(x_1, t_1) \leq 0, \\ \frac{\partial}{\partial t}(\varphi\omega)(x_1, t_1) \geq 0. \end{cases} \quad (2.13)$$

Thus, one has at (x_1, t_1)

$$\varphi\Box\omega + \omega\Box\varphi + 2\Gamma(\varphi, \omega) \leq 0. \quad (2.14)$$

Combining (2.12) with (2.14), one has at (x_1, t_1) ,

$$\begin{aligned} & -2K\varphi\omega - \varphi\frac{3\alpha|a|\omega}{\alpha-f} - \varphi\frac{C_0f^2}{\alpha(\alpha-f)} - \varphi\frac{3|b|\omega}{\alpha-f} - \varphi\frac{C_0}{\alpha-f} + 2\varphi(\alpha-f)\omega^2 \\ & - \varphi\frac{2(\alpha-1)}{\alpha-f}\nabla\omega\nabla f + \varphi\frac{2f\nabla\omega\nabla f}{\alpha-f} + (\Box\varphi)\omega + 2\nabla\varphi\nabla\omega \leq 0. \end{aligned} \quad (2.15)$$

We now estimate the seventh, eighth and ninth items in the left side of (2.15). By the Young's inequality, (2.4) and (2.13), we obtain at (x_1, t_1) ,

$$\begin{aligned} \varphi\frac{2(\alpha-1)}{\alpha-f}\nabla\omega\nabla f &= \frac{2t_1(1-\alpha)}{\alpha-f}(\nabla f\nabla\psi)\omega \\ &\leq 2t_1|1-\alpha||\nabla\psi|\omega^{3/2} \\ &\leq \frac{(\alpha-f)\varphi\omega^2}{2} + 2T\frac{|1-\alpha|^2}{\alpha-f}\frac{|\nabla\psi|^2}{\psi}\omega, \end{aligned} \quad (2.16)$$

$$\begin{aligned} -\varphi\frac{2f\nabla\omega\nabla f}{\alpha-f} &= \frac{2f\nabla f\nabla\varphi}{\alpha-f}\omega \\ &\leq 2|f||\nabla\varphi|\omega^{3/2} \\ &\leq \frac{\varphi(\alpha-f)\omega^2}{2} + \frac{2Tf^2}{\alpha-f}\frac{|\nabla\psi|^2}{\psi}\omega \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} (\Box\varphi)\omega &= (L\varphi - \partial_t\varphi)\omega \\ &= t_1(L\psi)\omega - \psi\omega \\ &\geq -\frac{TmC}{R^2}\omega - \frac{T\sqrt{(m-1)KC}}{R}\omega - \omega. \end{aligned} \quad (2.18)$$

Substituting (2.16)–(2.18) into (2.15) at (x_1, t_1) , one then has

$$\begin{aligned} 0 &\geq -2K\varphi\omega - \frac{3\alpha|a|\varphi\omega}{\alpha-f} - \frac{C_0f^2\varphi}{\alpha(\alpha-f)} - \frac{3|b|\varphi\omega}{\alpha-f} - \frac{C_0\varphi}{\alpha-f} \\ &\quad + (\alpha-f)\varphi\omega^2 - 2T\frac{|1-\alpha|^2}{\alpha-f}\frac{|\nabla\psi|^2}{\psi}\omega - \frac{2Tf^2}{\alpha-f}\frac{|\nabla\psi|^2}{\psi}\omega \\ &\quad - \frac{TmC}{R^2}\omega - T\frac{\sqrt{(m-1)KC}}{R}\omega - \omega - \frac{2T|\nabla\psi|^2}{\psi}\omega. \end{aligned} \quad (2.19)$$

Multiplying both sides of (2.19) by $\frac{\varphi}{\alpha-f}$ and noting that $0 \leq \psi \leq 1$, one obtains at (x_1, t_1) ,

$$\begin{aligned} 0 \geq & (\varphi\omega)^2 - T \left(\frac{2K}{\alpha-f} + \frac{3\alpha|a|}{(\alpha-f)^2} + \frac{3|b|}{(\alpha-f)^2} + \frac{2|1-\alpha|^2}{(\alpha-f)^2} \frac{|\nabla\psi|^2}{\psi} \right. \\ & + \frac{2f^2}{(\alpha-f)^2} \frac{|\nabla\psi|^2}{\psi} + \frac{mC}{R^2(\alpha-f)} + \frac{\sqrt{(m-1)KC}}{R(\alpha-f)} + \frac{1}{(\alpha-f)T} \\ & \left. + \frac{2|\nabla\psi|^2}{(\alpha-f)\psi} \right) \varphi\omega - \frac{C_0 f^2 T^2}{\alpha(\alpha-f)^2} - \frac{C_0 T^2}{(\alpha-f)^2}. \end{aligned} \quad (2.20)$$

Set $\gamma = \max\{1, |\alpha-1|\}$. We observe that if $f \leq 0$, $\frac{f^2}{(\alpha-f)^2} \leq 1$ and if $f \geq 0$, $\frac{f^2}{(\alpha-f)^2} \leq (\alpha-1)^2$. Using these together with (2.3) and the assumptions on a and b , (2.20) then becomes at (x_1, t_1)

$$\begin{aligned} 0 \geq & (\varphi\omega)^2 - T \left(2K + 3\alpha C_0 + 3C_0 + \frac{2\gamma C^2}{R^2} + \frac{2C^2\gamma^2}{R^2} + \frac{mC}{R^2} \right. \\ & \left. + \frac{\sqrt{(m-1)KC}}{R} + \frac{1}{T} + \frac{2C^2}{R^2} \right) \varphi\omega - \frac{C_0 T^2 \gamma^2}{\alpha} - C_0 T^2, \end{aligned}$$

where C_0, C are general constants independent of the dimension n of M .

Since $\psi = 1$ on $B(p, R)$ and (x_1, t_1) in $B(p, 2R) \times [0, T]$ is the maximum point of $\varphi\omega$, then one has

$$\begin{aligned} \omega(x, T) \leq & \left(2K + 3\alpha C_0 + 3C_0 + \frac{\sqrt{C_0}\gamma}{\sqrt{\alpha}} + \sqrt{C_0} \right) + \left(\frac{2\gamma C^2}{R^2} + \frac{mC}{R^2} \right. \\ & \left. + \frac{2C^2\gamma^2}{R^2} + \frac{2C^2}{R^2} \right) + \frac{\sqrt{(m-1)KC}}{R} + \frac{1}{T} \end{aligned}$$

$\forall x \in B(p, R)$.

As T is arbitrary, $\omega = |\nabla \ln(\alpha-f)|^2$ and $f = \ln u$, one has at $(x, t) \in B(p, R) \times [0, T_0]$,

$$\begin{aligned} \left| \frac{\nabla u}{u} \right| \leq & \left(\sqrt{2K} + C_1 \sqrt{\alpha} + \frac{C_1 \sqrt{\gamma}}{R} + \frac{C_1 \sqrt{m}}{R} + \frac{C_1 \gamma}{R} \right. \\ & \left. + \frac{C_1 \sqrt{\gamma}}{\sqrt{\alpha}} + \frac{C_1 \sqrt[4]{(m-1)K}}{\sqrt{R}} + \frac{1}{\sqrt{t}} \right) (\alpha - \ln u). \end{aligned}$$

This completes the proof of Theorem 1.3.

3 Some Further Remarks

From the proof of Theorem 1.3, we see that the curvature condition of $\widetilde{\text{Ric}} \geq -K$ is only used in the proof of the inequality (2.6) and the Laplacian comparison theorem of the distance function (see Lemma 2.1). On the other hand, again by the Bochner formula, one still has

$$\Gamma_2(u) = |\nabla^2 u|^2 + \text{Ric}_\infty(\nabla u, \nabla u).$$

So if the curvature condition of $\widetilde{\text{Ric}} \geq -K$ is replaced by $\text{Ric}_\infty \geq -K$, the same inequality as (2.6) still works. However, the Laplacian comparison theorem of the distance function (see Lemma 2.1) does not work anymore under the curvature condition of $\text{Ric}_\infty \geq -K$.

Fortunately, following Wei and Wylie's idea (see [11]), we can get that if $\text{Ric}_\infty \geq -K$ and $|\phi| \leq k$, then the similar Laplacian comparison theorem of the distance function still holds for the diffusion operator $L = \Delta + \nabla\phi \cdot \nabla$: $L\rho(x) \leq \frac{m+4k-1}{\rho(x)} + \sqrt{(m+4k-1)K}$. Thus, by using the same argument as above, we have the following global gradient estimate for the equation (1.1).

Theorem 3.1 *Let (M, g) be an any-dimensional complete non-compact Riemannian manifold with $\text{Ric}_\infty \geq -K$ and $|\phi| \leq k$ for some positive constants K and k . Suppose that $u(x, t)$ is a positive smooth solution to the diffusion equation (1.1) on $M \times [0, T]$. Let $\alpha = \sup_{(x,t) \in M \times [0,T]} \ln u + 1$ and assume that $\frac{|\nabla a|^2}{|a|} \leq C_0$, $\frac{|\nabla b|^2}{|b|} \leq C_0$, $|a| \leq C_0$, $|b| \leq C_0$ for some positive constant C_0 . Then one has in $M \times [0, T]$,*

$$\frac{|\nabla u|}{u} \leq C_1 \left(\sqrt{K} + \sqrt{\alpha} + \sqrt{\gamma} + \frac{1}{\sqrt{t}} \right) (\alpha - \ln u),$$

where $\gamma := \max\{1, |\alpha - 1|\}$ and C_1 is a positive constant depending only on C_0 (in particular, independent of the dimension n of M).

If both $a(x)$ and $b(x)$ in the equation (1.1) are constant functions, then we have the following global gradient estimate.

Corollary 3.1 *Let (M, g) be an any-dimensional complete non-compact Riemannian manifold with $\text{Ric}_\infty \geq -K$ and $|\phi| \leq k$ for some positive constants K and k . Suppose that $u(x, t)$ is a positive smooth solution on $M \times [0, T]$. Let $\alpha = \sup_{(x,t) \in M \times [0,T]} \ln u + 1$. Then one has in $M \times [0, T]$,*

$$\frac{|\nabla u|}{u} \leq \left(\sqrt{2K} + \sqrt{3|a|\alpha} + \sqrt{3|b|} + \frac{1}{\sqrt{t}} \right) (\alpha - \ln u).$$

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