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On the Dual Orlicz Mixed Volumes*

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Abstract In this paper, the authors define a harmonic Orlicz combination and a dual Orlicz mixed volume of star bodies, and then establish the dual Orlicz-Minkowski mixed-volume inequality and the dual Orlicz-Brunn-Minkowksi inequality.

Keywords Convex body, Harmonic Orlicz combination, Dual Orlicz mixed volume, Dual Orlicz-Brunn-Minkowski inequality
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1 Introduction

The dual mixed volumes, as a core concept in the dual Brunn-Minkowski theory, were firstly introduced by Lutwak [1], and played an important role in convex geometry. They are closely related to such important bodies as: Intersection bodies (see [2]), centroid bodies (see [3]), and projection bodies (see [4]). In [5], Gardner gave some stability results of these inequalities about dual mixed volumes. In [6], Klain presented a classification theorem for homogeneous valuations on star-sharped sets by dual mixed volumes.

Quite recently, Gardner, Hug and Weil [7] constructed a general framework for the Orlicz-Brunn-Minkowski theory, which was introduced by Lutwak, Yang and Zhang (see [8–11]), and they made clear for the first time its relation to Orlicz spaces and norms. In [7], Gardner, Hug and Weil gave a reasonable definition of Orlicz addition, then obtained the Orlicz-Brunn-Minkowksi inequality, and in the end gave the Orlicz mixed volume of convex bodies which contain the origin in their interiors and get the Orlicz mixed volume inequality. In [12], Xi, Jin and Leng also obtained the Orlicz-Brunn-Minkowski inequality by Steiner symmetry and the Orlicz Minkowksi mixed volume inequality.

$\mathbb{R}^n$ denotes the usual n-dimensional Euclidean space. A set $A \subseteq \mathbb{R}^n$ is said to be star-shaped, if $0 \in A$, and for each line l passing through the origin in $\mathbb{R}^n$, the set $A \cap l$ is a closed interval. Denote by $\mathcal{S}_0^n$ the set of all star bodies in $\mathbb{R}^n$, i.e., the set of all star-shaped sets with a positive and continuous radial function.

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In this paper, we define the harmonic Orlicz sum $K\widehat{+}_\varphi L$ of star bodies K and L in $\mathcal{S}_0^n$ implicitly by

$$\varphi\Big(\frac{\rho(K\widehat{+}_\varphi L,x)}{\rho(K,x)},\frac{\rho(K\widehat{+}_\varphi L,x)}{\rho(L,x)}\Big)=1 \tag{1.1}$$

for $x\in\mathbb{R}^n$. Here $\varphi\in\Phi_2$, and we have the set of convex functions $\varphi:[0,\infty)^2\to[0,\infty)$ that are strictly increasing in each variable and satisfy $\varphi(0,0)=0$ and $\varphi(1,0)=\varphi(0,1)=1$.

In Section 2, we introduce a new notion of the Orlicz harmonic combination $\widehat{+}_\varphi(K,L,\alpha,\beta)$ by means of an appropriate modification of (1.1). The particular instance of interest corresponds to using (1.1) with $\varphi(x_1,x_2)=\varphi_1(x_1)+\epsilon\varphi_2(x_2)$ for $\epsilon>0$ and $\varphi_1,\varphi_2\in\Phi$, the set of strictly increasing convex functions φ: $[0,\infty)\to[0,\infty)$ that satisfy $\varphi(0)=0$ and $\varphi(1)=1$, in which case we write $K\widehat{+}_{\varphi,\varepsilon}L$ instead of $K\widehat{+}_\varphi L$, and we obtain the following equation.

Theorem 1.1 *Suppose $\varphi\in\Phi_2$. For all $K,L\in\mathcal{S}_0^n$, we have*

$$-\frac{(\varphi_1)_l'(1)}{n}\lim_{\epsilon\downarrow 0}\frac{V(K\widehat{+}_{\varphi,\epsilon}L)-V(K)}{\epsilon}=\frac{1}{n}\int_{S^{n-1}}\varphi_2\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big)\rho_K(u)^n\mathrm{d}S(u), \tag{1.2}$$

where $\epsilon\downarrow 0$ means that ϵ is decreasing and tends to 0.

The integral on the right-hand side of (1.2) with φ_2 replaced by φ, a new dual Orlicz mixed volume $V_\varphi(K,L)$ is introduced, and we see that either side of the equation (1.2) is equal to $V_{\varphi_2}(K,L)$ and establish the following dual Orlicz-Minkowski inequality and the harmonic Orlicz addition version of the Brunn-Minkowski inequality.

Theorem 1.2 *Suppose $\varphi\in\Phi$, $K,L\in\mathcal{S}_0^n$, and then*

$$\widetilde{V}_\varphi(K,L)\ge V(K)\varphi\Big(\frac{V(K)^{\frac{1}{n}}}{V(L)^{\frac{1}{n}}}\Big)$$

with equality if and only if K and L are dilates.

Theorem 1.3 *Suppose $\varphi\in\Phi_2$ such that $\varphi(x_1,x_2)=\varphi_1(x_1)+\varphi_2(x_2)$ and $\varphi_i\in\Phi$, $i=1,2$, $x_i\in\mathbb{R}$. If $K,L\in\mathcal{S}_0^n$, then*

$$1\ge\varphi_1\Big(\frac{V(K\widehat{+}_\varphi L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\Big)+\varphi_2\Big(\frac{V(K\widehat{+}_\varphi L)^{\frac{1}{n}}}{V(L)^{\frac{1}{n}}}\Big)$$

with equality if and only if K and L are dilates.

2 Preliminaries

We shall denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ by S^{n-1}. A star body K is determined uniquely by its radial function $\rho_K=\rho(K,\cdot):S^{n-1}\to R$, defined for $u\in S^{n-1}$ by

$$\rho(K,u)=\max\{\lambda\ge 0:\lambda u\in K\}.$$

Suppose that $K,L\in\mathcal{S}_0^n$, and the radial Hausdorff metric $\widetilde{\delta}(K,L)$ is defined by

$$\widetilde{\delta}(K,L)=\max_{u\in S^{n-1}}|\rho_K(u)-\rho_L(u)|.$$

If $K \in \mathcal{S}_0^n$, then the polar coordinate formula for volume $V(K)$ is

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n \mathrm{d}S(u), \tag{2.1}$$

where $\mathrm{d}S(u)$ is the spherical Lebesgue measure of S^{n-1}.

Throughout the paper, $\Phi_m,\ m \in \mathbb{N}$, and denote the set of convex functions $\varphi : [0,\infty)^m \to [0,\infty)$ that are strictly increasing in each variable and satisfy $\varphi(0) = 0$ and $\varphi(e_j) = 1 > 0,\ j = 1,\cdots,m$. When $m = 1$, we shall write Φ instead of Φ_1.

Let $m \geq 2$ and $K_j \in \mathcal{S}_0^n$. The harmonic Orlicz sum of $K_1,\cdots,K_m$, denoted by $\widehat{+}_\varphi(K_1,\cdots,K_m)$ is defined by

$$\rho_{\widehat{+}_\varphi(K_1,\cdots,K_m)}(x) = \sup\Big\{\lambda > 0 : \varphi\Big(\frac{\lambda}{\rho_{K_1}(x)},\cdots,\frac{\lambda}{\rho_{K_m}(x)}\Big) \leq 1\Big\} \tag{2.2}$$

for all $x \in \mathbb{R}$.

Equivalently, the harmonic Orlicz sum $\widehat{+}_\varphi(K_1,\cdots,K_m)$ can be defined implicitly (and uniquely) by

$$\varphi\Big(\frac{\rho_{\widehat{+}_\varphi(K_1,\cdots,K_m)}(x)}{\rho_{K_1}(x)},\cdots,\frac{\rho_{\widehat{+}_\varphi(K_1,\cdots,K_m)}(x)}{\rho_{K_m}(x)}\Big) = 1. \tag{2.3}$$

An important special case is obtained when

$$\varphi(x_1,\cdots,x_m) = \sum_{j=1}^m \varphi_j(x_j)$$

for some fixed $\varphi_j \in \Phi,\ j = 1,\cdots,m$ such that $\varphi_1(1) = \cdots = \varphi_m(1) = 1$. We then write $\widehat{+}_\varphi(K_1,\cdots,K_m) = K_1\widehat{+}_\varphi\cdots\widehat{+}_\varphi K_m$. This means that $K_1\widehat{+}_\varphi\cdots\widehat{+}_\varphi K_m$ is defined either by

$$\rho_{\widehat{+}_\varphi(K_1,\cdots,K_m)} = \sup\Big\{\lambda > 0 : \varphi_1\Big(\frac{\lambda}{\rho_{K_1}(x)}\Big) + \cdots + \varphi_m\Big(\frac{\lambda}{\rho_{K_m}(x)}\Big) \leq 1\Big\} \tag{2.4}$$

for all $x \in \mathbb{R}$, or by the corresponding special case of (2.3).

Let $m = 2$ and $\varphi(x_1,x_2) = x_1^p + x_2^p,\ p \geq 1$, and we get the harmonic L_p sum $K\widehat{+}_pL$ (see [13]).

Suppose that $\alpha_j \geq 0$ and $\varphi_j \in \Phi,\ j = 1,\cdots,m$. If $K_j \in \mathcal{S}_0^n,\ j = 1,\cdots,m$, we define the Orlicz linear combination $\widehat{+}_\varphi(K_1,\cdots,K_m,\alpha_1,\cdots,\alpha_m)$ by

$$\rho_{\widehat{+}_\varphi(K_1,\cdots,K_m,\alpha_1,\cdots,\alpha_m)}(x) = \sup\Big\{\lambda > 0 : \sum_{j=1}^m \alpha_j\varphi_j\Big(\frac{\lambda}{\rho_{K_j}(x)}\Big) \leq 1\Big\} \tag{2.5}$$

for all $x \in \mathbb{R}$. Unlike the L_{-p} case, it is not generally possible to isolate a harmonic Orlicz scalar multiplication, since there is a dependence not just on one coefficient α_j, but on all $K_1,\cdots,K_m$ and $\alpha_1,\cdots,\alpha_m$.

For our purposes, it suffices to focus on the case $m = 2$. The harmonic Orlicz combination $\widehat{+}_\varphi(K,L,\alpha,\beta)$ for $K,L \in \mathcal{S}_0^n$ and $\alpha,\beta \geq 0$ (not both zero), is defined equivalently via the implicit equation

$$\alpha\varphi_1\Big(\frac{\rho(\widehat{+}_\varphi(K,L,\alpha,\beta),x)}{\rho(K,x)}\Big) + \beta\varphi_2\Big(\frac{\rho(\widehat{+}_\varphi(K,L,\alpha,\beta),x)}{\rho(L,x)}\Big) = 1 \tag{2.6}$$

for all $x \in \mathbb{R}^n$.

It is easy to verify that when $\varphi_1(t) = \varphi_2(t) = t^p,\ p \geq 1$, we get that the harmonic Orlicz linear combination $\widehat{+}_\varphi(K, L, \alpha, \beta)$ equals the harmonic L_p combination $\alpha\Diamond K\widehat{+}_p\beta\Diamond L$ (see [13]). In [14], He and Leng got a strong law of large numbers on the harmonic L_p combination.

Henceforth we shall write $K\widehat{+}_{\varphi,\epsilon}L$ instead of $\widehat{+}_\varphi(K, L, 1, \epsilon)$, and assume throughout that this is defined by (2.6), where $\alpha = 1,\ \beta = \epsilon$, and $\varphi_j \in \Phi,\ j = 1, 2$.

The left derivative of a real-valued function f is denoted by f'_l.

Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\varphi : I \to \mathbb{R}$ is a convex function, then

$$\int_X \varphi(g(x))\mathrm{d}\mu(x) \geq \varphi\Big(\int_X g(x)\mathrm{d}\mu(x)\Big).$$

If φ is a strictly convex, the equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$.

3 Dual Orlicz Mixed Volume

We firstly give some properties of the harmonic Orlicz addition.

Lemma 3.1 *Let $\varphi \in \Phi_2$. If $K, K_i, L, L_i \in \mathcal{S}_0^n$, then the harmonic Orlicz addition $\widehat{+}_\varphi$: $(\mathcal{S}_0^n)^2 \to \mathcal{S}_0^n$ has the following properties:*

(1) (*Continuity*) *$K_i\widehat{+}_\varphi L_i \to K\widehat{+}_\varphi L$, i.e., $\lim\limits_{i\to\infty} \rho(K_i\widehat{+}_\varphi L_i, x) = \rho(K\widehat{+}_\varphi L, x)$ for all $x \in \mathbb{R}^n$, as $K_i \to K$ and $L_i \to L$ in the radial Hausdorff metric.*

(2) (*Monotonicity*) *$K_1\widehat{+}_\varphi L_1 \subset K_2\widehat{+}_\varphi L_2$ as $K_1 \subset K_2$ and $L_1 \subset L_2$.*

(3) (*$GL(n)$ Covariance*) *$A(K\widehat{+}_\varphi L) = AK\widehat{+}_\varphi AL,\ A \in GL(n)$.*

Proof (1) Since $\varphi\big(\frac{\rho(K_i\widehat{+}_\varphi L_i,x)}{\rho(K_i,x)}, \frac{\rho(K_i\widehat{+}_\varphi L_i,x)}{\rho(L_i,x)}\big) = 1$, by the continuity of φ, we have

$$\varphi\Big(\frac{\lim\limits_{i\to\infty}\rho(K_i\widehat{+}_\varphi L_i, x)}{\rho(K, x)}, \frac{\lim\limits_{i\to\infty}\rho(K_i\widehat{+}_\varphi L_i, x)}{\rho(L, x)}\Big) = 1.$$

Hence, $\lim\limits_{i\to\infty} \rho(K_i\widehat{+}_\varphi L_i, x) = \rho(K\widehat{+}_\varphi L, x)$.

(2) By the monotonicity of φ, (2) is easy to get.

(3) Since $\varphi\big(\frac{\rho(AK\widehat{+}_\varphi AL,x)}{\rho(AK,x)}, \frac{\rho(AK\widehat{+}_\varphi AL,x)}{\rho(AL,x)}\big) = 1$, we have

$$\varphi\Big(\frac{\rho(A^{-1}(AK\widehat{+}_\varphi AL), A^{-1}x)}{\rho(K, A^{-1}x)}, \frac{\rho(A^{-1}(AK\widehat{+}_\varphi AL), A^{-1}x)}{\rho(L, A^{-1}x)}\Big) = 1.$$

Set $A^{-1}x = y$. Then

$$\varphi\Big(\frac{\rho(A^{-1}(AK\widehat{+}_\varphi AL), y)}{\rho(K, y)}, \frac{\rho(A^{-1}(AK\widehat{+}_\varphi AL), y)}{\rho(L, y)}\Big) = 1.$$

So $A^{-1}(AK\widehat{+}_\varphi AL) = K\widehat{+}_\varphi L$.

Proof of Theorem 1.1 Set $K_\epsilon = K\widehat{+}_{\varphi,\epsilon}L$. It is easy to see that $\rho_{K_\epsilon}(u) \to \rho_K(u)$ for each $u \in S^{n-1}$. Then

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{\rho_{K_\epsilon}(u)^n - \rho_K(u)^n}{\epsilon} \\
&= \lim_{\epsilon\downarrow 0} n\rho_K(u)^{n-1} \frac{\rho_{K_\epsilon}(u) - \rho_K(u)}{\epsilon} \\
&= \lim_{\epsilon\downarrow 0} n\rho_K(u)^n \Big(\frac{\frac{\rho_{K_\epsilon}(u)}{\rho_K(u)} - 1}{\epsilon} \Big).
\end{aligned}$$

From the definition of $K\widehat{+}_{\varphi,\epsilon}L$, we have $\frac{\rho_{K_\epsilon}(u)}{\rho_K(u)} = \varphi_1^{-1}\big(1 - \epsilon\varphi_2\big(\frac{\rho_{K_\epsilon}(u)}{\rho_L(u)}\big)\big)$. Then

$$\frac{\frac{\rho_{K_\epsilon}(u)}{\rho_K(u)} - 1}{\epsilon} = -\varphi_2\Big(\frac{\rho_{K_\epsilon}(u)}{\rho_L(u)}\Big) \frac{\varphi_1^{-1}\Big(1 - \epsilon\varphi_2\Big(\frac{\rho_{K_\epsilon}(u)}{\rho_L(u)}\Big)\Big) - 1}{1 - \epsilon\varphi_2\Big(\frac{\rho_{K_\epsilon}(u)}{\rho_L(u)}\Big) - 1}.$$

Let $z = \varphi_1^{-1}\big(1 - \epsilon\varphi_2\big(\frac{\rho_{K_\epsilon}(u)}{\rho_L(u)}\big)\big)$ and note that $z \to 1-$ as $\epsilon \downarrow 0$. Consequently,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{\rho_{K_\epsilon}(u)^n - \rho_K(u)^n}{\epsilon} \\
&= -\lim_{\epsilon\downarrow 0} n\rho_K(u)^n \varphi_2\Big(\frac{\rho_{K_\epsilon}(u)}{\rho_L(u)}\Big) \frac{z - 1}{\varphi_1(z) - \varphi_1(1)} \\
&= -\frac{n}{(\varphi_1)'_l(1)} \varphi_2\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big) \rho_K(u)^n.
\end{aligned}$$

Since $\rho_{K_\epsilon}(u)$ is monotonic with respect to ϵ, by Lemma 3.1 (2) and Theorem 7.11 of [15], we have $\lim_{\epsilon\downarrow 0} \rho_{K_\epsilon}(u) = \rho_K(u)$ uniformly on S^{n-1}. Hence,

$$\lim_{\epsilon\downarrow 0} \frac{\rho_{K_\epsilon}(u)^n - \rho_K(u)^n}{\epsilon} = -\frac{n}{(\varphi_1)'_l(1)} \varphi_2(\frac{\rho_K(u)}{\rho_L(u)}) \rho_K(u)^n \quad \text{uniformly on } S^{n-1}.$$

Therefore, by the polar coordinate formula for volume (2.1), we obtain

$$-\frac{(\varphi_1)'_l(1)}{n} \lim_{\epsilon\downarrow 0} \frac{V(K\widehat{+}_{\varphi,\epsilon}L) - V(K)}{\epsilon} = \frac{1}{n} \int_{S^{n-1}} \varphi_2\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big) \rho_K(u)^n \mathrm{d}S(u).$$

For $\varphi \in \Phi$, the dual Orlicz mixed volume $\widetilde{V}_\varphi(K,L)$ is defined by

$$\widetilde{V}_\varphi(K,L) = \frac{1}{n} \int_{S^{n-1}} \varphi\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big) \rho_K(u)^n \mathrm{d}S(u) \tag{3.1}$$

for all $K, L \in \mathcal{S}_0^n$. If $\varphi(t) = t^p$, $p \geq 1$, we get the dual mixed volume $\widetilde{V}_{-p}(K,L)$ (see [13, Proposition 1.9]). The following theorem gives a connection between the dual Orlicz mixed volume and the harmonic Orlicz combination.

Theorem 3.1 *Suppose $\varphi_i \in \Phi$, $i = 1, 2$ and $\varphi(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2)$. For all $K, L \in \mathcal{S}_0^n$, we have*

$$\widetilde{V}_{\varphi_2}(K,L) = -\frac{(\varphi_1)'_l(1)}{n} \lim_{\epsilon\downarrow 0} \frac{V(K\widehat{+}_{\varphi,\epsilon}L) - V(K)}{\epsilon}. \tag{3.2}$$

The following theorems show that the dual Orlicz volume is SL(n) invariant, continuous.

Theorem 3.2 *Suppose $\varphi \in \Phi$. If $K, L \in \mathcal{S}_0^n$ and $A \in SL(n)$, then*

$$\widetilde{V}_\varphi(AK, AL) = \widetilde{V}_\varphi(K, L).$$

Proof By the same method of the proof of Lemma 3.1(3), we get $AK \widehat{+}_{\varphi',\epsilon} AL = K \widehat{+}_{\varphi',\epsilon} L$ for $\varphi' \in \Phi_2$. By Theorem 3.2, we get $\widetilde{V}_\varphi(AK, AL) = \widetilde{V}_\varphi(K, L)$.

By the continuity of φ, we have the following.

Theorem 3.3 *Suppose $\varphi \in \Phi$. If $K, L \in \mathcal{S}_0^n$ and $K_i \to K$, $L_i \to L$ in the radial Hausdorff metric, then $\widetilde{V}_\varphi(K_i, L_i) \to \widetilde{V}_\varphi(K, L)$.*

Theorem 3.4 *Suppose $K, L \in \mathcal{S}_0^n$. If $\varphi_i, \varphi \in \Phi$ and $\varphi_i \to \varphi$, i.e., $\max_{t \in I} |\varphi_i(t) - \varphi(t)| \to 0$ for every compact interval $I \subset \mathbb{R}$, then $\widetilde{V}_{\varphi_i}(K, L) \to \widetilde{V}_\varphi(K, L)$.*

Proof Since $K, L \in \mathcal{S}_0^n$ and $\varphi_i \to \varphi$, we have $\varphi_i\big(\frac{\rho_K(u)}{\rho_L(u)}\big)\rho_K(u)^n \to \varphi\big(\frac{\rho_K(u)}{\rho_L(u)}\big)\rho_K(u)^n$ uniformly on S^{n-1}. By (3.1), we obtain that $\widetilde{V}_{\varphi_i}(K, L) \to \widetilde{V}_\varphi(K, L)$.

4 Inequality of the Dual Orlicz Mixed Volume

In [13], Lutwak proved the dual L_{-p}-mixed volume inequality: If $p \geq 1$, and $K, L \in \mathcal{S}_0^n$, then

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}$$

with equality if and only if K and L are dilates.

For the dual Orlicz mixed volumes, we also establish the dual Orlicz mixed volume inequalities.

The following lemma will be needed in the proof of Theorem 1.2, which is easy to be obtained by the Hölder inequality and the polar coordinate formula for volumes.

Lemma 4.1 (see [13, Proposition 1.10]) *If $K, L \in \mathcal{S}_0^n$, then*

$$\frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+1} \rho_L(u)^{-1} \mathrm{d}S(u) \geq V(K)^{\frac{n+1}{n}} V(L)^{-\frac{1}{n}}$$

with equality if and only if K and L are dilates.

Proof of Theorem 1.2 By Jensen's inequality and Lemma 4.1, we have

$$\begin{aligned}
\widetilde{V}_\varphi(K, L) &= \frac{1}{n} \int_{S^{n-1}} \varphi\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big) \rho_K(u)^n \mathrm{d}S(u) \\
&\geq V(K) \varphi\Big(\frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+1} \rho_L(u)^{-1} \mathrm{d}S(u)\Big) \\
&\geq V(K) \varphi\Big(\frac{V(K)^{\frac{1}{n}}}{V(L)^{\frac{1}{n}}}\Big).
\end{aligned}$$

If the equality holds, by the equality conditions of Jensen's inequality and Lemma 4.1, we have that K and L are dilates. Conversely, if K and L are dilates, it is easy to see that the equality holds.

In Theorem 1.2, if we set $\varphi(t) = t^p,\ p \geq 1$, it leads to the Lutwak's result of the dual L_{-p}-mixed volume.

By Theorem 1.2, we have the following uniqueness result which is the Orlicz version of Proposition 1.11 of [13].

Proposition 4.1 *Suppose $\varphi \in \Phi$, and $\mathcal{M} \subset \mathcal{S}_0^n$ such that $K, L \in \mathcal{M}$. If*

$$\frac{\widetilde{V}_\varphi(K,Q)}{V(K)} = \frac{\widetilde{V}_\varphi(L,Q)}{V(L)} \quad \text{for all } Q \in \mathcal{M},$$

then $K = L$.

Proof Taking $Q = L$ gives $\frac{\widetilde{V}_\varphi(K,L)}{V(K)} = \frac{\widetilde{V}_\varphi(L,L)}{V(L)} = 1$. Now Theorem 1.2 gives $V(L) \geq V(K)$, with equality if and only if K and L are dilates. Taking $Q = K$, we get $V(K) \geq V(L)$. Hence, $V(K) = V(L)$, and K and L must be dilates. Thus $K = L$.

In the following, we give the dual Orlicz-Brunn-Minkowski inequality for the harmonic Orlicz addition.

Proof of Theorem 1.3 By Theorem 1.2, we have

$$\begin{aligned} V(K\widehat{+}_\varphi L) &= \frac{1}{n}\int_{S^{n-1}} \Big(\varphi_1\Big(\frac{\rho(K\widehat{+}_\varphi L, x)}{\rho(K,x)}\Big) + \varphi_2\Big(\frac{\rho(K\widehat{+}_\varphi L, x)}{\rho(L,x)}\Big)\Big)\rho_{K\widehat{+}_\varphi L}(u)^n \mathrm{d}S(u) \\ &= \widetilde{V}_{\varphi_1}(K\widehat{+}_\varphi L, K) + \widetilde{V}_{\varphi_2}(K\widehat{+}_\varphi L, L) \\ &\geq V(K\widehat{+}_\varphi L)\Big(\varphi_1\Big(\frac{V(K\widehat{+}_\varphi L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\Big) + \varphi_2\Big(\frac{V(K\widehat{+}_\varphi L)^{\frac{1}{n}}}{V(L)^{\frac{1}{n}}}\Big)\Big). \end{aligned}$$

If the equality holds, by the equality condition of Theorem 1.2, we have that K and L are dilates. Conversely, if K and L are dilates, it is easy to check that the equality holds.

Let $\varphi(x_1, x_2) = x_1^p + x_2^p$, and we have the following result.

Corollary 4.1 (see [13, Proposition 1.12]) *Suppose $K, L \in \mathcal{S}_0^n$. If $p \geq 1$, then*

$$V(K\widehat{+}_p L)^{-\frac{p}{n}} \geq V(K)^{-\frac{p}{n}} + V(L)^{-\frac{p}{n}}$$

with equality if and only if K and L are dilates.

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