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On the First Hochschild Cohomology of Admissible Algebras*

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Abstract The aim of this paper is to investigate the first Hochschild cohomology of admissible algebras which can be regarded as a generalization of basic algebras. For this purpose, the authors study differential operators on an admissible algebra. Firstly, differential operators from a path algebra to its quotient algebra as an admissible algebra are discussed. Based on this discussion, the first cohomology with admissible algebras as coefficient modules is characterized, including their dimension formula. Besides, for planar quivers, the k-linear bases of the first cohomology of acyclic complete monomial algebras and acyclic truncated quiver algebras are constructed over the field k of characteristic 0.

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1 Introduction

The Hochschild cohomology of algebras is invariant under Morita equivalence. Hence it is enough to consider basic connected algebras when the algebras are Artinian. Let $\Gamma = (V, E)$ be a finite connected quiver, where V (resp. E) is the set of vertices (resp. arrows) in Γ. Let k be an arbitrary field and $k\Gamma$ be the corresponding path algebra. Denote by R the two-sided ideal of $k\Gamma$ generated by E. Recall that an ideal I is called admissible if there exists $m \geq 2$ such that $R^m \subseteq I \subseteq R^2$ (see [2]). According to the Gabriel theorem, a finite dimensional basic k-algebra over an algebraically closed field k is in the form of $k\Gamma/I$ for a finite quiver Γ and an admissible idea I.

An Artinian algebra is called a monomial algebra (see [3]) if it is isomorphic to a quotient $k\Gamma/I$ of a path algebra $k\Gamma$ for a finite quiver Γ and an idea I of $k\Gamma$ generated by some paths in Γ. In particular, denote by $k^n\Gamma$ the ideal of $k\Gamma$ generated by all paths of length n. Then the monomial algebra $k\Gamma/k^n\Gamma$ is called the n-truncated quiver algebra.

The study of Hochschild cohomology of quiver related algebras started with the paper of Happel in 1989 (see [11]), who gave the dimensions of Hochschild cohomology of arbitrary

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orders of path algebras for acyclic quivers. Afterwards, there have been extensive studies on the Hochschild cohomology of quiver-related algebras such as truncated quiver algebras, monomial algebras, schurian algebras and 2-nilpotent algebras (see [1, 4, 6–7, 12–15, 17–19]). In [11], a minimal projective resolution of a finite-dimensional algebra A over its enveloping algebra is described in terms of the combinatorics when the field k is an algebraically closed field. In these papers listed above, the authors used this kind of projective resolution or its improving version to compute the Hochschild cohomology.

In [10], the authors applied an explicit and combinatorial method to study $HH^1(k\Gamma)$. In this paper, we improve the method in [10] to the case of algebras with relations in order to study the $HH^1(k\Gamma/I)$, where $k\Gamma/I$ is an admissible algebra. This way does not depend on projective resolution and the requirement of k being an algebraically closed field. Using this method, we can obtain some structural results which would not arise by the classical method in the above listed papers.

If $I \subseteq R^2$ holds for a two-sided ideal I, we call $k\Gamma/I$ an admissible algebra (see Definition 2.1). So finite-dimensional basic algebras are always admissible algebras. We will give Proposition 2.1, which shows that admissible algebras, including basic algebras, possess the similar characterization of monomial algebras and truncated quiver algebras, although they are not graded. From this point of view, the admissible algebra is motivated to unify and generalize the basic algebra and the monomial algebra.

In the following, we always assume that $k\Gamma/I$ is an admissible algebra. This paper includes three sections except for the introduction. In Section 2, we introduce the basic definitions which are used in this paper. In particular, we define the notion of an acyclic admissible algebra, which can be thought as a generalization of the notion of an acyclic quiver. A sufficient and necessary condition is obtained for a linear operator from $k\Gamma$ to $k\Gamma/I$ to be a differential operator. Next, we give a standard basis of $\mathrm{Diff}(k\Gamma, k\Gamma/I)$.

In Section 3, we investigate $H^1(k\Gamma, k\Gamma/I)$. In (3.3), a dimension-formula of $H^1(k\Gamma, k\Gamma/I)$ is given for a finite-dimensional admissible algebra. Moreover, in Theorem 3.1, we construct a basis of $H^1(k\Gamma, k\Gamma/I)$ when Γ is planar and $k\Gamma/I$ is an acyclic admissible algebra.

In Section 4, we characterize $HH^1(k\Gamma/I)$. In (4.2), we give the dimension-formula of $HH^1(k\Gamma/I)$ for any finite-dimensional admissible algebras $k\Gamma/I$. Moreover, we apply this method to complete monomial algebras and truncated quiver algebras. In Theorems 4.1–4.2, we construct k-linear bases of their first cohomology groups under certain conditions. The Hochschild cohomology of monomial algebras and truncated quiver algebras has been studied in [7, 12, 16, 18–19]. Our results in Section 4 can be seen as the generalization of those corresponding conclusions in the listed references above. In the same section, two examples of admissible algebras are given which are not monomial algebras. Their first Hochschild cohomology is characterized using our theory.

2 The k-Linear Basis of $\mathrm{Diff}(k\Gamma, k\Gamma/I)$

We always assume $\Gamma = (V, E)$ to be a finite connected quiver, where V (resp. E) is the set of vertices (resp. arrows) in Γ. For a path p, denote its starting vertex by $t(p)$, called the tail of p, and the ending point by $h(p)$, called the head of p. For two paths p and q, if $t(p) = t(q)$ and $h(p) = h(q)$, we say p and q are parallel, denoted by $p \parallel q$. Denote by $\mathcal{P} = \mathcal{P}_\Gamma$ the set of paths in a quiver Γ including its vertices; denote by $\mathcal{P}_A$ the set of its acyclic paths. Trivially, Γ is acyclic if and only if $\mathcal{P}_\Gamma \backslash V = \mathcal{P}_A$. Throughout this paper, we always assume quivers to be finite and connected.

Definition 2.1 *Suppose $\Gamma = (V, E)$ is a quiver and I is a two-sided ideal of $k\Gamma$. We call the quotient algebra $k\Gamma/I$ an admissible algebra if $I \subsetneqq R^2$, where R denotes the two-sided ideal of $k\Gamma$ generated by E.*

Proposition 2.1 *Suppose $k\Gamma/I$ is an admissible algebra, and then there exists a subset $\mathcal{P}'$ of $\mathcal{P}$ such that $V \cup E \subseteq \mathcal{P}'$ and $\mathcal{Q} = \{\overline{x} \mid x \in \mathcal{P}'\}$ forms a basis of $k\Gamma/I$ for $\overline{x} = x + I$.*

Proof Let X be a k-linear basis of I. Denote by $\mathcal{P}_{\geq 2}$ the set of all paths of length ≥ 2. Define

$$T := \{Y \subseteq k\Gamma : Y \text{ is linearly independent in } k\Gamma \text{ satisfying } X \subseteq Y \subseteq X \cup \mathcal{P}_{\geq 2}\}.$$

T becomes a partial set due to the order of inclusion between subsets of $k\Gamma$. It is easy to see that $T \neq \emptyset$ and T satisfies the upper-bound condition of chains. So by the famous Zorn's lemma, T has a maximal element, denoted by Z.

We claim that Z is linearly equivalent to $\mathcal{P}_{\geq 2}$. Otherwise, there exists $p \in \mathcal{P}_{\geq 2}$ such that p can not be linearly expressed by Z, and then $Z \cup \{p\}$ is linearly independent in $k\Gamma$, which contradicts the maximal property of Z.

Since Z is linearly equivalent to $\mathcal{P}_{\geq 2}$, it follows that $V \cup E \cup Z$ is linearly equivalent to $\mathcal{P} = V \cup E \cup \mathcal{P}_{\geq 2}$. By the definition of T, $Z \subseteq X \cup \mathcal{P}_{\geq 2}$. I is generated by X. Hence $V \cup E \cup (Z \backslash X)$ forms a basis of the complement space of I in $k\Gamma$. It means that $\mathcal{Q} = \{\overline{x} : x \in V \cup E \cup (Z \backslash X)\}$ forms a basis of $k\Gamma/I$. It is clear that $V \cup E \cup (Z \backslash X) \subseteq \mathcal{P}$ and it is the $\mathcal{P}'$ we want.

When $I \subseteq R^2$ is finite dimensional, we have an explicit way to determine the $\mathcal{P}'$. Concretely, suppose that $\{x_1, x_2, \cdots, x_m\}$ is a basis of I. Then there exists a finite subset $\{p_1, p_2, \cdots, p_n\}$ of $\mathcal{P}$ such that x_i can be expressed by the linear combinations of p_j. Suppose $x_i = \sum\limits_{j=1}^{n} a_{ij} p_j$ for $i = 1, 2, \cdots, m$, and then we obtain an $m \times n$ matrix $A = (a_{ij})$. We can transform the matrix A into a row-ladder matrix $B = (b_{ij})$ through only row transformations. Suppose that $b_{i,c(i)}$ is the first nonzero number of the i-th row of B. Since B is a row-ladder matrix, we have $c_i \neq c_k$ for $i \neq k$. Then $\{x_1, x_2, \cdots, x_m\} \cup \{p_l \mid l \neq c_1, c_2, \cdots, c_m\}$ is linearly equivalent to $\{p_1, p_2, \cdots, p_n\}$. Hence $(\mathcal{P} \backslash \{p_1, p_2, \cdots, p_n\}) \cup \{p_l \mid l \neq c_1, c_2, \cdots, c_m\}$ is a basis of the complement space of I in $k\Gamma$. Then the residue classes in $k\Gamma/I$ of all elements in this basis form a basis of $k\Gamma/I$.

On the other hand, in some special cases, e.g., when $k\Gamma/I$ is a monomial algebra, even if I is not finite dimensional, the choice of $\mathcal{P}'$ is also given in the same way. If $k\Gamma/I$ is a monomial algebra and I is generated by a set of paths of length ≥ 2, the set of paths that does not belong to I is just the $\mathcal{P}'$ required.

Definition 2.2 *Let A be a k-algebra and M an A-bimodule. A differential operator (or say, derivation) from A into M is a k-linear map $D : A \to M$ such that*

$$D(xy) = D(x)y + xD(y). \tag{2.1}$$

In particular, when $M = A$, this coincides with the differential operator of algebras.

Lemma 2.1 *Suppose that D is a differential operator from $k\Gamma$ into $k\Gamma/I$. Then D is determined by its action on the set V of vertices of Γ and the set E of arrows of Γ.*

Lemma 2.2 *Let Γ be a quiver. Denote by kV (resp. kE) the linear space spanned by the set V of the vertices of Γ (resp. the set E of the arrows of Γ). Assume that we have a pair of linear maps $D_0 : kV \to k\Gamma/I$ and $D_1 : kE \to k\Gamma/I$ satisfying that*

$$D_0(x)x + xD_0(x) = D_0(x), \quad x \in V, \tag{2.2}$$

$$D_0(x)y + xD_0(y) = 0, \quad x, y \in V,\ x \neq y, \tag{2.3}$$

$$D_0(x)q + xD_1(q) = D_1(q), \quad x \in V,\ q \in E,\ t(q) = x, \tag{2.4}$$

$$D_1(q)y + qD_0(y) = D_1(q), \quad y \in V,\ q \in E,\ h(q) = y. \tag{2.5}$$

Then, the pair of linear maps (D_0, D_1) can be uniquely extended to a differential operator $D : k\Gamma \to\ k\Gamma/I$ satisfying that

$$D(p) := \sum_{i=1}^{l} p_1 \cdots p_{i-1} D_1(p_i) \cdots p_l \tag{2.6}$$

for any path $p = p_1 p_2 \cdots p_l,\ p_i \in E,\ 1 \le i \le l,\ l \ge 2$.

Proof One only need to prove that D is indeed a differential operator. For this, we need to check (2.1) in the next four cases:

(a) $x, y \in V$; (b) $x \in V,\ y \in \mathcal{P}\backslash V$; (c) $x \in \mathcal{P}\backslash V,\ y \in V$; (d) $x, y \in \mathcal{P}\backslash V$.

However, the checking process is routine, so we omit it here.

In the sequel, we always suppose $k\Gamma/I$ to be an admissible algebra for the given ideal I and the notations in Definition 2.1 are used. From Definition 2.1 and Proposition 2.1, there exists a basis of $k\Gamma/I$ which consists of residue classes of some paths including that of V and E. Denote the fixed basis of $k\Gamma/I$ by $\mathcal{Q}$. Suppose that $D : k\Gamma \to k\Gamma/I$ is a linear operator, then for any $p \in \mathcal{P}$, $D(p)$ is a unique combination of the basis $\mathcal{Q}$ of $k\Gamma/I$. Write this linear combination by

$$D(p) = \sum_{\overline{q} \in \mathcal{Q}} c^p_q \overline{q}, \tag{2.7}$$

where all $c^p_q \in k$. We will use this notation throughout this paper. As a convention, for the empty set $\emptyset$, we say $\sum_{\overline{q} \in \emptyset} c^p_q \overline{q} = 0$.

Lemma 2.3 *Suppose that $q_1, q_2 \in \mathcal{P},\ q_1, q_2 \notin I$ and $\overline{q_1} = \overline{q_2}$ in $k\Gamma/I$, then $t(q_1) = t(q_2),\ h(q_1) = h(q_2)$, i.e., $q_1 \parallel q_2$.*

Proof If $t(q_1) \neq t(q_2)$, then $\overline{q_1} = \overline{t(q_1)}\,\overline{q_1} = \overline{t(q_1)}\,\overline{q_2} = \overline{0}$, a contradiction, so $t(q_1) = t(q_2)$. Similarly, $h(q_1) = h(q_2)$.

According to the lemma above, for $\overline{p} \in \mathcal{Q}$, we can define $t(\overline{p}) := t(q)$ (resp. $h(\overline{p}) := h(q)$) for any path $q \in \mathcal{P}$ satisfying $\overline{q} = \overline{p}$ in $k\Gamma/I$. For a path $s \in \mathcal{P}$ and $\overline{p} \in \mathcal{Q}$, if $t(s) = t(\overline{p})$ and $h(s) = h(\overline{p})$, we say s and $\overline{p}$ are parallel, denoted by $s \parallel \overline{p}$.

Denote

$$\mathcal{Q}_A := \{\overline{p} \in \mathcal{Q} \mid t(\overline{p}) \neq h(\overline{p})\} \quad \text{and} \quad \mathcal{Q}_C := \{\overline{p} \in \mathcal{Q} \mid t(\overline{p}) = h(\overline{p})\}.$$

Moreover, $k\mathcal{Q}_A$ (resp. $k\mathcal{Q}_C$) denotes the subspace of $k\Gamma/I$ generated by $\mathcal{Q}_A$ (resp. $\mathcal{Q}_C$). Clearly, as k-linear spaces, $k\Gamma/I = k\mathcal{Q}_A \oplus k\mathcal{Q}_C$.

Definition 2.3 *Using the above notations, an admissible algebra $k\Gamma/I$ is called acyclic if*

$$\mathcal{Q}_C \backslash \{\overline{v} \mid v \in V\} = \emptyset.$$

It is easy to see from this definition that

(i) the fact whether the given $k\Gamma/I$ is acyclic is independent on the choice of $\mathcal{Q}$;

(ii) if the quiver Γ is acyclic, then $k\Gamma/I$ is acyclic; the converse is not true in general;

(iii) if $k\Gamma/I$ is acyclic, then it is finite dimensional; the converse is not true, e.g., $k\Gamma/k^n\Gamma$ if Γ is a loop for $n \geq 2$.

Proposition 2.2 *Let $D : k\Gamma \to k\Gamma/I$ be a k-linear operator.*

(i) *If D is a differential operator, then*

(a) *for $v \in V$,*

$$D(v) = \sum_{\substack{\overline{q} \in \mathcal{Q} \\ t(\overline{q}) = v \\ h(\overline{q}) \neq v}} c_{\overline{q}}^{v} \overline{q} + \sum_{\substack{\overline{q} \in \mathcal{Q} \\ h(\overline{q}) = v \\ t(\overline{q}) \neq v}} c_{\overline{q}}^{v} \overline{q}; \tag{2.8}$$

(b) *for $p \in E$,*

$$D(p) = \sum_{\substack{\overline{q} \in \mathcal{Q} \\ h(\overline{q}) = t(p) \\ t(\overline{q}) \neq t(p)}} c_{q}^{t(p)} \overline{qp} + \sum_{\substack{\overline{q} \in \mathcal{Q} \\ \overline{q} \parallel p}} c_{\overline{q}}^{p} \overline{q} + \sum_{\substack{\overline{q} \in \mathcal{Q}, \\ t(\overline{q}) = h(p) \\ h(\overline{q}) \neq h(p)}} c_{\overline{q}}^{h(p)} \overline{pq}, \tag{2.9}$$

where the coefficients are subject to the following condition: For any path $\overline{q} \in \mathcal{Q}$ such that $t(\overline{q}) \neq h(\overline{q})$,

$$c_{\overline{q}}^{h(\overline{q})} + c_{\overline{q}}^{t(\overline{q})} = 0. \tag{2.10}$$

(ii) *Conversely, assume that the linear map D from $kV \oplus kE$ to $k\Gamma/I$ satisfies* (2.8)–(2.10), *then D can be uniquely extended linearly to a differential operator as* (2.6).

Proof (i) For a given $v \in V$, since $vv = v$, we have

$$D(v) = D(vv) = D(v)v + vD(v).$$

So by the direct computation, we can get

$$D(v) = \sum_{\substack{\overline{q}\in\mathcal{Q}\\ h(\overline{q})=v}} c^v_{\overline{q}}\overline{q} + \sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})=v}} c^v_{\overline{q}}\overline{q}.$$

Moreover,

$$D(v) = D(v)v + vD(v) = (D(v)v + vD(v))v + vD(v) = D(v)v + vD(v)v + vD(v).$$

So we have $vD(v)v = 0$. That means $\sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})=h(\overline{q})=v}} c^v_{\overline{q}}\overline{q} = 0$. So we get (2.8).

Also, for a given $p \in E$, we have

$$\begin{aligned} D(p) &= D(t(p)ph(p)) \\ &= D(t(p))ph(p) + t(p)D(p)h(p) + t(p)pD(h(p)) \\ &= D(t(p))p + \sum_{\substack{\overline{q}\in\mathcal{Q}\\ \overline{q}\parallel p}} c^p_{\overline{q}}\overline{q} + pD(h(p)). \end{aligned}$$

Since $t(p), h(p) \in V$, by (2.8), we can easily get (2.9).

Let $x, y \in V$, and $x \neq y$. By (2.8),

$$\begin{aligned} D(xy) &= D(x)y + xD(y) \\ &= \sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})=x\\ h(\overline{q})=y}} c^x_{\overline{q}}\overline{q} + \sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})=x\\ h(\overline{q})=y}} c^y_{\overline{q}}\overline{q} \\ &= \sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})=x\\ h(\overline{q})=y}} (c^x_{\overline{q}} + c^y_{\overline{q}})\overline{q}. \end{aligned}$$

But $D(xy) = 0$ since $xy = 0$. So $\sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})=x\\ h(\overline{q})=y}} (c^x_{\overline{q}} + c^y_{\overline{q}})\overline{q} = 0$.

For a path $\overline{q} \in \mathcal{Q}$ such that $t(\overline{q}) \neq h(\overline{q})$, substituting x and y respectively with $t(\overline{q})$ and $h(\overline{q})$, we get (2.10).

(ii) We only need to verify that the conditions of Lemma 2.2 are satisfied. Because the process is straightforward, we leave it to the readers.

Next, we apply Proposition 2.2 to display a standard basis of differential operators from $k\Gamma$ to $k\Gamma/I$, for any admissible algebra $k\Gamma/I$.

Proposition 2.3 (Differential operator $D_{r,\overline{s}}$) *For a quiver $\Gamma = (V, E)$, let $r \in E$ and $s \in \mathcal{P}$ with $r \parallel s$. Define the k-linear operator $D_{r,\overline{s}} : kV \oplus kE \to k\Gamma/I$ satisfying*

$$D_{r,\overline{s}}(p) = \begin{cases} \overline{s}, & p = r \quad \textit{for} \quad p \in E, \\ 0, & p \neq r \quad \textit{for} \quad p \in E \cup V. \end{cases} \tag{2.11}$$

Then, the conditions of Lemma 2.2 are satisfied and thus, $D_{r,\overline{s}}$ can be uniquely extended to a differential operator from $k\Gamma$ to $k\Gamma/I$, denoted still by $D_{r,\overline{s}}$ for convenience.

Proof (2.2)–(2.5) can be checked easily by the definition of $D_{r,\overline{s}}$.

For a given $s \in \mathcal{P}$, we have the corresponding inner differential operator:

$$D_{\overline{s}} : k\Gamma \to k\Gamma/I, \quad D_{\overline{s}}(q) = \overline{sq} - \overline{qs}, \quad \forall q \in \mathcal{P}. \tag{2.12}$$

Theorem 2.1 *Let $\Gamma = (V, E)$ be a quiver and I be an ideal such that $k\Gamma/I$ is an admissible algebra. Then the set*

$$\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \tag{2.13}$$

is a basis of the k-linear space of differential operators from $k\Gamma$ to $k\Gamma/I$, where

$$\mathfrak{B}_1 := \{D_{\overline{s}} \mid \overline{s} \in \mathcal{Q}_A\}, \quad \mathfrak{B}_2 := \{D_{r,\overline{s}} \mid r \in E, \overline{s} \in \mathcal{Q}, r \parallel \overline{s}\}. \tag{2.14}$$

Proof We only need to verify that the operators in $\mathfrak{B}$ are linearly independent and any differential operators can be generated k-linearly by $\mathfrak{B}$.

Step 1 $\mathfrak{B}$ is linearly independent Suppose that there are $c_{\overline{p}}, c_{r,\overline{s}} \in k$ such that

$$\sum_{\substack{\overline{p}\in\mathcal{Q} \\ h(\overline{p})\neq t(\overline{p})}} c_{\overline{p}} D_{\overline{p}} + \sum_{\substack{r\in E \\ \overline{s}\in\mathcal{Q} \\ r\parallel\overline{s}}} c_{r,\overline{s}} D_{r,\overline{s}} = 0. \tag{2.15}$$

Then for any given $\overline{p_0} \in \mathcal{Q}$, $h(\overline{p_0}) \neq t(\overline{p_0})$, by the definitions of $D_{\overline{p}}$ and $D_{r,\overline{s}}$, we have

$$\begin{aligned} 0 &= \sum_{\substack{\overline{p}\in\mathcal{Q} \\ t(\overline{p})\neq h(\overline{p})}} c_{\overline{p}} D_{\overline{p}}(h(\overline{p_0})) + \sum_{\substack{r\in E \\ \overline{s}\in\mathcal{Q} \\ r\parallel\overline{s}}} c_{r,\overline{s}} D_{r,\overline{s}}(h(\overline{p_0})) \\ &= \sum_{\substack{\overline{p}\in\mathcal{Q} \\ t(\overline{p})\neq h(\overline{p})}} c_{\overline{p}} (\overline{ph(p_0)} - \overline{h(p_0)p}) + 0 \\ &= \sum_{\substack{\overline{p}\in\mathcal{Q} \\ t(\overline{p})\neq h(\overline{p})=h(\overline{p_0})}} c_{\overline{p}}\overline{p} - \sum_{\substack{\overline{q}\in\mathcal{Q} \\ h(\overline{q})\neq t(\overline{q})=h(\overline{p_0})}} c_{\overline{q}}\overline{q}. \end{aligned}$$

In the last formula above, $\overline{p} \neq \overline{q}$ always holds. Thus, their coefficients are all zero. In particular, $c_{\overline{p_0}} = 0$ for any $\overline{p_0} \in \mathcal{Q}$ with $h(\overline{p_0}) \neq t(\overline{p_0})$.

Thus, from (2.15), we get that

$$\sum_{\substack{r\in E \\ \overline{s}\in\mathcal{Q} \\ r\parallel\overline{s}}} c_{r,\overline{s}} D_{r,\overline{s}} = 0.$$

Further, for any given $r_0 \in E$, $\overline{s} \in \mathcal{Q}$ with $\overline{s} \parallel r_0$, we have

$$\sum_{\substack{r\in E \\ \overline{s}\in\mathcal{Q} \\ r\parallel\overline{s}}} c_{r,\overline{s}} D_{r,\overline{s}}(r_0) = 0 \Longrightarrow \sum_{\substack{\overline{s}\in\mathcal{Q} \\ r_0\parallel\overline{s}}} c_{r_0,\overline{s}}\overline{s} = 0.$$

It follows that $c_{r_0,\overline{s}} = 0$ for any $r_0 \in E$, $\overline{s} \in \mathcal{Q}$ with $r \parallel \overline{s}$.

Hence, $\mathfrak{B}$ is k-linearly independent.

Step 2 $\mathfrak{B}$ is the set of k-linear generators Let $D : k\Gamma \to k\Gamma/I$ be any differential operator. Then for $v \in V$ and $p \in E$, by (2.8)–(2.10), we have

$$D(v) = \sum_{\substack{\overline{q}\in\mathcal{Q}\\ h(\overline{q})\neq t(\overline{q})=v}} c_{\overline{q}}^{v}\overline{q} + \sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})\neq h(\overline{q})=v}} c_{\overline{q}}^{v}\overline{q}, \tag{2.16}$$

$$D(p) = -\sum_{\substack{\overline{q}\in\mathcal{Q}\\ t(\overline{q})\neq h(\overline{q})=t(p)}} c_{\overline{q}}^{t(\overline{q})}\overline{qp} + \sum_{\substack{\overline{q}\in\mathcal{Q}\\ \overline{q}\|p}} c_{\overline{q}}^{p}\overline{q} + \sum_{\substack{\overline{q}\in\mathcal{Q}\\ h(\overline{q})\neq t(\overline{q})=h(p)}} c_{\overline{q}}^{t(\overline{q})}\overline{pq}. \tag{2.17}$$

We claim that D agrees with the differential operator $\overline{D}$ defined by the linear combination

$$\overline{D} = -\sum_{\substack{\overline{s}\in\mathcal{Q}\\ t(\overline{s})\neq h(\overline{s})}} c_{\overline{s}}^{t(\overline{s})}D_{\overline{s}} + \sum_{\substack{r\in E\\ \overline{s}\in\mathcal{Q}\\ \overline{s}\|r}} c_{\overline{s}}^{r}D_{r,\overline{s}}, \tag{2.18}$$

where $c_{\overline{s}}^{t(\overline{s})}$ and $c_{\overline{s}}^{r}$ come from (2.16)–(2.17). Any path in $\mathcal{P}$ is either a vertex or a product of arrows. Thus by the product rule of differential operators, to show $D = \overline{D}$, we only need to verify that $D(q) = \overline{D}(q)$ for each $q = v \in V$ and $q = p \in E$. The verification is straightforward, so we omit it.

We call the set $\mathfrak{B}$ in Theorem 2.1 the standard basis of the k-linear space $\mathrm{Diff}(k\Gamma, k\Gamma/I)$ generated by all differential operators from $k\Gamma$ to $k\Gamma/I$.

From this theorem, we get $\mathrm{Diff}(k\Gamma, k\Gamma/I) = \mathfrak{D}_1 \oplus \mathfrak{D}_2$, where $\mathfrak{D}_i$ is the k-linear space generated by $\mathfrak{B}_i$ for $i = 1, 2$ in (2.14).

For any $p \in E$, $D_{p,\overline{p}} \in \mathfrak{B}_2$ is called the arrow differential operator from $k\Gamma$ to $k\Gamma/I$. Let $\mathfrak{B}_E := \{D_{p,\overline{p}} \mid p \in E\}$ and $\mathfrak{D}_E := k\mathfrak{B}_E$ is called the space of arrow differential operators.

3 $H^1(k\Gamma, k\Gamma/I)$ for an Admissible Algebra $k\Gamma/I$

Proposition 3.1 *Let $q \in \mathcal{P}$ be such that $h(q) = t(q) = v_0$. We have*

$$D_{\overline{q}} = \sum_{\substack{p\in E\\ t(p)=v_0}} D_{p,\overline{qp}} - \sum_{\substack{r\in E\\ h(r)=v_0}} D_{r,\overline{rq}}. \tag{3.1}$$

Proof Note that both sides of (3.1) are k-linearly generated by differential operators. So, by the product formula of differential operators, we only need to verify that the both sides always agree when they act on the elements of V and E. Since the computation is direct, we omit it here.

Remark 3.1 For $v \in V$, it is clear that $t(v) = h(v) = v$. From Proposition 3.1, we have

$$D_{\overline{v}} = \sum_{\substack{p\in E\\ t(p)=v}} D_{p,\overline{p}} - \sum_{\substack{r\in E\\ h(r)=v}} D_{r,\overline{r}}. \tag{3.2}$$

We call $D_{\overline{v}}$ the vertex differential operator from $k\Gamma$ to $k\Gamma/I$. Let $\mathfrak{D}_V$ denote the linear space spanned by $\{D_{\overline{v}} \mid v \in V\}$, called the space of vertex differential operators. It is clear that $\mathfrak{D}_V$ is a subspace of $\mathfrak{D}_E$.

Lemma 3.1 *Let $p \in \mathcal{P}$, and then $\overline{p}$ is always in the k-subspace $k\{\overline{q} \in \mathcal{Q} \mid \overline{q} \parallel p\}$ generated by $\overline{q}$ with $\overline{q} \parallel p$.*

Proof Suppose $\overline{p} = \sum\limits_{\overline{q} \in \mathcal{Q}} c_{\overline{q}}\overline{q}$, and then $\overline{t(p)ph(p)} = \sum\limits_{\overline{q} \in \mathcal{Q}} c_{\overline{q}}\overline{t(p)qh(p)} = \sum\limits_{\substack{\overline{q} \in \mathcal{Q} \\ \overline{q} \parallel p}} c_{\overline{q}}\overline{q}$.

Corollary 3.1 *Let $q \in \mathcal{P}$ be such that $h(q) = t(q)$. Then $D_{\overline{q}} \in k\mathfrak{B}_2 = \mathfrak{D}_2$.*

Proof For $r \in E$, $r \parallel s \in \mathcal{P}$; by Lemma 3.1, suppose $\overline{s} = \sum\limits_{\substack{\overline{q} \in \mathcal{Q} \\ \overline{q} \parallel s}} c_{\overline{q}}\overline{q}$, and it is clear that $D_{r,\overline{s}} = \sum\limits_{\substack{\overline{q} \in \mathcal{Q} \\ \overline{q} \parallel s}} c_{\overline{q}} D_{r,\overline{q}}$, so then we use Proposition 3.1.

Remark 3.2 For $\overline{q} \in \mathcal{Q}$, $t(\overline{q}) = h(\overline{q})$, from Theorem 2.1 and Corollary 3.1, we know that $D_{\overline{q}} \in k\mathfrak{B}_2 = \mathfrak{D}_2$, but not in $k\mathfrak{B}_1 = \mathfrak{D}_1$. Denote $\mathfrak{D}_C := k\{D_{\overline{q}} \mid \overline{q} \in \mathcal{Q},\ t(\overline{q}) = h(\overline{q})\}$. Then $\mathfrak{D}_C \subseteq \mathfrak{D}_2$ and $\mathfrak{D}_C \cap \mathfrak{D}_1 = 0$.

Denote by Inn-Diff$(k\Gamma, k\Gamma/I)$ the linear space consisting of inner differential operators from $k\Gamma$ to $k\Gamma/I$. Then, Inn-Diff$(k\Gamma, k\Gamma/I) = \mathfrak{D}_1 + \mathfrak{D}_C$. Thus, we have

$$\begin{aligned} H^1(k\Gamma, k\Gamma/I) &= \mathrm{Diff}(k\Gamma, k\Gamma/I)/\text{Inn-Diff}(k\Gamma, k\Gamma/I) \\ &= (\mathfrak{D}_1 + \mathfrak{D}_2)/(\mathfrak{D}_1 + \mathfrak{D}_C) \\ &\cong \mathfrak{D}_2/(\mathfrak{D}_2 \cap \mathfrak{D}_C) \\ &\cong \mathfrak{D}_2/\mathfrak{D}_C. \end{aligned}$$

Since the basis of $k\Gamma/I$ given in Proposition 2.1 contains the residue classes of V and E, we can see that the center of $k\Gamma/I$ as a $k\Gamma$-bimodule and the center of $k\Gamma/I$ as an algebra are the same, denoted by $Z(k\Gamma/I)$.

Proposition 3.2 *Let $k\Gamma/I$ be a finite-dimensional admissible algebra, and then*

$$\dim_k H^1(k\Gamma, k\Gamma/I) = |\mathfrak{B}_2| + \dim_k Z(k\Gamma/I) - |\mathcal{Q}_C|. \tag{3.3}$$

Proof By the discussion above, $\dim_k HH^1(k\Gamma, k\Gamma/I) = |\mathfrak{B}_2| - \dim_k \mathfrak{D}_C$. Then

$$\begin{aligned} \mathfrak{D}_1 \oplus \mathfrak{D}_C = \text{Inn-Diff}(k\Gamma, k\Gamma/I) &\cong (k\Gamma/I)/Z(k\Gamma/I) \\ &\cong (k\mathcal{Q}_C \oplus k\mathcal{Q}_A)/Z(k\Gamma/I) \\ &\cong k\mathcal{Q}_C/(Z(k\Gamma/I)) \oplus k\mathcal{Q}_A \\ &\cong k\mathcal{Q}_C/(Z(k\Gamma/I)) \oplus \mathfrak{D}_1, \end{aligned}$$

where the first isomorphism is assured by (2.12), the second and fourth isomorphisms are trivial, and the third is because of the facts that $Z(k\Gamma/I) \subseteq k\mathcal{Q}_C$ and $Z(k\Gamma/I) \cap k\mathcal{Q}_A = 0$. So $\mathfrak{D}_C \cong k\mathcal{Q}_C/Z(k\Gamma/I)$ as k-linear spaces, and it follows that

$$\dim_k H^1(k\Gamma, k\Gamma/I) = \dim_k \mathfrak{D}_2 - \dim_k \mathfrak{D}_C = |\mathfrak{B}_2| + \dim_k Z(k\Gamma/I) - |\mathcal{Q}_C|. \tag{3.4}$$

If $k\Gamma/I$ is acyclic, then $Z(k\Gamma/I) \cong k$ and $|\mathcal{Q}_C| = |V|$. Thus, we have the following corollary.

Corollary 3.2 *If $k\Gamma/I$ is an acyclic admissible algebra (in particular, if Γ is an acyclic quiver), then*

$$\dim_k H^1(k\Gamma, k\Gamma/I) = |\mathfrak{B}_2| + 1 - |V|. \tag{3.5}$$

On the other hand, when Γ is a planar quiver and $k\Gamma/I$ is an acyclic admissible algebra, we can apply the approach of [10] to give a basis of $HH^1(k\Gamma, k\Gamma/I)$. A planar quiver is a quiver with a fixed embedding into the plane $\mathbb{R}^2$. The set F of faces of a planar quiver Γ is the set of connected component of $\mathbb{R}^2\backslash\Gamma$.

We will need the famous Euler formula on the planar graph (see [5, 9]), which states that for any finite connected planar graph (which can be thought as the underlying graph of a quiver Γ), we have

$$|V| - |E| + |F| = 2. \tag{3.6}$$

For each face of Γ, its boundary is called a primitive cycle. Let $\imath_0$ denote the boundary of the unique unbounded face f_0 of Γ. Let $\Gamma_{\mathbb{P}}$ denote the set of primitive cycles of Γ and $\Gamma_{\mathbb{P}}^- := \Gamma_{\mathbb{P}}\backslash\imath_0$. Then clearly, the set $\Gamma_{\mathbb{P}}$ of primitive cycles of Γ is in bijection with the set F of the faces of Γ. So $|F| = |\Gamma_{\mathbb{P}}|$.

For a face $f \in F$, denote by $\imath_f$ the corresponding primitive cycle of f. Suppose that $\imath_f$ is comprised of an ordered list of arrows $p_1, \cdots, p_s \in E$, and define an operator from $k\Gamma$ to $k\Gamma/I$ by

$$D_{\imath_f} := \pm D_{p_1,\overline{p_1}} \pm \cdots \pm D_{p_s,\overline{p_s}}, \tag{3.7}$$

where a $\pm D_{p_i,\overline{p_i}}$ is $+D_{p_i,\overline{p_i}}$ if p_i is in clockwise direction when viewed from the interior of the face of $\imath_f$ and is $-D_{p_i,\overline{p_i}}$ otherwise. We call $D_{\imath_f}$ a face differential operator from $k\Gamma$ to $k\Gamma/I$. Let $\mathfrak{D}_{\mathbb{P}}$ denote the linear space spanned by $\{D_{\mathbb{P}} \mid \imath \in \Gamma_{\mathbb{P}}\}$, called the space of face differential operators.

The next lemma is similar to Theorem 4.9 in [10].

Lemma 3.2 *Let Γ be a planar quiver with the ground field k of characteristic 0, and then*
(a) $\dim \mathfrak{D}_V = |V| - 1$*;*
(b) $\dim \mathfrak{D}_{\mathbb{P}} = |F| - 1 = |\Gamma_{\mathbb{P}}^-|$*;*
(c) $\mathfrak{D}_V$ *and* $\mathfrak{D}_{\mathbb{P}}$ *are linearly disjoint subspaces of* $\mathfrak{D}_E$*.*

Proof (a) Denote $\gamma_0 = |V|$. Since $\overline{e} = \sum\limits_{i=1}^{\gamma_0} \overline{v_i}$ is the identity of $k\Gamma/I$, which clearly lies in the center of $k\Gamma/I$, we have

$$D_{\overline{e}} = \sum_{i=1}^{\gamma_0} D_{\overline{v_i}} = 0. \tag{3.8}$$

So $\dim \mathfrak{D}_V \le \gamma_0 - 1$. We next prove that $\dim \mathfrak{D}_V \ge \gamma_0 - 1$. We may assume that $\gamma_0 \ge 2$.

We claim that any $\gamma_0 - 1$ elements of $\{D_{\overline{v_i}} \mid i = 1, \cdots, \gamma_0\}$ are linearly independent. In fact, suppose $\sum\limits_{i=1}^{\gamma_0-1} a_i D_{\overline{v_i}} = 0$, where $a_i \in k$, which means that $\sum\limits_{i=1}^{\gamma_0-1} a_i\overline{v_i}$ is in the center of $k\Gamma/I$.

Since Γ is connected, let the vertex v_{γ_0} be connected to v_i by an arrow p for $i \neq \gamma_0$. We may assume that $t(p) = v_i$ and $h(p) = v_{\gamma_0}$. We have

$$a_i \overline{p} = \Big(\sum_{i=1}^{\gamma_0 - 1} a_i \overline{v_i} \Big) \overline{p} = \overline{p} \Big(\sum_{i=1}^{\gamma_0 - 1} a_i \overline{v_i} \Big) = \overline{0}.$$

So $a_i = 0$. Note that Γ is connected, and we can repeat this process to get $a_j = 0$ for any j.

(b) Let $|F| = \gamma_2$. Through simple observation of the planar quiver, we can see that if $p \in E$ is in the boundary, then it is at most in the boundary of two primitive cycles. Note that if $p \in E$ is in the boundary of two primitive cycles $\mathbb{P}_1$ and $\mathbb{P}_2$, then the signs of $D_{p,\overline{p}}$ in $D_{\mathbb{P}_1}$ and $D_{\mathbb{P}_2}$ are opposite. If $p \in E$ is in the boundary of only one primitive cycle $\mathbb{P}$, then $D_{p,\overline{p}}$ occurs twice in $D_{\mathbb{P}}$ with an opposite sign. Thus we have

$$\sum_{j=0}^{\gamma_2 - 1} D_{\mathbb{P}_i} = 0, \tag{3.9}$$

where $\mathbb{P}_0$ denotes the primite cycle corresponding to f_0 as above. So $\dim \mathfrak{D}_{\mathbb{P}} \leq |F| - 1$.

We next prove that $\dim \mathfrak{D}_{\mathbb{P}} \geq |F| - 1$. We may assume that $|F| \geq 2$. Suppose

$$\sum_{j=1}^{\gamma_2 - 1} b_j D_{\mathbb{P}_j} = 0, \tag{3.10}$$

where $b_j \in k$. If $p \in E$ is in the boundary of $\mathbb{P}_0$ and $\mathbb{P}_j$ for $j > 0$, then $\overline{0} = \sum\limits_{j=1}^{\gamma_2 - 1} b_j D_{\mathbb{P}_j}(p) = \pm b_j \overline{p}$. So we have $b_j = 0$. This means that if $\mathbb{P}_j$ and $\mathbb{P}_0$ have a common $p \in E$ in their boundary, then $b_j = 0$. Replace $\mathbb{P}_0$ by $\mathbb{P}_j$, and repeat this process. Since the quiver is connected, we can get $b_j = 0$ for any $j > 0$.

(c) From [10] and Theorem 2.1, we know that $\mathfrak{B}_E^o := \{D_{p,p} \mid p \in E\}$ and $\mathfrak{B}_E := \{D_{p,\overline{p}} \mid p \in E\}$ are k-linearly independent sets in $\mathrm{Diff}(k\Gamma)$ and $\mathrm{Diff}(k\Gamma, k\Gamma/I)$ respectively. Based on this, $D_{\overline{v_i}}$ and $D_{\shortmid_f}$ can be linearly expressed by using $\mathfrak{B}_E$, which is similar to the fact that D_{v_i} and $D_{\mathfrak{c}_f}$ can be linearly expressed by $\mathfrak{B}_E^o$ in [10]. Under this correspondence, referring to Theorem 4.9 in [10] in the same process, we obtain that $\mathfrak{D}_V$ and $\mathfrak{D}_{\mathbb{P}}$ are linearly disjoint subspaces of $\mathfrak{D}_E$.

By this lemma, $\mathfrak{B}_{\mathbb{P}} := \{D_{\mathbb{P}} \mid \shortmid \in \Gamma_{\mathbb{P}}^-\}$ is a basis of $\mathfrak{D}_{\mathbb{P}}$.

Theorem 3.1 *Let Γ be a planar quiver and $k\Gamma/I$ be an acyclic admissible algebra with the ground field k of characteristic* 0*. Then the union set*

$$(\mathfrak{B}_2 \backslash \mathfrak{B}_E) \cup \mathfrak{B}_{\mathbb{P}}$$

is a basis of $H^1(k\Gamma, k\Gamma/I)$.

Proof By the Euler formula and Lemma 3.2, we can get $\mathfrak{D}_E = \mathfrak{D}_V \oplus \mathfrak{D}_{\mathbb{P}}$. Because $k\Gamma/I$ is acyclic, we have $\mathfrak{D}_C = \mathfrak{D}_V$, and then

$$H^1(k\Gamma, k\Gamma/I) \cong \mathfrak{D}_2 / \mathfrak{D}_C$$

$$
\begin{aligned}
&\cong (\mathfrak{D}_E \oplus k\{\mathfrak{B}_2 \backslash \mathfrak{B}_E\})/\mathfrak{D}_V \\
&\cong \mathfrak{D}_{\mathbb{P}} \oplus k\{\mathfrak{B}_2 \backslash \mathfrak{B}_E\} \\
&\cong k\mathfrak{B}_{\mathbb{P}} \oplus k\{\mathfrak{B}_2 \backslash \mathfrak{B}_E\}.
\end{aligned}
$$

4 $HH^1(k\Gamma/I)$ for an Admissible Algebra $k\Gamma/I$

Lemma 4.1 *A differential operator of $k\Gamma/I$ can naturally induce a differential operator from $k\Gamma$ to $k\Gamma/I$. Conversely, a differential operator D from $k\Gamma$ to $k\Gamma/I$ satisfying $D(I) = \overline{0}$ can induce a differential operator of $k\Gamma/I$.*

Proof Denote by p the canonical map from $k\Gamma$ to $k\Gamma/I$. Given a differential operator D of $k\Gamma/I$, we claim that the composition Dp is a differential operator from $k\Gamma$ to $k\Gamma/I$. Note that the canonical map from $k\Gamma$ to $k\Gamma/I$ is an algebra homomorphism, and it can be directly verified. The converse result can be shown directly, too.

For a differential operator D from $k\Gamma$ to $k\Gamma/I$ satisfying $D(I) = \overline{0}$, we denote by $\overline{D}$ the induced differential operator on $k\Gamma/I$. Write

$$\mathfrak{F}(I) := \{D \mid D \in \mathrm{Diff}(k\Gamma, k\Gamma/I),\ D(I) = \overline{0}\}, \quad \mathfrak{F}_i(I) := \{D \mid D \in \mathfrak{D}_i,\ D(I) = \overline{0}\} \text{ for } i = 1, 2.$$

It is clear that $D_{\overline{s}}(I) = \overline{0}$ for $s \in \mathcal{P}$. So $\mathfrak{F}_1(I)$=$\mathfrak{D}_1$ and $\mathfrak{F}(I) = \mathfrak{D}_1 \oplus \mathfrak{F}_2(I)$.

Lemma 4.2 $\mathfrak{F}(I) \cong \mathrm{Diff}(k\Gamma/I)$ *as k-linear spaces.*

Proof The map from $\mathfrak{F}(I)$ to $\mathrm{Diff}(k\Gamma/I)$ is as follows:

$$\mathfrak{F}(I) \to \mathrm{Diff}(k\Gamma/I), \quad D \longmapsto \overline{D}.$$

The proof of Lemma 4.1 assures that the map from $\mathfrak{F}(I)$ to $\mathrm{Diff}(k\Gamma/I)$ is surjective. As for the injectivity, suppose $D_1, D_2 \in \mathfrak{F}(I)$ and $D_1 \neq D_2$, so according to Lemma 2.1, there exists a path $p \in V \cup E$ such that $D_1(p) \neq D_2(p)$. Since $\overline{0} \neq \overline{p} \in k\Gamma/I$, $\overline{D}_1(\overline{p}) \neq \overline{D}_2(\overline{p})$.

By this lemma, we can think $\mathrm{Diff}(k\Gamma/I)$ as a k-subspace of $\mathrm{Diff}(k\Gamma, k\Gamma/I)$.

From Lemma 4.2, we have

$$HH^1(k\Gamma/I) \cong \mathfrak{F}(I)/(\mathfrak{D}_1 \oplus \mathfrak{D}_C) \cong (\mathfrak{D}_1 \oplus \mathfrak{F}_2(I))/(\mathfrak{D}_1 \oplus \mathfrak{D}_C) \cong \mathfrak{F}_2(I)/\mathfrak{D}_C \tag{4.1}$$

as linear spaces. This means that $HH^1(k\Gamma/I)$ can be embedded into $H^1(k\Gamma, k\Gamma/I) \cong \mathfrak{D}_2/\mathfrak{D}_C$. Moreover, we have the next proposition.

Proposition 4.1 *Suppose that $k\Gamma/I$ is a finite-dimensional admissible algebra, and then*

$$\dim_k HH^1(k\Gamma/I) = \dim_k \mathfrak{F}_2(I) + \dim_k Z(k\Gamma/I) - |\mathcal{Q}_C|. \tag{4.2}$$

Proof Note that $k\{\mathcal{Q}_C\}/(Z(k\Gamma/I)) \cong \mathfrak{D}_C$ are linear spaces. By (4.2), we have

$$\dim_k HH^1(k\Gamma/I) = \dim_k \mathfrak{F}_2(I) - \dim_k \mathfrak{D}_C = \dim_k \mathfrak{F}_2(I) + \dim_k Z(k\Gamma/I) - |\mathcal{Q}_C|.$$

Corollary 4.1 *If $k\Gamma/I$ is an acyclic admissible algebra (in particular, if Γ is an acyclic quiver), then*

$$\dim_k HH^1(k\Gamma/I) = \dim_k \mathfrak{F}_2(I) + 1 - |V|. \tag{4.3}$$

If $k\Gamma/I$ is an acyclic admissible algebra, we have a standard procedure to compute $\dim_k \mathfrak{F}_2(I)$. First note that for a differential operator D from $k\Gamma$ to $k\Gamma/I$, $D(I) = \overline{0}$ if and only if $D(r_i) = \overline{0}$ where $\{r_1, \cdots, r_i, \cdots, r_n\}$ is a minimal set of generators of I. This property follows easily from the Leibniz rule of differential operators. Since $k\Gamma/I$ is acyclic, $|\mathfrak{B}_2|$ is finite for $\mathfrak{B}_2$ as given in Theorem 2.1. Suppose that $\sum\limits_{D_{r,\overline{s}} \in \mathfrak{B}_2} c_{r,\overline{s}} D_{r,\overline{s}}(r_i) = \overline{0}$ for $i = 1, \cdots, n$. This means that the coefficients $c_{r,\overline{s}}$ satisfy the system of these homogeneous linear equations. So $\dim \mathfrak{F}_2(I)$ is equal to the dimension of the solution space of the system of homogeneous linear equations.

Now we give two examples of admissible algebras that are not monomial algebras or truncated quiver algebras, and characterize their first Hochschild cohomology.

Example 4.1 Let $\Gamma = (V, E)$ be the quiver

$$\begin{tikzcd} & \cdot \arrow[dl, "\alpha_2"'] & \\ \cdot & & \cdot \arrow[ul, "\alpha_1"'] \arrow[dl, "\beta_1"] \\ & \cdot \arrow[ul, "\beta_2"] & \end{tikzcd}$$

and $I = \langle \alpha_1\alpha_2 - \beta_1\beta_2 \rangle$.

In this case, $\mathfrak{B}_2 = \{D_{\alpha_1,\overline{\alpha_1}}, D_{\alpha_2,\overline{\alpha_2}}, D_{\beta_1,\overline{\beta_1}}, D_{\beta_2,\overline{\beta_2}}\}$. So we have

$$\dim_k H^1(k\Gamma, k\Gamma/I) = |\mathfrak{B}_2| + \dim_k Z(k\Gamma/I) - |\mathcal{Q}_C| = 4 + 1 - 4 = 1.$$

Suppose that

$$(aD_{\alpha_1,\overline{\alpha_1}} + bD_{\alpha_2,\overline{\alpha_2}} + cD_{\beta_1,\overline{\beta_1}} + dD_{\beta_2,\overline{\beta_2}})(\alpha_1\alpha_2 - \beta_1\beta_2) = (a + b - c - d)\overline{\alpha\beta} = \overline{0},$$

and then we get $a + b - c - d = 0$. Hence $\dim_k \mathfrak{F}_2(I) = 3$ and $\dim_k HH^1(k\Gamma/I) = 0$.

Example 4.2 Let Γ be the quiver having one vertex with two loops, or equivalently, $k\Gamma = k\langle x, y\rangle$. Suppose the ideal $I = \langle xy - yx \rangle$. Then $k\Gamma/I = k[x, y]$.

In this case, $\mathfrak{B}_1 = \emptyset$ and $\mathfrak{B}_2 = \{D_{x,x^m y^n}, D_{y,x^m y^n} \mid m, n \geq 0\}$ are the basis of $\mathrm{Diff}(k\langle x, y\rangle, k[x, y])$, where $x^m y^n$ means the multiplication in $k[x, y]$.

Since $k[x, y]$ is commutative, we get that

$$\text{Inn-Diff}(k\langle x, y\rangle, k[x, y]) = 0, \quad H^1(k\langle x, y\rangle, k[x, y]) = \mathrm{Diff}(k\langle x, y\rangle, k[x, y]).$$

Moreover, note that $D_{x,x^m y^n}(xy - yx) = 0$ and $D_{y,x^m y^n}(xy - yx) = 0$. Thus we obtain the basis of $HH^1(k[x, y])$ to be

$$\{D_{x,x^m y^n}, D_{y,x^m y^n} \mid m, n \geq 0\}.$$

Similarly, we can obtain the first Hochschild cohomology for $k[x_1, x_2, \cdots, x_n]$.

Assume that $k\Gamma/I$ is a monomial algebra. The residue classes of paths that do not belong to I form a basis of $k\Gamma/I$. For convenience, we also denote by $\mathcal{Q}$ the basis of $k\Gamma/I$ when $k\Gamma/I$ is a monomial algebra.

Definition 4.1 *A monomial algebra $k\Gamma/I$ is called complete if for any parallel paths p, p' in Γ, $p \in I$ implies $p' \in I$.*

Proposition 4.2 *Suppose that $k\Gamma/I$ is a complete monomial algebra with $I \subseteq R^2$. Then the following set*

$$\overline{\mathfrak{B}} = \overline{\mathfrak{B}}_1 \cup \overline{\mathfrak{B}}_2 \tag{4.4}$$

is a basis of $\mathrm{Diff}(k\Gamma/I)$, *where*

$$\overline{\mathfrak{B}}_1 := \{\overline{D}_{\overline{s}} \mid \overline{s} \in \mathcal{Q}, h(\overline{s}) \neq t(\overline{s})\}, \quad \overline{\mathfrak{B}}_2 := \{\overline{D}_{r,\overline{s}} \mid r \in E, \overline{s} \in \mathcal{Q}, r \parallel \overline{s}\}. \tag{4.5}$$

Proof Since $k\Gamma/I$ is complete, we have $D_{r,\overline{s}}(p) = \overline{0}$ for any $D_{r,\overline{s}} \in \mathfrak{B}_2$, where p is any path in I. Then $\mathfrak{F}_2(I) = \mathfrak{D}_2$. It follows that $\mathrm{Diff}(k\Gamma, k\Gamma/I) \cong \mathrm{Diff}(k\Gamma/I)$ as k-linear spaces. Thus due to Theorem 2.1, the result follows.

Corollary 4.2 *Suppose that $k\Gamma/I$ is an acyclic complete monomial algebra with $I \subseteq R^2$. Then*

$$\dim_k HH^1(k\Gamma/I) = |\overline{\mathfrak{B}}_2| + 1 - |V|. \tag{4.6}$$

Proof By the proof of Proposition 4.2, $\dim_k \mathfrak{F}_2(I) = \dim_k \mathfrak{D}_2 = \dim_k \overline{\mathfrak{D}}_2 = |\overline{\mathfrak{B}}_2|$. By Corollary 4.1, we get the required result.

In [15], the author gave a characterization of the first Hochschild cohomology of an acyclic complete monomial algebra through a projective resolution. However, its k-linear basis has not been constructed so far. Here, we want to reach this aim by our method.

Theorem 4.1 *Let Γ be a planar quiver, and $k\Gamma/I$ be an acyclic complete monomial algebra with $I \subseteq R^2$ over the field k of characteristic 0. Then the union set*

$$(\overline{\mathfrak{B}}_2 \backslash \overline{\mathfrak{B}}_E) \cup \overline{\mathfrak{B}}_{\mathbb{P}}$$

is a basis of $HH^1(k\Gamma/I)$, where $\overline{\mathfrak{B}}_E = \{\overline{D}_{p,\overline{p}} \mid p \in E\}$ and $\overline{\mathfrak{B}}_{\mathbb{P}} = \{\overline{D}_{\imath} \mid \imath \in \Gamma_{\mathbb{P}}^{-}\}$.

Proof By (4.1) and $\mathfrak{F}_2(I) = \mathfrak{D}_2$, we have $HH^1(k\Gamma/I) \cong H^1(k\Gamma, k\Gamma/I)$ in this case. So from Theorem 3.1, we can directly get this theorem.

For a truncated quiver algebra $k\Gamma/k^n\Gamma$ with $n \geq 2$, we can give a standard basis of $\mathrm{Diff}(k\Gamma/k^n\Gamma)$. $k\Gamma/k^n\Gamma$ has the basis formed by the residue classes of the paths of length $\leq n-1$, denoted also by $\mathcal{Q}$.

Proposition 4.3 *Let $\Gamma = (V, E)$ be a quiver and the field k be of characteristic 0. A basis of* $\mathrm{Diff}(k\Gamma/k^n\Gamma)$ *for any truncated quiver algebra $k\Gamma/k^n\Gamma$ with $n \geq 2$ is given by the set*

$$\overline{\mathfrak{B}} = \overline{\mathfrak{B}}_1 \cup \overline{\mathfrak{B}}_2, \tag{4.7}$$

where

$$\overline{\mathfrak{B}}_1 := \{\overline{D}_{\overline{s}} \mid \overline{s} \in \mathcal{Q}, h(\overline{s}) \neq t(\overline{s})\}, \quad \overline{\mathfrak{B}}_2 := \{\overline{D}_{r,\overline{s}} \mid r \in E, \overline{s} \in \mathcal{Q}, s \notin V, r \parallel \overline{s}\}. \tag{4.8}$$

Proof It is clear that $D_{\overline{s}}(k^n\Gamma) = \overline{0}$ for $\overline{s} \in \mathcal{Q}$, $h(\overline{s}) \neq t(\overline{s})$ and $D_{r,\overline{s}}(k^n\Gamma) = \overline{0}$ for $r \in E, \overline{s} \in \mathcal{Q}$, $s \notin V$, $\overline{s} \parallel r$. Note that when r is a loop of Γ, $D_{r,\overline{h(r)}} \in \mathrm{Diff}(k\Gamma, k\Gamma/k^n\Gamma)$, but $D_{r,\overline{h(r)}}(r^n) = nr^{n-1} \neq \overline{0}$. Moreover, for all loops $r_1, \cdots, r_s$ of Γ and $c_1, \cdots, c_s$ not all 0, we claim that $\sum c_i D_{r_i,\overline{h(r_i)}}(k^n\Gamma) \neq \overline{0}$. Without loss of generality, we can assume $c_1 \neq 0$. So we have

$$\sum c_i D_{r_i,\overline{h(r_i)}}(r_1^n) = nc_1 r_1^{n-1} \neq \overline{0}.$$

Then by Theorem 2.1, the union set

$$\{D_{\overline{s}} \mid \overline{s} \in \mathcal{Q}, h(\overline{s}) \neq t(\overline{s})\} \cup \{D_{r,\overline{s}} \mid r \in E, \overline{s} \in \mathcal{Q}, s \notin V, r \parallel \overline{s}\}$$

forms a basis of the linear space $\mathfrak{F}_2(k^n\Gamma)$ for $I = k^n\Gamma$. By Lemma 4.2, we have

$$\mathfrak{F}_2(k^n\Gamma) \cong \mathrm{Diff}(k\Gamma/k^n\Gamma).$$

Noting the map from $\mathfrak{F}_2(k^n\Gamma)$ to $\mathrm{Diff}(k\Gamma/k^n\Gamma)$ in Lemma 4.2, we can see that the union set $\overline{\mathfrak{B}} = \overline{\mathfrak{B}}_1 \cup \overline{\mathfrak{B}}_2$ is a k-linear basis of $\mathrm{Diff}(k\Gamma/k^n\Gamma)$.

Thus $\mathrm{Diff}(k\Gamma/k^n\Gamma) = \overline{\mathfrak{D}}_1 \oplus \overline{\mathfrak{D}}_2$, where $\overline{\mathfrak{D}}_i$ is the k-linear space generated by $\overline{\mathfrak{B}}_i$ for $i = 1, 2$.

Corollary 4.3 *Let $\Gamma = (V, E)$ be a quiver and the field k be of characteristic 0. Then*

$$\dim_k HH^1(k\Gamma/k^n\Gamma) = |\overline{\mathfrak{B}}_2| + \dim_k Z(k\Gamma/k^n\Gamma) - |\mathcal{Q}_C|.$$

Proof By the proof of Proposition 4.3 and the definition of $\mathfrak{F}_2(I)$, we can see that $\{D_{r,\overline{s}} \mid r \in E, \overline{s} \in \mathcal{Q}, s \notin V, r \parallel \overline{s}\}$ is a basis of $\mathfrak{F}_2(k^n\Gamma)$ for $I = k^n\Gamma$. By Proposition 4.3,

$$\overline{\mathfrak{B}}_2 := \{\overline{D}_{r,\overline{s}} \mid r \in E, \overline{s} \in \mathcal{Q}, s \notin V, r \parallel \overline{s}\}.$$

Then by Proposition 4.1 and the correspondence between $D_{r,\overline{s}}$ and $\overline{D}_{r,\overline{s}}$ for each pair $(r, \overline{s})$, we get the required result.

This corollary has indeed been given as Theorem 1 in [12] and Theorem 2 in [18]. The method we obtain here is different from that in [12, 18].

Moreover, when $k\Gamma/k^n\Gamma$ is acyclic, we can get a basis of $HH^1(k\Gamma/k^n\Gamma)$ as in Theorem 4.1.

Theorem 4.2 *Let Γ be a planar quiver, and $k\Gamma/k^n\Gamma$ for $n \geq 2$ be acyclic over the field k of characteristic 0. Then the union set*

$$(\overline{\mathfrak{B}}_2 \backslash \overline{\mathfrak{B}}_E) \cup \overline{\mathfrak{B}}_{\mathbb{P}}$$

is a basis of $HH^1(k\Gamma/k^n\Gamma)$, where $\overline{\mathfrak{B}}_E = \{\overline{D}_{p,\overline{p}} \mid p \in E\}$ and $\overline{\mathfrak{B}}_{\mathbb{P}} = \{\overline{D}_{\imath} \mid \imath \in \Gamma_{\mathbb{P}}^{-}\}$.

Proof Since $k\Gamma/I$ is acyclic, $\mathfrak{D}_C = \mathfrak{D}_V$. By Lemma 4.2, $\overline{\mathfrak{D}}_C \cong \mathfrak{D}_C$, $\overline{\mathfrak{D}}_E \cong \mathfrak{D}_E$, $\overline{\mathfrak{D}}_{\mathbb{P}} \cong \mathfrak{D}_{\mathbb{P}}$. So the result can be obtained in the same way as the proof of Theorem 3.1.

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