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The Zero Mach Number Limit of the Three-Dimensional Compressible Viscous Magnetohydrodynamic Equations*

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Abstract This paper is concerned with the zero Mach number limit of the three-dimensional compressible viscous magnetohydrodynamic equations. More precisely, based on the local existence of the three-dimensional compressible viscous magnetohydrodynamic equations, first the convergence-stability principle is established. Then it is shown that, when the Mach number is sufficiently small, the periodic initial value problems of the equations have a unique smooth solution in the time interval, where the incompressible viscous magnetohydrodynamic equations have a smooth solution. When the latter has a global smooth solution, the maximal existence time for the former tends to infinity as the Mach number goes to zero. Moreover, the authors prove the convergence of smooth solutions of the equations towards those of the incompressible viscous magnetohydrodynamic equations with a sharp convergence rate.

Keywords Compressible viscous MHD equation, Mach number limit, Convergence-stability principle, Incompressible viscous MHD equation, Energy-type error estimate
2000 MR Subject Classification 76W05, 35B40

1 Introduction

This paper is concerned with the isentropic compressible viscous magnetohydrodynamic (MHD for short) equations with a small Mach number (see [18–19]). These equations model the dynamics of compressible quasineutrally ionized fluids under the influence of electromagnetic fields and cover very wide applications of physical objects from liquid metals to cosmic plasmas. In a suitable nondimensional form (see, e.g., [10]), the compressible viscous magnetohydrodynamic equations for an isentropic fluid read as

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\dfrac{\nabla p}{\varepsilon^2}=H\cdot\nabla H-\dfrac{1}{2}\nabla|H|^2+\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u,\\ \partial_t H+u\cdot\nabla H+\operatorname{div}uH-H\cdot\nabla u=\nu\Delta H,\\ \operatorname{div}H=0\end{cases}\tag{1.1}$$

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for $(x,t) \in \Omega \times [0,+\infty)$. Throughout this paper, Ω is assumed to be the 3-dimensional torus. Here the unknown functions are the density ρ, the velocity $u \in \mathbb{R}^3$ and the magnetic field $H \in \mathbb{R}^3$. The pressure $p = p(\rho)$ is a given strictly increasing smooth function of $\rho > 0$. The constants μ and λ are the shear and bulk viscosity coefficients of the flow, satisfying $\mu > 0$ and $2\mu + 3\lambda \geq 0$, respectively, the constant $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and ε is proportional to the Mach number. Note that, when $H = 0$, (1.1) reduces to the compressible Navier-Stokes equation.

It is well-known that the incompressible limit of compressible fluid dynamical equations is an important mathematical problem. Much effort was made for the limit of the compressible Navier-Stokes equations and related models (see [1–3, 6–7, 14], etc.). Recently, Hu and Wang [8] discussed the convergence of weak solutions of the full compressible MHD flows (1.1) to the weak solutions of the incompressible viscous MHD equations in the whole space and the periodic domains, as the Mach number tends to zero. Jiang, Ju and Li [9] employed the modulated energy method to verify the limit of weak solutions of the compressible MHD equation (1.1) in the torus to the strong solutions of the incompressible viscous or partial viscous MHD equation (the shear viscosity coefficient is zero, but the magnetic diffusion coefficient is a positive constant). The authors of [9] also derived the ideal incompressible MHD equation from the compressible MHD equation (1.1) in the whole space $\mathbb{R}^d$ ($d = 2$ or $d = 3$) with general initial data in [10]. That is, when the viscosities (including the shear viscosity coefficient and the magnetic diffusion coefficient) go to zero, they proved that the weak solutions of the compressible MHD equation (1.1) converge to the smooth solutions of the ideal incompressible MHD equation. We remak that these results are all about the weak solutions.

In this paper, we analyze the incompressible limit for smooth solutions of the compressible magnetohydrodynamic equations (1.1). The result can be roughly stated as follows. Suppose that the initial data for (1.1) are smooth and have the form

$$\rho^\varepsilon(x,0) = 1 + O(\varepsilon^2), \quad u^\varepsilon(x,0) = u_0 + O(\varepsilon), \quad H^\varepsilon(x,0) = H_0 + O(\varepsilon)$$

with $u^\varepsilon(x,0)$ and $H^\varepsilon(x,0)$ solenoidal. Let $[0,T_0]$ be a (finite) time interval where the incompressible magnetohydrodynamic equations

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = H^0 \cdot \nabla H^0 - \frac{1}{2}\nabla |H^0|^2 + \mu \Delta u^0, \\ \partial_t H^0 + u^0 \cdot \nabla H^0 - H^0 \cdot \nabla u^0 = \nu \Delta H^0, \\ \mathrm{div} u^0 = 0, \quad \mathrm{div} H^0 = 0 \end{cases} \tag{1.2}$$

with initial data

$$u^0(x,0) = u_0, \quad H^0(x,0) = H_0$$

have a smooth solution. Then, for ε sufficiently small, the compressible magnetohydrodynamic equations have a unique smooth solution defined for $(x,t) \in \Omega \times [0,T_0]$ and satisfying

$$\rho^\varepsilon = 1 + O(\varepsilon^2), \quad u^\varepsilon = u^0 + O(\varepsilon), \quad H^\varepsilon = H^0 + O(\varepsilon).$$

Unlike those in [8–10], our result contains a sharp convergence rate and the existence time interval for (1.1) is optimal. Our analysis is guided by the spirit of the convergence-stability principle developed in [20–21] for singular limit problems of symmetrizable hyperbolic systems.

In this approach, we will not derive any ε-uniform a priori estimate. Instead, we only need to obtain the error estimate in Theorem 2.3. Finally, we also thank the anonymous referees for telling us about the paper [11–12]. Indeed, we completed our manuscript in 2012. Comparing with [11–12], we consider the convergence of solutions to be on the time interval where a smooth solution of the limit equations exists, and we also obtain the sharp convergence rate. These were pointed out by the anonymous referees. Moreover, our method here is different from that of [11–12].

The paper is organized as follows. Our main ideas and results are outlined in Section 2. All required (error) estimates are obtained in Section 3.

Notation 1.1 $|U|$ denotes some norm of a vector or matrix U. For a nonnegative integer k, $H^k = H^k(\Omega)$ denotes the usual L^2-type Sobolev space of order k. We write $\|\cdot\|_k$ for the standard norm of H^k, and $\|\cdot\|$ for $\|\cdot\|_0$. When U is a function of another variable t as well as $x \in \Omega$, we write $\|U(\cdot,t)\|$ to recall that the norm is taken with respect to x, while t is viewed as a parameter. In addition, we denote by $C([0,T],\mathbf{X})$ (resp. $L^2([0,T],\mathbf{X})$) the space of continuous (resp. square integrable) functions on $[0,T]$ with values in a Banach space $\mathbf{X}$.

2 Main Ideas and Results

Our analysis is guided by the spirit of the convergence-stability principle developed in [20–21] for singular limit problems of symmetrizable hyperbolic systems.

To explain the main ideas, we firstly reformulate the compressible MHD equations (1.1) in terms of the pressure p, the velocity u and the magnetic field H. Since $p = p(\rho)$ is strictly increasing, it has an inverse $\rho = \rho(p)$. Set

$$q(p) = [\rho(p)p'(\rho(p))]^{-1}.$$

Then the compressible MHD equation for smooth solutions is equivalent to

$$\begin{cases} q(p)(p_t + u\cdot\nabla p) + \mathrm{div}u = 0, \\ \rho(p)(u_t + u\cdot\nabla u) + \varepsilon^{-2}\nabla p = H\cdot\nabla H - \dfrac{1}{2}\nabla|H|^2 + \mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u, \\ \partial_t H + u\cdot\nabla H + H\mathrm{div}u - H\cdot\nabla u = \nu\Delta H, \\ \mathrm{div}H = 0. \end{cases}$$

Moreover, we introduce

$$\widetilde{p} = \frac{p - p_0}{\varepsilon}, \quad \widetilde{u} = u, \quad \widetilde{H} = H$$

with $p_0 > 0$ being constant. Then the above equation can be rewritten as

$$\begin{cases} q(p_0 + \varepsilon\widetilde{p})(\widetilde{p}_t + \widetilde{u}\cdot\nabla\widetilde{p}) + \varepsilon^{-1}\mathrm{div}\widetilde{u} = 0, \\ \rho(p_0 + \varepsilon\widetilde{p})(\widetilde{u}_t + \widetilde{u}\cdot\nabla\widetilde{u}) + \varepsilon^{-1}\nabla\widetilde{p} = \widetilde{H}\cdot\nabla\widetilde{H} - \dfrac{1}{2}\nabla|\widetilde{H}|^2 + \mu\Delta\widetilde{u} + (\mu+\lambda)\nabla\mathrm{div}\widetilde{u}, \\ \partial_t\widetilde{H} + \widetilde{u}\cdot\nabla\widetilde{H} + \widetilde{H}\mathrm{div}\widetilde{u} - \widetilde{H}\cdot\nabla\widetilde{u} = \nu\Delta\widetilde{H}, \\ \mathrm{div}\widetilde{H} = 0. \end{cases} \tag{2.1}$$

For (2.1), we have the following local existence of the classical solution of the initial value problem.

Lemma 2.1 *Let $p = p(\rho)$ be a smooth function. Assume $(\overline{p}, \overline{u}, \overline{H}) = (\overline{p}, \overline{u}, \overline{H})(x) \in H^3$. Then there exists a positive constant $T_0 > 0$, such that* (2.1) *with initial data $(\overline{p}, \overline{u}, \overline{H})$ has a unique classical solution $(\widetilde{p}, \widetilde{u}, \widetilde{H}) = (\widetilde{p}, \widetilde{u}, \widetilde{H})(x, t)$, satisfying $\varepsilon\widetilde{p}(x, t) + p_0 > 0$ for all $(x, t) \in \mathbb{R}^3 \times [0, T]$ and*

$$\widetilde{p} \in C([0, T], H^3), \quad \widetilde{u}, \widetilde{H} \in C([0, T], H^3) \cap L^2([0, T], H^4).$$

The proof of Lemma 2.1 is similar to that in [16–17] for the compressible Navier-Stokes equation and the details can be found in [13].

Now we fix $\varepsilon \in (0, 1]$. According to Lemma 2.1, there is a time interval $[0, T]$, such that (2.1) with initial data $(\overline{p}, \overline{u}, \overline{H})(x, \varepsilon)$ has a unique solution $(p^\varepsilon, u^\varepsilon, H^\varepsilon)$ satisfying $\varepsilon p^\varepsilon + p_0 > 0$ for all $(x, t) \in \mathbb{R}^3 \times [0, T]$ and

$$p^\varepsilon \in C([0, T], H^3), \quad u^\varepsilon, H^\varepsilon \in C([0, T], H^3) \cap L^2([0, T], H^4).$$

Define

$$\begin{aligned} T_\varepsilon = \sup\Big\{ & T > 0 : p^\varepsilon \in C([0, T], H^3),\ u^\varepsilon, H^\varepsilon \in C([0, T], H^3) \cap L^2([0, T], H^4), \\ & -\frac{1}{2}p_0 \le \varepsilon p^\varepsilon(x, t) \le 2p_0,\ \forall (x, t) \in \mathbb{R}^3 \times [0, T]\Big\}. \end{aligned} \tag{2.2}$$

(Here the "2" can be replaced by any positive number larger than 1.) Namely, $[0, T_\varepsilon)$ is the maximal time interval of H^3-existence. Note that T_ε may tend to 0 as ε goes to 0.

In order to show that $\lim\limits_{\varepsilon\to 0} T_\varepsilon > 0$, we follow the convergence-stability principle [21] and seek a formal approximation of $(p^\varepsilon, u^\varepsilon, H^\varepsilon)$. To this end, we consider the initial-value problem (IVP for short) of the incompressible viscous magnetohydrodynamic equations:

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = H^0 \cdot \nabla H^0 - \frac{1}{2}\nabla |H^0|^2 + \mu\Delta u^0, \\ \partial_t H^0 + u^0 \cdot \nabla H^0 - H^0 \cdot \nabla u^0 = \nu\Delta H^0, \\ \mathrm{div} u^0 = 0, \quad \mathrm{div} H^0 = 0, \\ u^0(x, 0) = u_0(x), \quad H^0(x, 0) = H_0(x). \end{cases} \tag{2.3}$$

Since $(u_0, H_0) \in H^4$, we know from [5, 19] that the following lemma holds.

Lemma 2.2 *There exists $T_0 \in (0, +\infty)$, such that the IVP* (2.3) *of the incompressible viscous magnetohydrodynamic equations has a unique smooth solution*

$$(u^0, H^0, p^0) \in C([0, T_0], H^4),$$

satisfying

$$\sup_{0\le t\le T_0} \|(u^0, H^0, p^0)(\cdot, t)\|_4 + \|(\partial_t u^0, \partial_t p^0)(\cdot, t)\|_2 < \infty.$$

In the next section, we will prove the following theorem.

Theorem 2.1 *Suppose that $p = p(\rho)$ is smooth and satisfies $p'(\rho) > 0$ for $\rho > 0$, and $u_0, H_0 \in H^4$ are both divergence-free. Then there exist constants $K = K(T_0)$ and $\varepsilon_0 = \varepsilon_0(T_0)$, such that for all $\varepsilon \le \varepsilon_0$,*

$$\|p^\varepsilon(\cdot, t) - \varepsilon p^0(\cdot, t)\|_3 + \|u^\varepsilon(\cdot, t) - u^0(\cdot, t)\|_3 + \|H^\varepsilon(\cdot, t) - H^0(\cdot, t)\|_3 \le K\varepsilon$$

for $t \in [0, \min\{T_0, T_\varepsilon\})$.

Having this theorem, we slightly modify the arguments in [20] to prove the following result.

Theorem 2.2 *Under the conditions of Theorem* 2.1, *there exists a constant* $\varepsilon_0 = \varepsilon_0(T_0)$, *such that for all* $\varepsilon \le \varepsilon_0$,

$$T_\varepsilon > T_0.$$

Proof Otherwise, there is a sequence $\{\varepsilon_k\}_{k\ge 1}$, such that $\lim\limits_{k\to\infty} \varepsilon_k = 0$ and $T_{\varepsilon_k} \le T_0$. Thanks to the error estimate in Theorem 2.1, there exists a k, such that $4p^{\varepsilon_k}(x,t) \in (-p_0, 5p_0)$ for all x and t. Next we deduce from

$$\begin{aligned}
&\|p^{\varepsilon_k}(\cdot,t)\|_3 + \|u^{\varepsilon_k}(\cdot,t)\|_3 + \|H^{\varepsilon_k}(\cdot,t)\|_3 \\
&\le \|p^{\varepsilon_k}(\cdot,t) - \varepsilon p^0(\cdot,t)\|_3 + \|\varepsilon p^0(\cdot,t)\|_3 + \|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_3 \\
&\quad + \|u^0(\cdot,t)\|_3 + \|H^{\varepsilon_k}(\cdot,t) - H^0(\cdot,t)\|_3 + \|H^0(\cdot,t)\|_3
\end{aligned}$$

and Lemma 2.2 that $\|p^{\varepsilon_k}(\cdot,t)\|_3 + \|u^{\varepsilon_k}(\cdot,t)\|_3 + \|H^{\varepsilon_k}(\cdot,t)\|_3$ is bounded uniformly with respect to $t \in [0, T_{\varepsilon_k})$. Now we could use Lemma 2.1, beginning at a time t less than T_{ε_k}, to continue this solution beyond T_{ε_k}. This contradicts the definition of T_ε in (2.2).

By combining Theorems 2.1 and 2.2, we achieve our main result as follows.

Theorem 2.3 *Suppose that* $p = p(\rho)$ *is smooth and satisfies* $p'(\rho) > 0$ *for* $\rho > 0$, *and that* $u_0, H_0 \in H^4$ *are both divergence-free. Denote by* $T_0 > 0$ *the life-span of the unique classical solution* $(u^0, H^0)(x,t) \in C([0,T_0], H^4)$ *to the incompressible viscous magnetohydrodynamic equations* (1.2) *with initial data* (u_0, H_0). *If* $T_0 < \infty$, *then, for* ε *sufficiently small, the compressible magnetohydrodynamic equation* (1.1) *with initial data*

$$\rho^\varepsilon(x,0) = 1, \quad u^\varepsilon(x,0) = u_0, \quad H^\varepsilon(x,0) = H_0$$

has a unique solution $(\rho^\varepsilon, u^\varepsilon, H^\varepsilon)(x,t)$ *satisfying*

$$\rho^\varepsilon - 1 \in C([0,T_0], H^3), \quad u^\varepsilon, H^\varepsilon \in C([0,T_0], H^3).$$

Moreover, there exists a constant $K > 0$, *independent of* ε *but dependent on* T_0, *such that*

$$\sup_{t\in[0,T_0]} (\|(\rho^\varepsilon - 1)(\cdot,t)\|_3 + \|(u^\varepsilon - u^0)(\cdot,t)\|_3 + \|(H^\varepsilon - H^0)(\cdot,t)\|_3) \le K\varepsilon. \tag{2.4}$$

In the case $T_0 = \infty$, *the maximal existence time* T_ε *of* $(\rho^\varepsilon, u^\varepsilon, H^\varepsilon)$ *tends to infinity as* ε *goes to zero.*

Remark 2.1 The initial data

$$\rho^\varepsilon(x,0) = 1, \quad u^\varepsilon(x,0) = u_0, \quad H^\varepsilon(x,0) = H_0$$

can be relaxed as

$$\rho^\varepsilon(x,0) = 1 + O(\varepsilon^2), \quad u^\varepsilon(x,0) = u_0 + O(\varepsilon), \quad H^\varepsilon(x,0) = H_0 + O(\varepsilon)$$

without changing our arguments.

We conclude this section with the following interesting remarks, which is a by-product of our approach.

Remark 2.2 The proof of Theorem 2.1 requires $T_0 < \infty$. However, when the IVP (2.3) of the incompressible viscous magnetohydrodynamic equations has a global-in-time regular solution, T_0 can be any positive number. Hence we have an almost global-in-time existence result for (2.1) as follows:

$$\lim_{\varepsilon \to 0} T_\varepsilon = +\infty.$$

Remark 2.3 In terms of the formal expansion, $\varepsilon p^0, u^0$ and H^0 are the zero-order profile of the solutions $p^\varepsilon, u^\varepsilon$ and H^ε, respectively. Therefore, the convergence rate in (2.4) is sharp and optimal.

3 Error Estimate

In this section, we prove the error estimate in Theorem 2.1. For this purpose, we need the following classical calculus inequalities in Sobolev spaces (see [14]).

Lemma 3.1 (i) *For $s \ge 2$, $H^s = H^s(\mathbb{R}^3)$ is an algebra, namely, for $f, g \in H^s$, it holds that $fg \in H^s$ and*

$$\|fg\|_s \le C_s \|f\|_s \|g\|_s.$$

(ii) *For $s \ge 3$, let $f \in H^s$ and $g \in H^{s-1}$. Then for all multi-indices α with $|\alpha| \le s$, it holds that $[\partial_x^\alpha, f]g \in L^2$ and*

$$\|[\partial_x^\alpha, f]g\| \le C_s \|\nabla f\|_{s-1} \|g\|_{s-1}.$$

Here C_s is a generic constant depending only on s.

We notice that, with u^0, H^0 and p^0 as constructed in Lemma 2.2,

$$(p_\varepsilon, u_\varepsilon, H_\varepsilon) := (\varepsilon p^0, u^0, H^0)$$

satisfies

$$\begin{cases} q(p_0 + \varepsilon p_\varepsilon)(p_{\varepsilon t} + u_\varepsilon \cdot \nabla p_\varepsilon) + \varepsilon^{-1} \mathrm{div} u_\varepsilon = \varepsilon R_1, \\ \rho(p_0 + \varepsilon p_\varepsilon)(u_{\varepsilon t} + u_\varepsilon \cdot \nabla u_\varepsilon) + \varepsilon^{-1} \nabla p_\varepsilon \\ \quad = H_\varepsilon \cdot \nabla H_\varepsilon - \frac{1}{2} \nabla |H_\varepsilon|^2 + \mu \Delta u_\varepsilon + (\mu + \lambda) \nabla \mathrm{div} u_\varepsilon + R_2, \\ \partial_t H_\varepsilon + u_\varepsilon \cdot \nabla H_\varepsilon + H_\varepsilon \mathrm{div} u_\varepsilon - H_\varepsilon \cdot \nabla u_\varepsilon = \nu \Delta H_\varepsilon \end{cases} \tag{3.1}$$

with

$$\begin{aligned} R_1 &= q(p_0 + \varepsilon^2 p^0)(p_t^0 + u^0 \cdot \nabla p^0), \\ R_2 &= (\rho(p_0 + \varepsilon^2 p^0) - \rho(p_0))(u_t^0 + u^0 \cdot \nabla u^0). \end{aligned}$$

From Lemma 2.2, it follows that

$$\begin{aligned} \max_{t \in [0, T_0]} \|q^{-1}(p_0 + \varepsilon^2 p^0) R_1(\cdot, t)\|_3 \le C, \\ \max_{t \in [0, T_0]} \|\rho^{-1}(p_0 + \varepsilon^2 p^0) R_2(\cdot, t)\|_3 \le C\varepsilon^2. \end{aligned} \tag{3.2}$$

Here and below C is a generic positive constant.

Set

$$P = p^\varepsilon - p_\varepsilon, \quad U = u^\varepsilon - u_\varepsilon, \quad \mathcal{H} = H^\varepsilon - H_\varepsilon.$$

Note that u_ε is divergence-free. We deduce from (2.1) and (3.1) that

$$P_t + u^\varepsilon \cdot \nabla P + U \cdot \nabla p_\varepsilon + \varepsilon^{-1} q^{-1}(p_0 + \varepsilon p^\varepsilon)\mathrm{div} U = f_1, \tag{3.3}$$

$$\begin{aligned} &U_t + u^\varepsilon \cdot \nabla U + U \cdot \nabla u_\varepsilon + \varepsilon^{-1}\rho^{-1}(p_0 + \varepsilon p^\varepsilon)\nabla P \\ &- \rho^{-1}(p_0 + \varepsilon p^\varepsilon)(\mu \Delta U + (\mu + \lambda)\nabla \mathrm{div} U) \\ = {}& \rho^{-1}(p_0 + \varepsilon p^\varepsilon)\Big(\mathcal{H} \cdot \nabla H_\varepsilon + H^\varepsilon \cdot \nabla \mathcal{H} - \frac{1}{2}\nabla(|H^\varepsilon|^2 - |H_\varepsilon|^2)\Big) + f_2 \end{aligned} \tag{3.4}$$

and

$$\partial_t \mathcal{H} + u^\varepsilon \cdot \nabla \mathcal{H} + (\mathrm{div} U)H^\varepsilon - H^\varepsilon \cdot \nabla U + U \cdot \nabla H_\varepsilon - \mathcal{H} \cdot \nabla u_\varepsilon = \nu \Delta \mathcal{H}, \tag{3.5}$$

where f_1 and f_2 are given by

$$f_1 = -q^{-1}(p_0 + \varepsilon p_\varepsilon)\varepsilon R_1$$

and

$$\begin{aligned} f_2 = {}& -\rho^{-1}(p_0 + \varepsilon p_\varepsilon)R_2 - \varepsilon^{-1}(\rho^{-1}(p_0 + \varepsilon p^\varepsilon) - \rho^{-1}(p_0 + \varepsilon p_\varepsilon))\nabla p_\varepsilon \\ &+ \Big(H_\varepsilon \cdot \nabla H_\varepsilon - \frac{1}{2}\nabla |H_\varepsilon|^2\Big)(\rho^{-1}(p_0 + \varepsilon p^\varepsilon) - \rho^{-1}(p_0 + \varepsilon p_\varepsilon)) \\ &+ (\rho^{-1}(p_0 + \varepsilon p^\varepsilon) - \rho^{-1}(p_0 + \varepsilon p_\varepsilon))\mu \Delta u_\varepsilon, \end{aligned}$$

respectively. Let α be a multi-index with $|\alpha| \le 3$. Differentiating the two sides of the equations in (3.3)–(3.5) with ∂_x^α and setting

$$P_\alpha = \partial_x^\alpha P, \quad U_\alpha = \partial_x^\alpha U, \quad \mathcal{H}_\alpha = \partial_x^\alpha \mathcal{H}, \quad f_{i\alpha} = \partial_x^\alpha f_i, \quad i = 1, 2,$$

we obtain

$$\begin{aligned} &\partial_t P_\alpha + u^\varepsilon \cdot \nabla P_\alpha + \varepsilon^{-1} q^{-1}(p_0 + \varepsilon p^\varepsilon)\mathrm{div} U_\alpha \\ = {}& f_{1\alpha} - [\partial_x^\alpha, u^\varepsilon]\nabla P - \partial_x^\alpha(U \cdot \nabla p_\varepsilon) - \varepsilon^{-1}[\partial_x^\alpha, q^{-1}(p_0 + \varepsilon p^\varepsilon)]\mathrm{div} U, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\partial_t U_\alpha + u^\varepsilon \cdot \nabla U_\alpha + \varepsilon^{-1}\rho^{-1}(p_0 + \varepsilon p^\varepsilon)\nabla P_\alpha - \rho^{-1}(p_0 + \varepsilon p^\varepsilon)(\mu \Delta U_\alpha + (\mu + \lambda)\nabla \mathrm{div} U_\alpha) \\ = {}& \rho^{-1}(p_0 + \varepsilon p^\varepsilon)\Big(H^\varepsilon \cdot \nabla \mathcal{H}_\alpha - \frac{1}{2}\partial_x^\alpha \nabla(|H^\varepsilon|^2 - |H_\varepsilon|^2)\Big) + f_{2\alpha} - [\partial_x^\alpha, u^\varepsilon]\nabla U - \partial_x^\alpha(U \cdot \nabla u_\varepsilon) \\ &- \varepsilon^{-1}[\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon p^\varepsilon)]\nabla P + [\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon p^\varepsilon)](\mu \Delta U + (\mu + \lambda)\nabla \mathrm{div} U) \\ &+ [\partial_x^\alpha, \rho^{-1}(p_0 + \varepsilon p^\varepsilon)]\Big(\mathcal{H} \cdot \nabla H_\varepsilon + H^\varepsilon \cdot \nabla \mathcal{H} - \frac{1}{2}\nabla(|H^\varepsilon|^2 - |H_\varepsilon|^2)\Big) \\ &+ \rho^{-1}(p_0 + \varepsilon p^\varepsilon)(\partial_x^\alpha(\mathcal{H} \cdot \nabla H_\varepsilon) + [\partial_x^\alpha, H^\varepsilon]\nabla \mathcal{H}), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\partial_t \mathcal{H}_\alpha + u^\varepsilon \cdot \nabla \mathcal{H}_\alpha + (\mathrm{div} U_\alpha)H^\varepsilon - H^\varepsilon \cdot \nabla U_\alpha + \partial_x^\alpha(U \cdot \nabla H_\varepsilon) \\ &+ [\partial_x^\alpha, u^\varepsilon]\nabla \mathcal{H} + [\partial_x^\alpha, H^\varepsilon]\mathrm{div} U - [\partial_x^\alpha, H^\varepsilon]\nabla U - \partial_x^\alpha(\mathcal{H} \cdot \nabla u_\varepsilon) = \nu \Delta \mathcal{H}_\alpha. \end{aligned} \tag{3.8}$$

First, taking the inner product of (3.8) with $\mathcal{H}_\alpha$ over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |H_\alpha|^2\mathrm{d}x + \nu\int_\Omega |\nabla\mathcal{H}_\alpha|^2\mathrm{d}x \\
=& -\int_\Omega \mathcal{H}_\alpha u^\varepsilon\cdot\nabla\mathcal{H}_\alpha\mathrm{d}x + \int_\Omega \mathcal{H}_\alpha\Big(H^\varepsilon\cdot\nabla U_\alpha - (\mathrm{div}U_\alpha)H^\varepsilon\Big)\mathrm{d}x \\
& -\int_\Omega \mathcal{H}_\alpha\partial_x^\alpha(U\cdot\nabla H_\varepsilon - \mathcal{H}\cdot\nabla u_\varepsilon) + [\partial_x^\alpha, u^\varepsilon]\nabla\mathcal{H} - [\partial_x^\alpha, H^\varepsilon]\nabla U + [\partial_x^\alpha, H^\varepsilon]\mathrm{div}U)\mathrm{d}x \\
=:& \mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3.
\end{aligned}\tag{3.9}$$

Here we use integration by parts for the term $\nu\Delta\mathcal{H}_\alpha$. It is easy to see that

$$\mathrm{K}_1 = -\frac{1}{2}\int_\Omega u^\varepsilon\cdot\nabla|\mathcal{H}_\alpha|^2\mathrm{d}x = \frac{1}{2}\int_\Omega |\mathcal{H}_\alpha|^2\mathrm{div}u^\varepsilon\mathrm{d}x \le |\mathrm{div}u^\varepsilon|_{L^\infty}\|\mathcal{H}_\alpha\|^2, \tag{3.10}$$

$$\mathrm{K}_2 \le \delta(\|\mathrm{div}U_\alpha\|^2 + \|\nabla U_\alpha\|^2) + C\|\mathcal{H}_\alpha\|^2, \tag{3.11}$$

where δ is a small positive number to be determined.

For $|\mathrm{div}u^\varepsilon|_{L^\infty}$ and other terms in the sequel, we follow [20–21] and formulate the following lemma.

Lemma 3.2 *Set*

$$D = D(t) = \sqrt{\|P(\cdot,t)\|_3^2 + \|U(\cdot,t)\|_3^2 + \|\mathcal{H}(\cdot,t)\|_3^2}$$

for $t\in[0,\min\{T_0,T_\varepsilon\})$. *Then for multi-indices* β *satisfying* $|\beta|\le 1$, *it holds that*

$$|\partial_x^\beta u^\varepsilon| + |\partial_x^\beta p^\varepsilon| + |\partial_x^\beta H^\varepsilon| \le C(1+D).$$

Proof It is obvious from Lemma 2.2 and the Sobolev inequality that

$$|\partial_x^\beta u^\varepsilon| \le |\partial_x^\beta(u^\varepsilon - u_\varepsilon)| + |\partial_x^\beta u^0| \le CD + C.$$

The other estimates can be showed similarly. This completes the proof.

For K_3, we use Lemmas 3.1 and 3.2 to deduce that

$$\begin{aligned}
\mathrm{K}_3 \le\ & C\|\mathcal{H}_\alpha\|(\|\partial_x^\alpha(U\cdot\nabla H_\varepsilon)\| + \|\partial_x^\alpha(\mathcal{H}\cdot\nabla u_\varepsilon)\| \\
& + \|[\partial_x^\alpha, u^\varepsilon]\nabla\mathcal{H}\| + \|[\partial_x^\alpha, H^\varepsilon]\nabla U\| + \|[\partial_x^\alpha, H^\varepsilon]\mathrm{div}U\|) \\
\le\ & C\|\mathcal{H}_\alpha\|(\|U\|_3\|\nabla H_\varepsilon\|_3 + \|\mathcal{H}\|_3\|\nabla u_\varepsilon\|_3 \\
& + \|\nabla u^\varepsilon\|_2\|\nabla\mathcal{H}\|_2 + \|\nabla H^\varepsilon\|_2\|\nabla U\|_2 + \|\nabla H^\varepsilon\|_2\|\mathrm{div}U\|_2) \\
\le\ & C(1+D)(\|\mathcal{H}\|_3^2 + \|U\|_3^2).
\end{aligned}\tag{3.12}$$

Therefore, putting (3.10)–(3.12) into (3.9) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H_\alpha\|^2 + 2\nu\|\nabla\mathcal{H}_\alpha\|^2 \le 2\delta(\|\mathrm{div}U_\alpha\|^2 + \|\nabla U_\alpha\|^2) + C(1+D)(\|\mathcal{H}\|_3^2 + \|U\|_3^2). \tag{3.13}$$

Next we take the inner product of (3.6) and (3.7) with $q(p_0+\varepsilon p^\varepsilon)P_\alpha$ and $\rho(p_0+\varepsilon p_\varepsilon)U_\alpha$, respectively, and sum up the resultant equalities to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (q(p_0+\varepsilon p^\varepsilon)P_\alpha^2+\rho(p_0+\varepsilon p^\varepsilon)|U_\alpha|^2)\mathrm{d}x+\int_\Omega (\mu|\nabla U_\alpha|^2+(\mu+\lambda)|\mathrm{div}U_\alpha|^2)\mathrm{d}x\\
=&\int_\Omega \Big(\frac{1}{2}q(p_0+\varepsilon p^\varepsilon)_t P_\alpha^2+\frac{1}{2}\rho(p_0+\varepsilon p^\varepsilon)_t|U_\alpha|^2\\
&-q(p_0+\varepsilon p^\varepsilon)P_\alpha(u^\varepsilon\cdot\nabla)P_\alpha-\rho(p_0+\varepsilon p^\varepsilon)U_\alpha(u^\varepsilon\cdot\nabla)U_\alpha\Big)\mathrm{d}x\\
&+\int_\Omega \Big(H^\varepsilon\cdot\nabla\mathcal{H}_\alpha-\frac{1}{2}\partial_x^\alpha\nabla(|H^\varepsilon|^2-|H_\varepsilon|^2)\Big)U_\alpha\mathrm{d}x\\
&+\int_\Omega ((\partial_x^\alpha(\mathcal{H}\cdot\nabla H_\varepsilon)+[\partial_x^\alpha,H^\varepsilon]\nabla\mathcal{H})U_\alpha-([\partial_x^\alpha,u^\varepsilon]\nabla P+\partial_x^\alpha(U\cdot\nabla p_\varepsilon))q(p_0+\varepsilon p^\varepsilon)P_\alpha\\
&-(\partial_x^\alpha(U\cdot\nabla u_\varepsilon)+[\partial_x^\alpha,u^\varepsilon]\nabla U)\rho(p_0+\varepsilon p^\varepsilon)U_\alpha)\mathrm{d}x\\
&-\frac{1}{\varepsilon}\int_\Omega (q(p_0+\varepsilon p^\varepsilon)P_\alpha[\partial_x^\alpha,q^{-1}(p_0+\varepsilon p^\varepsilon)]\mathrm{div}U+\rho(p_0+\varepsilon p^\varepsilon)U_\alpha[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon p^\varepsilon)]\nabla P)\mathrm{d}x\\
&+\int_\Omega \rho(p_0+\varepsilon p^\varepsilon)U_\alpha[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon p^\varepsilon)](\mu\Delta U+(\mu+\lambda)\nabla\mathrm{div}U)\mathrm{d}x\\
&+\int_\Omega \rho(p_0+\varepsilon p^\varepsilon)U_\alpha[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon p^\varepsilon)]\Big(H^\varepsilon\cdot\nabla\mathcal{H}+\mathcal{H}\cdot\nabla H_\varepsilon-\frac{1}{2}\nabla(|H^\varepsilon|^2-|H_\varepsilon|^2)\Big)\mathrm{d}x\\
&+\int_\Omega (q(p_0+\varepsilon p^\varepsilon)f_{1\alpha}P_\alpha+\rho(p_0+\varepsilon p^\varepsilon)f_{2\alpha}U_\alpha)\mathrm{d}x\\
=:&\ \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6+\mathrm{I}_7. \qquad (3.14)
\end{aligned}$$

Now we turn to estimate the I_i's in (3.14). Using integration by parts, $(2.1)_1$ and Lemma 3.2, we deduce that

$$\begin{aligned}
\mathrm{I}_1=&\frac{1}{2}\int_\Omega ([q'(p_0+\varepsilon p^\varepsilon)\varepsilon p_t^\varepsilon+q(p_0+\varepsilon p^\varepsilon)\mathrm{div}u^\varepsilon+u^\varepsilon\cdot\nabla q(p_0+\varepsilon p^\varepsilon)]P_\alpha^2\\
&+[\rho'(p_0+\varepsilon p^\varepsilon)\varepsilon p_t^\varepsilon+\rho(p_0+\varepsilon p^\varepsilon)\mathrm{div}u^\varepsilon+u^\varepsilon\cdot\nabla\rho(p_0+\varepsilon p^\varepsilon)]|U_\alpha|^2)\mathrm{d}x\\
=&\frac{1}{2}\int_\Omega (q(p_0+\varepsilon p^\varepsilon)\mathrm{div}u^\varepsilon+u^\varepsilon\cdot\nabla q(p_0+\varepsilon p^\varepsilon))P_\alpha^2\mathrm{d}x\\
&+\frac{1}{2}\int_\Omega (\rho(p_0+\varepsilon p^\varepsilon)\mathrm{div}u^\varepsilon+u^\varepsilon\cdot\nabla\rho(p_0+\varepsilon p^\varepsilon))|U_\alpha|^2\mathrm{d}x\\
&-\frac{\varepsilon}{2}\int_\Omega (q'(p_0+\varepsilon p^\varepsilon)P_\alpha^2+\rho'(p_0+\varepsilon p^\varepsilon)|U_\alpha|^2)\Big(u^\varepsilon\cdot\nabla p^\varepsilon+\frac{\mathrm{div}u^\varepsilon}{\varepsilon q(p_0+\varepsilon p^\varepsilon)}\Big)\mathrm{d}x\\
\le& C\|\mathrm{div}u^\varepsilon\|_{L^\infty}(\|U_\alpha\|^2+\|P_\alpha\|^2)+C\varepsilon\|u^\varepsilon\cdot\nabla p^\varepsilon\|_{L^\infty}(\|U_\alpha\|^2+\|P_\alpha\|^2)\\
\le& C(1+D^2)(\|U_\alpha\|^2+\|P_\alpha\|^2).
\end{aligned}$$

For I_2, it follows from Lemmas 3.1–3.2 that

$$\begin{aligned}
\mathrm{I}_2=&\int_\Omega H^\varepsilon\cdot\nabla\mathcal{H}_\alpha U_\alpha\mathrm{d}x+\frac{1}{2}\int_\Omega \partial_x^\alpha(|H^\varepsilon|^2-|H_\varepsilon|^2)\mathrm{div}U_\alpha\mathrm{d}x\\
\le&\delta\|\nabla\mathcal{H}_\alpha\|^2+C\|U_\alpha\|^2+\delta\|\mathrm{div}U_\alpha\|^2+C\|\partial_x^\alpha(\mathcal{H}(H^\varepsilon+H_\varepsilon))\|^2\\
\le&\delta\|\nabla\mathcal{H}_\alpha\|^2+\delta\|\mathrm{div}U_\alpha\|^2+C\|U_\alpha\|^2+C\|\mathcal{H}\|_3^2\|H^\varepsilon+H_\varepsilon\|_3^2\\
\le&\delta\|\nabla\mathcal{H}_\alpha\|^2+\delta\|\mathrm{div}U_\alpha\|^2+C(1+D^2)(\|\mathcal{H}\|_3^2+\|U_\alpha\|^2).
\end{aligned}$$

Like K_3, I_3 can be simply estimated as

$$\mathrm{I}_3\le C(\|\mathcal{H}\|_3^2+\|U\|_3^2+\|P\|_3^2).$$

In order to treat other terms, we compute that

$$\|\nabla\rho^{-1}(p_0+\varepsilon p^\varepsilon)\|_2,\ \|\nabla q^{-1}(p_0+\varepsilon p^\varepsilon)\|_2 \le C\varepsilon(1+D^3).$$

Thus, for I_4 we have

$$\begin{aligned}
\mathrm{I}_4 \le{}& \frac{1}{\varepsilon}(\|[\partial_x^\alpha, q^{-1}(p_0+\varepsilon p^\varepsilon)]\mathrm{div}U\|\|q(p_0+\varepsilon p^\varepsilon)P_\alpha\| \\
&+\|[\partial_x^\alpha, \rho^{-1}(p_0+\varepsilon p^\varepsilon)]\nabla P\|\|\rho(p_0+\varepsilon p^\varepsilon)U_\alpha\|) \\
\le{}& \frac{C}{\varepsilon}(\|\nabla q^{-1}(p_0+\varepsilon p^\varepsilon)\|_2\|\mathrm{div}U\|_2\|P_\alpha\| + \|\nabla\rho^{-1}(p_0+\varepsilon p^\varepsilon)\|_2\|\nabla P\|_2\|U_\alpha\|) \\
\le{}& C(1+D^3)(\|\mathrm{div}U\|_2\|P_\alpha\| + \|\nabla P\|_2\|U_\alpha\|) \\
\le{}& C(1+D^3)(\|U\|_3^2+\|P\|_3^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{I}_5 &\le C\|\nabla\rho^{-1}(p_0+\varepsilon p^\varepsilon)\|_2\|\mu\Delta U+(\mu+\lambda)\nabla\mathrm{div}U\|_2\|U_\alpha\| \\
&\le \delta\|\nabla U\|_3^2 + C(1+D^6)\|U_\alpha\|^2, \\
\mathrm{I}_6 &\le C(1+D^4)\|U_\alpha\|\|\mathcal{H}\|_3.
\end{aligned}$$

Finally, from the definitions of f_1 and f_2, we deduce that

$$\mathrm{I}_7 \le C\varepsilon^2 + C(1+D^6)(\|P\|_3^2+\|U\|_3^2) + \delta\|\nabla U_\alpha\|^2.$$

Putting the above estimates into (3.14), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (q(p_0+\varepsilon p^\varepsilon)P_\alpha^2 + \rho(p_0+\varepsilon p^\varepsilon)|U_\alpha|^2)\mathrm{d}x + \mu\|\nabla U_\alpha\|^2 \\
&\le C\varepsilon^2 + \delta(\|\nabla U\|_3^2 + \|\mathrm{div}U_\alpha\|^2 + \|\nabla\mathcal{H}_\alpha\|^2) + C(1+D^6)(\|U\|_3^2+\|P\|_3^2+\|\mathcal{H}\|_3^2).
\end{aligned}$$

Combining the last inequality with (3.13), we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|P\|_3^2+\|U\|_3^2+\|\mathcal{H}\|_3^2) + (\|\nabla U\|_3^2+\|\nabla\mathcal{H}\|_3^2) \\
&\le C(1+D^6)(\|P\|_3^2+\|U\|_3^2+\|\mathcal{H}\|_3^2) + C\varepsilon^2.
\end{aligned}$$

We integrate this inequality from 0 to T with $[0,T]\subset[0,\min\{T_\varepsilon,T_0\})$ to obtain

$$\begin{aligned}
&\|P\|_3^2+\|U\|_3^2+\|\mathcal{H}\|_3^2 + \int_0^T (\|\nabla U\|_3^2+\|\nabla\mathcal{H}\|_3^2)\mathrm{d}t \\
&\le CT\varepsilon^2 + C\int_0^T (1+D^6)(\|P\|_3^2+\|U\|_3^2+\|\mathcal{H}\|_3^2)\mathrm{d}t.
\end{aligned}$$

Here we use the fact that the initial data are in equilibrium. Furthermore, we apply the Gronwall's lemma to the last inequality to get

$$\|P\|_3^2+\|U\|_3^2+\|\mathcal{H}\|_3^2 \le CT_0\varepsilon^2\exp\Big[C\int_0^T(1+D^6)\mathrm{d}t\Big]. \tag{3.15}$$

Since $\|P\|_3^2+\|U\|_3^2+\|\mathcal{H}\|_3^2 = D^2$, it follows from (3.15) that

$$D(T)^2 \le CT_0\varepsilon^2\exp\Big[C\int_0^T(1+D^6)\mathrm{d}t\Big] \equiv Q(T). \tag{3.16}$$

Thus, it holds that

$$Q'(t) = C(1 + D^6)Q(t) \leq CQ(t) + CQ^4(t).$$

Applying the nonlinear Gronwall-type inequality in [20] to the last inequality yields

$$Q(t) \leq \mathrm{e}^{CT_0}$$

for $t \in [0, \min\{T_0, T_\varepsilon\})$ if we choose ε so small that

$$Q(0) = CT_0\varepsilon^2 \leq \mathrm{e}^{-CT_0}.$$

Thus, it follows from (3.16) that $D(T) \leq \mathrm{e}^{\frac{CT_0}{2}}$ for $T \in [0, \min\{T_0, T_\varepsilon\})$. Finally, Theorem 2.1 is concluded from (3.15). This completes the proof.

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