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Existence and Global Asymptotic Behavior of Positive Solutions for Sublinear and Superlinear Fractional Boundary Value Problems*

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Abstract In this paper, the authors aim at proving two existence results of fractional differential boundary value problems of the form

$$(P_{a,b}) \quad \begin{cases} D^\alpha u(x) + f(x, u(x)) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad D^{\alpha-3}u(0) = a, \quad u'(1) = -b, \end{cases}$$

where $3 < \alpha \leq 4$, D^α is the standard Riemann-Liouville fractional derivative and a, b are nonnegative constants. First the authors suppose that $f(x,t) = -p(x)t^\sigma$, with $\sigma \in (-1,1)$ and p being a nonnegative continuous function that may be singular at $x = 0$ or $x = 1$ and satisfies some conditions related to the Karamata regular variation theory. Combining sharp estimates on some potential functions and the Schäuder fixed point theorem, the authors prove the existence of a unique positive continuous solution to problem $(P_{0,0})$. Global estimates on such a solution are also obtained. To state the second existence result, the authors assume that a, b are nonnegative constants such that $a + b > 0$ and $f(x,t) = t\varphi(x,t)$, with $\varphi(x,t)$ being a nonnegative continuous function in $(0,1)\times[0,\infty)$ that is required to satisfy some suitable integrability condition. Using estimates on the Green's function and a perturbation argument, the authors prove the existence and uniqueness of a positive continuous solution u to problem $(P_{a,b})$, which behaves like the unique solution of the homogeneous problem corresponding to $(P_{a,b})$. Some examples are given to illustrate the existence results.

Keywords Fractional differential equation, Positive solution, Fractional Green's function, Karamata function, Perturbation arguments, Schäuder fixed point theorem
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1 Introduction

Fractional differential equations have extensive applications in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media,

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electromagnetism, and other fields, can be modeled by fractional differential equations. We refer the reader to [6–8, 10–14, 20–21, 24–25, 27–28, 30–31] and the references therein for discussions of various applications. The existence, uniqueness and global asymptotic behavior of a positive continuous solution is an essential problem for fractional two-point boundary value problems. Such problems have been extensively investigated by many researchers, and various forms of the equation and boundary conditions have been discussed (see, for example, [1–2, 4, 9, 11, 15–16, 18–19, 23, 26, 32–34] and the references therein). In particular, in [2], Alsaedi studied the existence of a unique positive continuous solution to the following fourth order two-point value problem:

$$\begin{cases} u^{(4)}(x) = p(x)u^{\sigma}(x), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $\sigma \in (-1, 1)$ and p is a nonnegative continuous function satisfying some conditions related to the Karamata regular variation theory. Motivated by the above work, it is natural to ask when we can extend his result to the fractional setting. More precisely, in the first part of this paper, we are concerned with the following sublinear fractional differential two-point boundary value problem:

$$\begin{cases} D^{\alpha}u(x) = p(x)u^{\sigma}(x), \quad x \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0, \end{cases} \tag{1.2}$$

where $3 < \alpha \leq 4$, $\sigma \in (-1, 1)$ and p is a nonnegative continuous function on $(0, 1)$ that may be singular at $x = 0$ or $x = 1$ and satisfies some appropriate assumptions related to the Karamata class $\mathcal{K}$ (see Definition 1.1 below). Using the Schäuder fixed point theorem, we prove the existence of a unique positive continuous solution to problem (1.2). Further, by applying the Karamata regular variation theory, we establish sharp estimates on such a solution. To state our first existence result, we need some notations. We first introduce the Karamata class $\mathcal{K}$.

Definition 1.1 *The class $\mathcal{K}$ is the set of Karamata functions L defined on $(0, \eta]$ by*

$$L(t) := c \exp\Big(\int_t^{\eta} \frac{z(s)}{s} \mathrm{d}s\Big)$$

for some $\eta > 1$, where $c > 0$ and $z \in C([0, \eta])$ such that $z(0) = 0$.

Remark 1.1 It is clear that a function L is in $\mathcal{K}$ if and only if L is a positive function in $\mathcal{C}^1((0, \eta])$ for some $\eta > 1$, such that $\lim\limits_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$.

As a typical example of functions belonging to the class $\mathcal{K}$, we quote

$$L(t) = \prod_{j=1}^{m} \Big(\log_j\Big(\frac{\delta}{t}\Big)\Big)^{\xi_j},$$

where ξ_j are real numbers, $\log_j x = \log \circ \log \circ \cdots \log x$ (j times) and δ is a sufficiently large positive real number such that L is defined and positive on $(0, \eta]$ for some $\eta > 1$. For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $\frac{1}{c} f(x) \leq g(x) \leq c f(x)$ for all $x \in S$. We denote $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$ and denote by $\mathcal{B}^+((0,1))$ the set of all nonnegative measurable functions on $(0,1)$. We denote by $C((0,1))$ (resp. $C([0,1])$) the set of all continuous functions in $(0,1)$ (resp. $[0,1]$). In the problem (1.2), we assume that p is a nonnegative function on $(0,1)$ satisfying the following condition:

(**H**) $p \in C((0,1))$ such that

$$p(x) \approx x^{-\lambda} L_1(x)(1-x)^{-\mu} L_2(1-x), \quad x \in (0,1), \tag{1.3}$$

where $\lambda \leq 3 + (\alpha - 3)\sigma$, $\mu \leq \alpha - 1 + \sigma$ and $L_1, L_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta t^{2+(\alpha-3)\sigma-\lambda} L_1(t) \mathrm{d}t < \infty \quad \text{and} \quad \int_0^\eta t^{\alpha-2+\sigma-\mu} L_2(t) \mathrm{d}t < \infty. \tag{1.4}$$

We define the function θ on $[0,1]$ by

$$\theta(x) := x^{\min(\alpha-2, \frac{\alpha-\lambda}{1-\sigma})} (\widetilde{L}_1(x))^{\frac{1}{1-\sigma}} (1-x)^{\min(2, \frac{\alpha-\mu}{1-\sigma})} (\widetilde{L}_2(1-x))^{\frac{1}{1-\sigma}}, \tag{1.5}$$

where

$$\widetilde{L}_1(x) := \begin{cases} 1, & \text{if } \lambda < 2 + (\alpha-2)\sigma, \\ \displaystyle\int_x^\eta \frac{L_1(s)}{s} \mathrm{d}s, & \text{if } \lambda = 2 + (\alpha-2)\sigma, \\ L_1(x), & \text{if } 2 + (\alpha-2)\sigma < \lambda < 3 + (\alpha-3)\sigma, \\ \displaystyle\int_0^x \frac{L_1(s)}{s} \mathrm{d}s, & \text{if } \lambda = 3 + (\alpha-3)\sigma \end{cases}$$

and

$$\widetilde{L}_2(x) := \begin{cases} 1, & \text{if } \mu < \alpha - 2(1-\sigma), \\ \displaystyle\int_x^\eta \frac{L_2(s)}{s} \mathrm{d}s, & \text{if } \mu = \alpha - 2(1-\sigma), \\ L_2(x), & \text{if } \alpha - 2(1-\sigma) < \mu < \alpha - 1 + \sigma, \\ \displaystyle\int_0^x \frac{L_2(s)}{s} \mathrm{d}s, & \text{if } \mu = \alpha - 1 + \sigma. \end{cases}$$

Our first existence result is the following.

Theorem 1.1 *Let $\sigma \in (-1,1)$ and assume that p satisfies* (H). *Then problem* (1.2) *has a unique positive solution $u \in C([0,1])$ satisfying for $x \in [0,1]$,*

$$u(x) \approx \theta(x). \tag{1.6}$$

This theorem extends the one obtained in [2] to the fractional setting.

In the second part of this paper, we are concerned with the following superlinear fractional boundary value problem:

$$\begin{cases} D^\alpha u(x) + u(x)\varphi(x, u(x)) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad D^{\alpha-3} u(0) = a, \quad u'(1) = -b, \end{cases} \tag{1.7}$$

where $3 < \alpha \leq 4$, a, b are nonnegative constants such that $a+b > 0$ and $\varphi(x,t)$ is a nonnegative continuous function in $(0,1) \times [0,\infty)$ that is required to satisfy some appropriate condition related to the following class $\mathcal{K}_\alpha$.

Definition 1.2 *Let $3 < \alpha \leq 4$. A Borel measurable function q in $(0,1)$ belongs to the class $\mathcal{K}_\alpha$ if q satisfies the following condition:*

$$\int_0^1 r^{\alpha-1}(1-r)^{\alpha-1}|q(r)|\mathrm{d}r < \infty. \tag{1.8}$$

To state our second existence result, we introduce the following notations. We let

$$h_1(x) = \frac{1}{\Gamma(\alpha-2)} x^{\alpha-3}(1-x)^2, \quad h_2(x) = x^{\alpha-2}(1-x), \quad x \in [0,1], \tag{1.9}$$

and $\omega(x) := ah_1(x) + bh_2(x)$ be the unique solution of the homogeneous problem

$$\begin{cases} D^\alpha u(x) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad D^{\alpha-3}u(0) = a, \quad u'(1) = -b. \end{cases} \tag{1.10}$$

We denote by $G(x,t)$ the Green's function of the operator $u \to D^\alpha u$ with boundary conditions $u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0$, which can be explicitly given by (see Lemma 2.2)

$$G(x,t) = \frac{1}{\Gamma(\alpha)}(x^{\alpha-2}(1-t)^{\alpha-2}[t - x + (\alpha-2)t(1-x)] + ((x-t)^+)^{\alpha-1}).$$

For each $q \in \mathcal{K}_\alpha$, we denote

$$\alpha_q := \sup_{x,t\in(0,1)} \int_0^1 \frac{G(x,r)G(r,t)}{G(x,t)}|q(r)|\mathrm{d}r \tag{1.11}$$

and we will prove that if $q \in \mathcal{K}_\alpha$, then $\alpha_q < \infty$.

We require a combination of the following assumptions on the term φ :

($\mathbf{H}_1$) φ is a nonnegative continuous function in $(0,1) \times [0,\infty)$.

($\mathbf{H}_2$) There exists a nonnegative function $q \in \mathcal{K}_\alpha \cap C((0,1))$ with $\alpha_q \leq \frac{1}{2}$ such that for each $x \in (0,1)$, the map $t \to t\,(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$.

($\mathbf{H}_3$) For each $x \in (0,1)$, the function $t \to t\varphi(x,t)$ is nondecreasing on $[0,\infty)$.

We will first prove that if q is a nonnegative function in $\mathcal{K}_\alpha \cap C((0,1))$ with $\alpha_q \leq \frac{1}{2}$ and ψ is a positive measurable function, then the following problem

$$\begin{cases} D^\alpha u(x) + q(x)u(x) = \psi(x), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0 \end{cases} \tag{1.12}$$

has a positive solution. It turns out to prove that problem (1.12) admits a positive Green's function $\mathcal{G}(x,t)$. Based on the construction of this Green's function and by using a perturbation argument, we prove the following theorem.

Theorem 1.2 *Assume* (H_1)–(H_2)*, and then problem* (1.7) *has a positive solution* u *in* $C([0,1])$ *satisfying*

$$c_0\omega(x) \leq u(x) \leq \omega(x), \quad x \in [0,1], \tag{1.13}$$

where c_0 *is a constant in* $(0,1)$*. Moreover, if hypothesis* (H_3) *is also satisfied, then the solution* u *to problem* (1.7) *satisfying* (1.13) *is unique.*

Corollary 1.1 *Let* f *be a nonnegative function in* $C^1([0,\infty))$ *such that the map* $t \to \theta(t) = tf(t)$ *is nondecreasing on* $[0,\infty)$*. Let* p *be a nonnegative continuous function on* $(0,1)$ *such that the function* $x \to \widetilde{p}(x) := p(x) \cdot \max\limits_{0\leq \xi \leq \omega(x)} \theta'(\xi)$ *belongs to the class* $\mathcal{K}_\alpha$*. Then for* $\lambda \in \left[0, \frac{1}{2\alpha_{\widetilde{p}}}\right]$*, the following problem*

$$\begin{cases} D^\alpha u(x) + \lambda p(x)u(x)f(u(x)) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad D^{\alpha-3}u(0) = a, \quad u'(1) = -b \end{cases} \tag{1.14}$$

has a unique positive solution u *in* $C([0,1])$ *satisfying*

$$(1 - \lambda\alpha_{\widetilde{p}})\omega(x) \leq u(x) \leq \omega(x), \quad x \in [0,1].$$

These results extend the ones obtained in [3] with $\alpha = 4$ to the fractional setting. Also observe that in Theorem 1.2, we obtain a positive solution u in $C([0,1])$ to problem (1.7), whose behavior is not affected by the perturbed term. That is, it behaves like the solution ω of the homogeneous problem (1.10). As a typical example of nonlinearity satisfying (H_1)–(H_3), we quote $\varphi(x,t) = \lambda p(x)t^\sigma$, for $\sigma \geq 0$, p being a positive continuous function on $(0,1)$ such that

$$\int_0^1 r^{(\alpha-1)+(\alpha-3)\sigma}(1-r)^{\alpha+\sigma-1}p(r)\mathrm{d}r < \infty$$

and $q(x) = \lambda\widetilde{p}(x) := \lambda(\sigma+1)p(x)(\omega(x))^\sigma \in \mathcal{K}_\alpha$ with $\lambda \in \left[0, \frac{1}{2\alpha_{\widetilde{p}}}\right)$.

The rest of this paper is organized as follows. In Section 2, we prove some sharp estimates on the Green's function $G(x,t)$, including the following inequality: For each $x, r, t \in (0,1)$,

$$\frac{G(x,r)G(r,t)}{G(x,t)} \leq \frac{k^2}{(\alpha-2)\Gamma(\alpha)} r^{\alpha-1}(1-r)^{\alpha-1},$$

where $k := \max((\alpha-2)^2, \alpha-1)$. In particular, we deduce from this inequality that for each $q \in \mathcal{K}_\alpha$, $\alpha_q < \infty$. In Section 3, we present some known results on functions belonging to the class $\mathcal{K}$ and we establish sharp estimates on some potential functions. Exploiting theses results, we prove Theorem 1.1 by means of the Schäuder fixed point theorem. In Section 4, for a given function $q \in \mathcal{K}_\alpha$ with $\alpha_q \leq \frac{1}{2}$, we construct the Green's function $\mathcal{G}(x,t)$ of the boundary value problem (1.12). Next, we establish some estimates on this function. In particular, we prove that for $(x,t) \in [0,1] \times [0,1]$, we have

$$(1-\alpha_q)G(x,t) \leq \mathcal{G}(x,t) \leq G(x,t).$$

Also we prove the following resolvent equation:

$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f) \quad \text{for } f \in \mathcal{B}^+((0,1)),$$

where the kernels V and V_q are defined on $\mathcal{B}^+((0,1))$ by

$$Vf(x) := \int_0^1 G(x,t)f(t)\mathrm{d}t, \quad V_q f(x) := \int_0^1 \mathcal{G}(x,t)f(t)\mathrm{d}t, \quad x \in [0,1].$$

By using the above results and a perturbation argument, we prove Theorem 1.2. Finally, we give some examples to illustrate our existence results.

2 Fractional Calculus and Estimates on the Green's Function

2.1 Fractional calculus

For the convenience of the reader, we recall in this section some basic definitions of fractional calculus (see [12, 25, 27]).

Definition 2.1 *The Riemann-Liouville fractional integral of order $\beta > 0$ for a measurable function $f : (0,\infty) \to \mathbb{R}$ is defined as*

$$I^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)\mathrm{d}t, \quad x > 0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here Γ is the Euler Gamma function.

Definition 2.2 *The Riemann-Liouville fractional derivative of order $\beta > 0$ for a measurable function $f : (0,\infty) \to \mathbb{R}$ is defined as*

$$D^\beta f(x) = \frac{1}{\Gamma(n-\beta)} \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n \int_0^x (x-t)^{n-\beta-1} f(t)\mathrm{d}t = \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n I^{n-\beta} f(x)$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here $n = [\beta]+1$ and $[\beta]$ denotes the integer part of the number β.

Lemma 2.1 *Let $\beta > 0$ and $u \in C((0,1)) \cap L^1((0,1))$. Then we have the following assertions:*

(i) *For $0 < \gamma < \beta$, $D^\gamma I^\beta u = I^{\beta-\gamma} u$ and $D^\beta I^\beta u = u$.*

(ii) *$D^\beta u(x) = 0$ if and only if $u(x) = c_1 x^{\beta-1} + c_2 x^{\beta-2} + \cdots + c_m x^{\beta-m}$, $c_i \in \mathbb{R}$, $i = 1, \cdots, m$, where m is the smallest integer greater than or equal to β.*

(iii) *Assume that $D^\beta u \in C((0,1)) \cap L^1((0,1))$, and then*

$$I^\beta D^\beta u(x) = u(x) + c_1 x^{\beta-1} + c_2 x^{\beta-2} + \cdots + c_m x^{\beta-m},$$

$c_i \in \mathbb{R}$, $i = 1, \cdots, m$, where m is the smallest integer greater than or equal to β.

2.2 Estimates on the Green's Function

In this section, we derive the corresponding Green's function for the homogeneous boundary value problem (1.2) and we prove some estimates on this function.

Lemma 2.2 *Let* $3 < \alpha \le 4$ *and* $f \in C([0,1])$, *and then the boundary-value problem*

$$\begin{cases} D^\alpha u(x) = f(x) \quad \text{in } (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(x) = \int_0^1 G(x,t) f(t) \mathrm{d}t, \tag{2.2}$$

where for $x, t \in [0,1]$,

$$G(x,t) = \frac{1}{\Gamma(\alpha)} (x^{\alpha-2}(1-t)^{\alpha-2}[t - x + (\alpha-2)t(1-x)] + ((x-t)^+)^{\alpha-1}) \tag{2.3}$$

is the Green's function of boundary-value problem (2.1).

Proof By means of Lemma 2.1, we can reduce equation $D^\alpha u(x) = f(x)$ to an equivalent integral equation

$$u(x) = I^\alpha f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + c_3 x^{\alpha-3} + c_4 x^{\alpha-4}, \tag{2.4}$$

where $(c_1, c_2, c_3, c_4) \in \mathbb{R}^4$. The boundary condition $u(0) = 0$ implies that $c_4 = 0$. Applying the operator $D^{\alpha-3}$ to both sides of (2.4) and using Lemma 2.1(i), we obtain that

$$D^{\alpha-3} u(x) = c_1 \frac{\Gamma(\alpha)}{6} x^2 + c_2 \frac{\Gamma(\alpha-1)}{2} x + c_3 \Gamma(\alpha-2) + I^3 f(x). \tag{2.5}$$

The boundary condition $D^{\alpha-3}u(0) = 0$ gives $c_3 = 0$. Hence

$$u(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + I^\alpha f(x). \tag{2.6}$$

Now, using (2.6) and the boundary conditions $u(1) = u'(1) = 0$, we obtain

$$c_1 = (\alpha-2) I^\alpha f(1) - I^{\alpha-1} f(1), \quad c_2 = I^{\alpha-1} f(1) - (\alpha-1) I^\alpha f(1).$$

Therefore the unique solution of problem (2.1) is

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)} \Big(\int_0^x (x-t)^{\alpha-1} f(t) \mathrm{d}t \\ &\quad + \int_0^1 (\alpha-1) x^{\alpha-2} (1-x)(1-t)^{\alpha-2} f(t) \mathrm{d}t \\ &\quad + \int_0^1 [(\alpha-2)x - (\alpha-1)] x^{\alpha-2} (1-t)^{\alpha-1} f(t) \mathrm{d}t \Big) \\ &= \int_0^1 G(x,t) f(t) \mathrm{d}t. \end{aligned}$$

Proposition 2.1 *Let $3 < \alpha \leq 4$ and $k = \max((\alpha-2)^2, \alpha-1)$. The Green's function $G(x,t)$ satisfies the following properties on $[0,1] \times [0,1]$:*

(i) $(\alpha-2)H(x,t) \leq \Gamma(\alpha)G(x,t) \leq kH(x,t)$, *where*

$$H(x,t) := x^{\alpha-3}(1-x)t(1-t)^{\alpha-3}\min(x,t)(1-\max(x,t)).$$

(ii) $(\alpha-2)x^{\alpha-2}(1-x)^2t^2(1-t)^{\alpha-2} \leq \Gamma(\alpha)G(x,t) \leq kx^{\alpha-3}(1-x)t^2(1-t)^{\alpha-2}$.

(iii) $G(x,t) = G(1-t, 1-x)$.

Proof (i) We divide the proof into two cases.

Case 1 $0 \leq t \leq x \leq 1$. First, we remark that

$$\begin{aligned}
&x^{\alpha-2}(1-t)^{\alpha-2}(t-x) + (x-t)^{\alpha-1} \\
&= -(x-t)((x-xt)^{\alpha-2} - (x-t)^{\alpha-2}) \\
&= -(x-t)(\alpha-2)\int_{x-t}^{x-xt} s^{\alpha-3}\mathrm{d}s.
\end{aligned}$$

This implies that

$$\Gamma(\alpha)G(x,t) = -(x-t)(\alpha-2)\int_{x-t}^{x-xt} s^{\alpha-3}\mathrm{d}s + (\alpha-2)(t-tx)(x-xt)^{\alpha-2}. \tag{2.7}$$

So, we get

$$\begin{aligned}
&\Gamma(\alpha)G(x,t) \\
&\geq -(x-t)(\alpha-2)(x-xt)^{\alpha-3}(t-tx) \\
&\quad + (\alpha-2)(t-tx)(x-xt)^{\alpha-2} \\
&= (\alpha-2)(x-xt)^{\alpha-3}(t-tx)^2.
\end{aligned}$$

That is

$$\Gamma(\alpha)G(x,t) \geq (\alpha-2)x^{\alpha-3}(1-t)^{\alpha-3}t^2(1-x)^2. \tag{2.8}$$

On the other hand, using (2.7), we obtain

$$\begin{aligned}
&\Gamma(\alpha)G(x,t) \\
&\leq -(x-t)(\alpha-2)(x-t)^{\alpha-3}(t-tx) \\
&\quad + (\alpha-2)(t-tx)(x-xt)^{\alpha-2} \\
&= (\alpha-2)(t-tx)((x-xt)^{\alpha-2} - (x-t)^{\alpha-2}) \\
&= (\alpha-2)^2(t-tx)\int_{x-t}^{x-xt} s^{\alpha-3}\mathrm{d}s \\
&\leq (\alpha-2)^2(t-tx)^2(x-xt)^{\alpha-3}.
\end{aligned}$$

That is

$$\Gamma(\alpha)G(x,t) \leq (\alpha-2)^2x^{\alpha-3}(1-t)^{\alpha-3}t^2(1-x)^2. \tag{2.9}$$

Combining (2.8) and (2.9), we get, for $0 \leq t \leq x \leq 1$,

$$(\alpha-2)H(x,t) \leq \Gamma(\alpha)G(x,t) \leq (\alpha-2)^2 H(x,t). \tag{2.10}$$

Case 2 $0 \leq x \leq t \leq 1$. We have

$$\Gamma(\alpha)G(x,t) = x^{\alpha-2}(1-t)^{\alpha-2}[(t-x)+(\alpha-2)t(1-x)]. \tag{2.11}$$

Since $t-x \geq 0$, we get

$$\Gamma(\alpha)G(x,t) \geq (\alpha-2)H(x,t). \tag{2.12}$$

Moreover, using (2.11) and the fact that $t-x \leq t(1-x)$, we obtain that

$$\Gamma(\alpha)G(x,t) \leq (\alpha-1)H(x,t). \tag{2.13}$$

From (2.12) and (2.13), we have, for $0 \leq x \leq t \leq 1$,

$$(\alpha-2)H(x,t) \leq \Gamma(\alpha)G(x,t) \leq (\alpha-1)H(x,t). \tag{2.14}$$

The assertion (i) holds immediately from (2.10) and (2.14).

(ii) The assertion follows from (i) and the fact that for $x,t \in [0,1]$, we have

$$xt \leq \min(x,t) \leq t, \quad (1-x)(1-t) \leq 1-\max(x,t) \leq 1-t.$$

(iii) The assertion follows from (2.3) and a simple computation.

Remark 2.1 Note that estimates on the Green's function $G(x,t)$ obtained in the previous proposition improve those obtained in [32, Lemma 2.4].

As an immediately consequence of the assertion (ii) of Proposition 2.1, we obtain the following.

Corollary 2.1 *Let $f \in \mathcal{B}^+((0,1))$, and then the function $x \to Vf(x)$ is continuous on $[0,1]$ if and only if the integral $\int_0^1 t^2(1-t)^{\alpha-2}f(t)\mathrm{d}t$ converges.*

Proposition 2.2 *Let $3<\alpha<4$ and f be a function such that the map $t \to t^2(1-t)^{\alpha-2}f(t)$ is continuous and integrable on $(0,1)$. Then Vf is the unique solution in $C([0,1])$ of the boundary value problem*

$$\begin{cases} D^\alpha u(x) = f(x), \quad x \in (0,1), \\ u(0) = D^{\alpha-3}u(0) = u(1) = u'(1) = 0. \end{cases} \tag{2.15}$$

Proof From Corollary 2.1, the function Vf is in $C([0,1])$. This implies that $I^{4-\alpha}(V|f|)$ is bounded on $[0,1]$. So by using Fubini's theorem, we obtain

$$\begin{aligned} &I^{4-\alpha}(Vf)(x) \\ =& \frac{1}{\Gamma(4-\alpha)}\int_0^x (x-t)^{3-\alpha}Vf(t)\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(4-\alpha)} \int_0^1 \Big(\int_0^x (x-t)^{3-\alpha} G(t,s)\mathrm{d}t \Big) f(s)\mathrm{d}s \\
&= \int_0^1 K(x,s) f(s)\mathrm{d}s,
\end{aligned}
$$

where $K(x,s) := \frac{1}{\Gamma(4-\alpha)} \int_0^x (x-t)^{3-\alpha} G(t,s)\mathrm{d}t$.

Next, we aim at giving an explicit expression of the kernel $K(x,s)$. To this end, observe that by making the substitution $t = s + (x-s)\theta$, we obtain, for $\gamma, \nu > -1$,

$$
\int_s^x (x-t)^{\gamma} (t-s)^{\nu} \mathrm{d}t = \frac{\Gamma(\gamma+1)\Gamma(\nu+1)}{\Gamma(\gamma+\nu+2)} (x-s)^{\gamma+\nu+1}. \tag{2.16}
$$

Using this fact and (2.3), we deduce that

$$
\begin{aligned}
&K(x,s) \\
&= \frac{(1-s)^{\alpha-2}}{\Gamma(4-\alpha)\Gamma(\alpha)} \Big[(\alpha-1)s \int_0^x (x-t)^{3-\alpha} t^{\alpha-2} \mathrm{d}t \\
&\quad - (1+(\alpha-2)s) \int_0^x (x-t)^{3-\alpha} t^{\alpha-1} \mathrm{d}t \Big] \\
&\quad + \frac{1}{\Gamma(4-\alpha)\Gamma(\alpha)} \int_0^x (x-t)^{3-\alpha} ((t-s)^+)^{\alpha-1} \mathrm{d}t \\
&= (1-s)^{\alpha-2} \Big[\frac{s}{2} x^2 - \frac{1+(\alpha-2)s}{6} x^3 \Big] \\
&\quad + \frac{1}{\Gamma(4-\alpha)\Gamma(\alpha)} \int_0^x (x-t)^{3-\alpha} ((t-s)^+)^{\alpha-1} \mathrm{d}t.
\end{aligned}
$$

Now, assume that $s \le x$, and then by (2.16) we have

$$
\begin{aligned}
&\int_0^x (x-t)^{3-\alpha} ((t-s)^+)^{\alpha-1} \mathrm{d}t \\
&= \int_s^x (x-t)^{3-\alpha} (t-s)^{\alpha-1} \mathrm{d}t \\
&= \frac{\Gamma(\alpha)\Gamma(4-\alpha)}{6} (x-s)^3.
\end{aligned} \tag{2.17}
$$

On the other hand, if $0 \le t \le x \le s$, we have

$$
\int_0^x (x-t)^{3-\alpha} ((t-s)^+)^{\alpha-1} \mathrm{d}t = 0. \tag{2.18}
$$

So, combining (2.17) and (2.18), we obtain

$$
K(x,s) = (1-s)^{\alpha-2} \Big[\frac{s}{2} x^2 - \frac{1+(\alpha-2)s}{6} x^3 \Big] + \frac{1}{6} ((x-s)^+)^3.
$$

Hence for $x \in [0,1]$, we have

$$
\begin{aligned}
&6 I^{4-\alpha}(Vf)(x) \\
&= 6 \int_0^1 K(x,s) f(s) \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= x^3\int_0^x [1-(1+(\alpha-2)s)(1-s)^{\alpha-2}]f(s)\mathrm{d}s \\
&\quad -\int_0^x s^3 f(s)\mathrm{d}s + 3x\int_0^x s^2 f(s)\mathrm{d}s \\
&\quad -3x^2\int_0^x (1-(1-s)^{\alpha-2})sf(s)\mathrm{d}s \\
&\quad +3x^2\int_x^1 s(1-s)^{\alpha-2}f(s)\mathrm{d}s \\
&\quad -x^3\int_x^1 (1+(\alpha-2)s)(1-s)^{\alpha-2}f(s)\mathrm{d}s \\
&:= \mathrm{J}_1(x)+\mathrm{J}_2(x)+\mathrm{J}_3(x)+\mathrm{J}_4(x)+\mathrm{J}_5(x)+\mathrm{J}_6(x).
\end{aligned}
$$

We claim that

$$D^\alpha(Vf)(x) := \frac{\mathrm{d}^4}{\mathrm{d}x^4}(I^{4-\alpha}(Vf))(x) = f(x) \quad \text{for } x\in(0,1).$$

Indeed, firstly note that from the hypothesis, the function $s\to s^2f(s)$ is continuous and integrable near 0 and the function $s\to(1-s)^{\alpha-2}f(s)$ is continuous and integrable near 1. This implies in particular that $\mathrm{J}_5(x)$ and $\mathrm{J}_6(x)$ are differentiable on $(0,1)$. On the other hand, since

$$[1-(1+(\alpha-2)s)(1-s)^{\alpha-2}] = O(s^2)$$

and $(1-(1-s)^{\alpha-2})s = O(s^2)$ near 0, it follows that $\mathrm{J}_1(x),\mathrm{J}_2(x),\mathrm{J}_3(x)$ and $\mathrm{J}_4(x)$ are differentiable on $(0,1)$.

So, by simple computation, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(6I^{4-\alpha}(Vf))(x) \\
&= 3x^2\int_0^x [1-(1+(\alpha-2)s)(1-s)^{\alpha-2}]f(s)\mathrm{d}s \\
&\quad +3\int_0^x s^2 f(s)\mathrm{d}s - 6x\int_0^x (1-(1-s)^{\alpha-2})sf(s)\mathrm{d}s \\
&\quad +6x\int_x^1 s(1-s)^{\alpha-2}f(s)\mathrm{d}s \\
&\quad -3x^2\int_x^1 (1+(\alpha-2)s)(1-s)^{\alpha-2}f(s)\mathrm{d}s.
\end{aligned}
$$

By similar arguments as above, we obtain

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}(I^{4-\alpha}(Vf))(x) = f(x) \quad \text{for } x\in(0,1).$$

Next, we need to verify that the function Vf satisfies the boundary conditions. By Proposition 2.1(ii), there exists a nonnegative constant c such that

$$|Vf(x)| \le cx^{\alpha-3}(1-x). \tag{2.19}$$

So, it follows that $Vf(0) = Vf(1) = 0$. On the other hand, by Proposition 2.1(i), for each $t \in [0,1]$, we have

$$\lim_{x\to 1} \frac{G(x,t)}{1-x} = 0, \quad 0 \leq \frac{G(x,t)}{1-x} \leq \frac{k}{\Gamma(\alpha)} t^2 (1-t)^{\alpha-2}.$$

This implies by the dominated convergence theorem that $(Vf)'(1) = 0$. It remains to prove

$$D^{\alpha-3}Vf(0) := \lim_{x\to 0} \frac{I^{4-\alpha}(Vf)(x) - I^{4-\alpha}(Vf)(0)}{x} = \lim_{x\to 0} \frac{I^{4-\alpha}(Vf)(x)}{x} = 0.$$

To this end, we only need to verify that

$$\lim_{x\to 0} \frac{\mathrm{J}_i(x)}{x} = 0 \quad \text{for } i = 1,2,3,4,5,6.$$

The assertion is clear for $\mathrm{J}_1(x)$, $\mathrm{J}_3(x)$ and $\mathrm{J}_4(x)$.

Now, since $\frac{|\mathrm{J}_2(x)|}{x} \leq \int_0^x s^2 |f(s)| \mathrm{d}s$, we deduce that $\lim\limits_{x\to 0} \frac{\mathrm{J}_2(x)}{x} = 0$.

By applying [17, Lemma 2.2], we conclude that $\lim\limits_{x\to 0} \frac{\mathrm{J}_i(x)}{x} = 0$ for $i = 5,6$. Hence $D^{\alpha-3}Vf(0) = 0$. Finally, we prove the uniqueness. Let $u, v \in C([0,1])$ be two solutions of (2.19) and put $w = u - v$. Then $w \in C([0,1])$ and $D^\alpha w = 0$. Hence, it follows from Lemma 2.1(iii) that $w(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + c_3 x^{\alpha-3} + c_4 x^{\alpha-4}$. Using the fact that $w(0) = D^{\alpha-3}w(0) = w(1) = w'(1) = 0$, we deduce that $w = 0$ and therefore $u = v$.

Remark 2.2 Note that the conclusion of Proposition 2.2 is also valid for $\alpha = 4$ (see [2]).

Proposition 2.3 *We have, for each $x, r, t \in (0,1)$,*

$$\frac{G(x,r)G(r,t)}{G(x,t)} \leq \frac{k^2}{(\alpha-2)\Gamma(\alpha)} r^{\alpha-1}(1-r)^{\alpha-1}, \tag{2.20}$$

where $k = \max((\alpha-2)^2, \alpha-1)$.

Proof Using Proposition 2.1(i), we have, for each $x, r, t \in (0,1)$,

$$\begin{aligned} \frac{G(x,r)G(r,t)}{G(x,t)} \leq & \frac{k^2}{(\alpha-2)\Gamma(\alpha)} r^{\alpha-2}(1-r)^{\alpha-2} \\ & \times \frac{\min(x,r)(1-\max(x,r))\min(r,t)(1-\max(r,t))}{\min(x,t)(1-\max(x,t))}. \end{aligned}$$

We claim that

$$\frac{\min(x,r)(1-\max(x,r))\min(r,t)(1-\max(r,t))}{\min(x,t)(1-\max(x,t))} \leq r(1-r). \tag{2.21}$$

Indeed, by symmetry, we may assume that $x \leq t$. Then we deduce that

$$\begin{aligned} \frac{\min(x,r)(1-\max(x,r))\min(r,t)(1-\max(r,t))}{x(1-t)} &\leq \min(r,t)(1-\max(r,t)) \\ &\leq r(1-r). \end{aligned}$$

Now, by using (2.21), we obtain the required result.

Next we recall that $\omega(x) := ah_1(x) + bh_2(x)$, where

$$h_1(x) = \frac{1}{\Gamma(\alpha-2)} x^{\alpha-3}(1-x)^2$$

and $h_2(x) = x^{\alpha-2}(1-x)$ for $x \in [0,1]$.

Proposition 2.4 *Let q be a function in $\mathcal{K}_\alpha$, and then we have that*

(i)

$$\alpha_q \leq \frac{k^2}{(\alpha-2)\Gamma(\alpha)} \int_0^1 r^{\alpha-1}(1-r)^{\alpha-1}\,|q(r)|\,\mathrm{d}r < \infty, \tag{2.22}$$

where $k = \max((\alpha-2)^2, \alpha-1)$ and α_q is given by (1.11);

(ii) *for $x \in [0,1]$,*

$$\int_0^1 G(x,t)h_1(t)|q(t)|\mathrm{d}t \leq \alpha_q h_1(x); \tag{2.23}$$

(iii) *for $x \in [0,1]$,*

$$\int_0^1 G(x,t)h_2(t)|q(t)|\mathrm{d}t \leq \alpha_q h_2(x). \tag{2.24}$$

In particular for $x \in [0,1]$,

$$\int_0^1 G(x,t)\omega(t)|q(t)|\mathrm{d}t \leq \alpha_q \omega(x). \tag{2.25}$$

Proof Let q be a function in $\mathcal{K}_\alpha$.

(i) The inequality (2.22) follows from (1.11) and (2.20).

(ii) Since for each $x, t \in (0,1)$, we have $\lim\limits_{r\to 0} \frac{G(t,r)}{G(x,r)} = \frac{h_1(t)}{h_1(x)}$, then we deduce by Fatou's lemma and (1.11) that

$$\int_0^1 G(x,t)\frac{h_1(t)}{h_1(x)}|q(t)|\mathrm{d}t \leq \liminf_{r\to 0} \int_0^1 G(x,t)\frac{G(t,r)}{G(x,r)}|q(t)|\mathrm{d}t \leq \alpha_q,$$

which implies that for $x \in [0,1]$,

$$\int_0^1 G(x,t)h_1(t)|q(t)|\mathrm{d}t \leq \alpha_q h_1(x).$$

(iii) Similarly, we prove inequality (2.24) by observing that

$$\lim_{r\to 1} \frac{G(t,r)}{G(x,r)} = \frac{h_2(t)}{h_2(x)}.$$

Inequality (2.25) follows from (2.23)–(2.24) and the fact that $\omega(x) = ah_1(x) + bh_2(x)$.

3 First Existence Result

In this section, we aim at proving Theorem 1.1.

3.1 Karamata class and sharp estimates on some potential functions

In this subsection, we recall some fundamental properties of functions belonging to the class $\mathcal{K}$ and we establish estimates on some potential functions.

Lemma 3.1 (see [22, 29]) *Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. Then we have that*

(i) *if $\gamma > -1$, then $\int_0^\eta s^\gamma L(s)\mathrm{d}s$ converges and $\int_0^t s^\gamma L(s)\mathrm{d}s \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma}$;*

(ii) *if $\gamma < -1$, then $\int_0^\eta s^\gamma L(s)\mathrm{d}s$ diverges and $\int_t^\eta s^\gamma L(s)\mathrm{d}s \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}$.*

Lemma 3.2 (see [5, 29]) (i) *Let $L \in \mathcal{K}$ and $\epsilon > 0$. So then we have*

$$\lim_{t\to 0^+} t^\epsilon L(t) = 0.$$

(ii) *Let L_1 and $L_2 \in \mathcal{K}$ defined on $(0,\eta]$ and $p \in \mathbb{R}$. Then functions*

$$L_1 + L_2,\ L_1L_2 \text{ and } L_1^p \text{ belong to the class } \mathcal{K}.$$

(iii) *Let $L \in \mathcal{K}$ defined on $(0,\eta]$. So then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\displaystyle\int_t^\eta \frac{L(s)}{s}\mathrm{d}s} = 0.$$

In particular,

$$t \to \int_t^\eta \frac{L(s)}{s}\mathrm{d}s \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}\mathrm{d}s$ converges, then we have $\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}\mathrm{d}s} = 0$.
In particular,

$$t \to \int_0^t \frac{L(s)}{s}\mathrm{d}s \in \mathcal{K}.$$

Next, we shall prove sharp estimates on the potential function $V(p\theta^\sigma)$, where p is a function satisfying (H) and θ is the function given by (1.5).

To this end, we need the following proposition.

Proposition 3.1 *Let $\gamma \le 3$, $\nu \le \alpha - 1$ and $L_3, L_4 \in \mathcal{K}$ such that*

$$\int_0^\eta t^{2-\gamma} L_3(t)\mathrm{d}t < \infty, \quad \int_0^\eta t^{\alpha-2-\nu} L_4(t)\mathrm{d}t < \infty. \tag{3.1}$$

Put

$$b(x) = x^{-\gamma} L_3(x)(1-x)^{-\nu} L_4(1-x) \quad \text{for } x \in (0,1).$$

Then we have, for $x \in [0,1]$,

$$Vb(x) \approx x^{\min(\alpha-2,\alpha-\gamma)} \widetilde{L}_3(x)(1-x)^{\min(2,\alpha-\nu)} \widetilde{L}_4(1-x),$$

where

$$\widetilde{L}_3(x) := \begin{cases} 1, & \textit{if } \gamma < 2, \\ \displaystyle\int_x^\eta \frac{L_3(s)}{s}\mathrm{d}s, & \textit{if } \gamma = 2, \\ L_3(x), & \textit{if } 2 < \gamma < 3, \\ \displaystyle\int_0^x \frac{L_3(s)}{s}\mathrm{d}s, & \textit{if } \gamma = 3 \end{cases}$$

and

$$\widetilde{L}_4(x) := \begin{cases} 1, & \textit{if } \nu < \alpha - 2, \\ \displaystyle\int_x^\eta \frac{L_4(s)}{s}\mathrm{d}s, & \textit{if } \nu = \alpha - 2, \\ L_4(x), & \textit{if } \alpha - 2 < \nu < \alpha - 1, \\ \displaystyle\int_0^x \frac{L_4(s)}{s}\mathrm{d}s, & \textit{if } \nu = \alpha - 1. \end{cases}$$

Proof For $x \in [0, 1]$, we have

$$Vb(x) = \int_0^1 G(x,t)b(t)\mathrm{d}t.$$

Using Proposition 2.1(i), we obtain that

$$\begin{aligned} Vb(x) \approx{}& (1-x)^2 x^{\alpha-3} \int_0^x t^{2-\gamma}(1-t)^{\alpha-3-\nu} L_3(t)L_4(1-t)\mathrm{d}t \\ &+ (1-x)x^{\alpha-2}\int_x^1 t^{1-\gamma}(1-t)^{\alpha-2-\nu}L_3(t)L_4(1-t)\mathrm{d}t. \end{aligned}$$

In what follows, we distinguish two cases.

Case 1 $0 \le x \le \frac{1}{2}$. In this case, we have $1 - x \approx 1$. So, we obtain

$$\begin{aligned} Vb(x) &\approx (1-x)^2 x^{\alpha-3} \int_0^x t^{2-\gamma}(1-t)^{\alpha-3-\nu} L_3(t)L_4(1-t)\mathrm{d}t \\ &\quad + (1-x)x^{\alpha-2}\Big(\int_x^{\frac{1}{2}} t^{1-\gamma}(1-t)^{\alpha-2-\nu}L_3(t)L_4(1-t)\mathrm{d}t \\ &\quad + \int_{\frac{1}{2}}^1 t^{1-\gamma}(1-t)^{\alpha-2-\nu}L_3(t)L_4(1-t)\mathrm{d}t\Big). \\ &\approx x^{\alpha-3}\int_0^x t^{2-\gamma}L_3(t)\mathrm{d}t \\ &\quad + x^{\alpha-2}\Big(\int_x^{\frac{1}{2}} t^{1-\gamma}L_3(t)\mathrm{d}t + \int_{\frac{1}{2}}^1 (1-t)^{\alpha-2-\nu}L_4(1-t)\mathrm{d}t\Big) \\ &= x^{\alpha-3}\int_0^x t^{2-\gamma}L_3(t)\mathrm{d}t \\ &\quad + x^{\alpha-2}\Big(\int_x^{\frac{1}{2}} t^{1-\gamma}L_3(t)\mathrm{d}t + \int_0^{\frac{1}{2}} t^{\alpha-2-\nu}L_4(t)\mathrm{d}t\Big). \end{aligned}$$

Since $\int_0^\eta t^{\alpha-2-\nu}L_4(t)\mathrm{d}t<\infty$, we deduce that

$$Vb(x)\approx x^{\alpha-3}\int_0^x t^{2-\gamma}L_3(t)\mathrm{d}t+x^{\alpha-2}\Big(1+\int_x^{\frac{1}{2}} t^{1-\gamma}L_3(t)\mathrm{d}t\Big).$$

Using Lemma 3.1 and hypothesis (3.1), we deduce that

$$\int_0^x t^{2-\gamma}L_3(t)\mathrm{d}t\approx\begin{cases} x^{3-\gamma}L_3(x), & \text{if} \quad \gamma<3,\\ \displaystyle\int_0^x \frac{L_3(s)}{s}\mathrm{d}s, & \text{if} \quad \gamma=3\end{cases}$$

and

$$1+\int_x^{\frac{1}{2}} t^{1-\gamma}L_3(t)\mathrm{d}t\approx\begin{cases} 1, & \text{if} \quad \gamma<2,\\ \displaystyle\int_x^\eta \frac{L_3(s)}{s}\mathrm{d}s, & \text{if} \quad \gamma=2,\\ x^{2-\gamma}L_3(x), & \text{if} \quad 2<\gamma\leq 3.\end{cases}$$

Hence, it follows by Lemmas 3.1–3.2 and hypothesis (3.1) that for $0\leq x\leq\frac{1}{2}$,

$$Vb(x)\approx\begin{cases} x^{\alpha-2}, & \text{if} \quad \gamma<2,\\ x^{\alpha-2}\displaystyle\int_x^\eta \frac{L_3(s)}{s}\mathrm{d}s, & \text{if} \quad \gamma=2,\\ x^{\alpha-\gamma}L_3(x), & \text{if} \quad 2<\gamma<3,\\ x^{\alpha-3}\displaystyle\int_0^x \frac{L_3(s)}{s}\mathrm{d}s, & \text{if} \quad \gamma=3.\end{cases}$$

That is

$$Vb(x)\approx x^{\min(\alpha-2,\alpha-\gamma)}\widetilde{L}_3(x). \tag{3.2}$$

Case 2 $\frac{1}{2}\leq x\leq 1$. In this case, we have $x\approx 1$. Therefore, we obtain

$$\begin{aligned} Vb(x)&\approx(1-x)^2x^{\alpha-3}\Big(\int_0^{\frac{1}{2}} t^{2-\gamma}(1-t)^{\alpha-3-\nu}L_3(t)L_4(1-t)\mathrm{d}t\\ &\quad+\int_{\frac{1}{2}}^x t^{2-\gamma}(1-t)^{\alpha-3-\nu}L_3(t)L_4(1-t)\mathrm{d}t\Big)\\ &\quad+(1-x)x^{\alpha-2}\int_x^1 t^{1-\gamma}(1-t)^{\alpha-2-\nu}L_3(t)L_4(1-t)\mathrm{d}t\\ &\approx(1-x)^2\Big(\int_0^{\frac{1}{2}} t^{2-\gamma}L_3(t)\mathrm{d}t+\int_{\frac{1}{2}}^x(1-t)^{\alpha-3-\nu}L_4(1-t)\mathrm{d}t\Big)\\ &\quad+(1-x)\int_x^1(1-t)^{\alpha-2-\nu}L_4(1-t)\mathrm{d}t.\\ &=(1-x)^2\Big(\int_0^{\frac{1}{2}} t^{2-\gamma}L_3(t)\mathrm{d}t+\int_{1-x}^{\frac{1}{2}} t^{\alpha-3-\nu}L_4(t)\mathrm{d}t\Big)\\ &\quad+(1-x)\int_0^{1-x} t^{\alpha-2-\nu}L_4(t)\mathrm{d}t. \end{aligned}$$

Since $\int_0^\eta t^{2-\gamma}L_3(t)\mathrm{d}t<\infty$, we deduce that

$$Vb(x)\approx(1-x)^2\Big(1+\int_{1-x}^{\frac{1}{2}}t^{\alpha-3-\nu}L_4(t)\mathrm{d}t\Big)+(1-x)\int_0^{1-x}t^{\alpha-2-\nu}L_4(t)\mathrm{d}t.$$

Using again Lemma 3.1 and hypothesis (3.1), we deduce that

$$\int_0^{1-x}t^{\alpha-2-\nu}L_4(t)\mathrm{d}t\approx\begin{cases}(1-x)^{\alpha-1-\nu}L_4(1-x), & \text{if } \nu<\alpha-1,\\ \displaystyle\int_0^{1-x}\frac{L_4(s)}{s}\mathrm{d}s, & \text{if } \nu=\alpha-1\end{cases}$$

and

$$1+\int_{1-x}^{\frac{1}{2}}t^{\alpha-3-\nu}L_4(t)\mathrm{d}t\approx\begin{cases}1, & \text{if } \nu<\alpha-2,\\ \displaystyle\int_{1-x}^{\eta}\frac{L_4(s)}{s}\mathrm{d}s, & \text{if } \nu=\alpha-2,\\ (1-x)^{2-\gamma}L_4(1-x), & \text{if } \alpha-2<\nu\le\alpha-1.\end{cases}$$

Hence, it follows by Lemmas 3.1–3.2 and hypothesis (3.1) that for $\frac{1}{2}\le x\le1$,

$$Vb(x)\approx\begin{cases}(1-x)^2, & \text{if } \nu<\alpha-2,\\ (1-x)^2\displaystyle\int_{1-x}^{\eta}\frac{L_4(s)}{s}\mathrm{d}s, & \text{if } \nu=\alpha-2,\\ (1-x)^{\alpha-\nu}L_4(1-x), & \text{if } \alpha-2<\nu<\alpha-1,\\ (1-x)\displaystyle\int_0^{1-x}\frac{L_4(s)}{s}\mathrm{d}s, & \text{if } \nu=\alpha-1.\end{cases}$$

That is

$$Vb(x)\approx(1-x)^{\min(2,\alpha-\nu)}\widetilde{L}_4(1-x). \tag{3.3}$$

This together with (3.2) implies that for $x\in[0,1]$, we have

$$Vb(x)\approx x^{\min(\alpha-2,\alpha-\gamma)}\widetilde{L}_3(x)(1-x)^{\min(2,\alpha-\nu)}\widetilde{L}_4(1-x).$$

This ends the proof.

The following proposition plays a crucial role in the proof of Theorem 1.1.

Proposition 3.2 *Let p be a function satisfying* (H). *Then we have, for $x\in[0,1]$,*

$$V(p\theta^\sigma)(x)\approx\theta(x).$$

Proof Let p be a function satisfying (H). Let $\gamma=\lambda-\sigma\min\big(\alpha-2,\frac{\alpha-\lambda}{1-\sigma}\big)$ and $\nu=\mu-\sigma\min\big(2,\frac{\alpha-\mu}{1-\sigma}\big)$, where the constants λ and μ are given by (H). Since $\lambda\le3+(\alpha-3)\sigma$ and $\mu\le\alpha-1+\sigma$, we verify that $\gamma\le3$ and $\nu\le\alpha-1$. On the other hand, by using (1.3) and (1.5), we get

$$p(x)\theta^\sigma(x)\approx x^{-\gamma}(1-x)^{-\nu}L_1(x)(\widetilde{L}_1(x))^{\frac{\sigma}{1-\sigma}}L_2(1-x)(\widetilde{L}_2(1-x))^{\frac{\sigma}{1-\sigma}}.$$

So, using Lemmas 3.1–3.2 and Proposition 3.1 with $L_3 = L_1(\widetilde{L}_1)^{\frac{\sigma}{1-\sigma}}$ and $L_4 = L_2(\widetilde{L}_2)^{\frac{\sigma}{1-\sigma}}$, we deduce that for each $x \in [0,1]$,

$$V(p\theta^\sigma)(x) \approx x^{\min(\alpha-2,\alpha-\gamma)}\widetilde{L}_3(x)(1-x)^{\min(2,\alpha-\nu)}\widetilde{L}_4(1-x).$$

Since

$$\min(\alpha-2,\alpha-\gamma) = \min\Big(\alpha-2, \frac{\alpha-\lambda}{1-\sigma}\Big), \quad \min(2,\alpha-\nu) = \min\Big(2, \frac{\alpha-\mu}{1-\sigma}\Big),$$

we conclude by elementary calculus that for $x \in [0,1]$,

$$V(p\theta^\sigma)(x) \approx x^{\min(\alpha-2,\alpha-\gamma)}\widetilde{L}_3(x)(1-x)^{\min(2,\alpha-\nu)}\widetilde{L}_4(1-x) \approx \theta(x).$$

This completes the proof.

3.2 Proof of Theorem 1.1

Let p be a function satisfying (H) and let θ be the function given by (1.5). By Proposition 3.2, there exists $M \geq 1$ such that for each $x \in [0,1]$,

$$\frac{1}{M}\theta(x) \leq V\left(p\theta^\sigma\right)(x) \leq M\theta(x).$$

We shall use a fixed point argument to construct a solution to problem (1.2). For this end, put $c = M^{\frac{1}{1-|\sigma|}}$ and consider the closed convex set given by

$$\Lambda = \Big\{u \in C([0,1]) : \frac{1}{c}\theta \leq u \leq c\theta\Big\}.$$

Obviously, the function θ belongs to $C([0,1])$ and so Λ is not empty. We define the operator T on Λ by

$$Tu = V(pu^\sigma).$$

For this choice of c, we can easily get that for $u \in \Lambda$, we have $\frac{1}{c}\theta \leq Tu \leq c\theta$. Now, since the function $(x,t) \mapsto G(x,t)$ is continuous on $[0,1]\times[0,1]$ and the function $t \mapsto t^2(1-t)^{\alpha-2}p(t)\theta^\sigma(t)$ is integrable on $(0,1)$, we deduce that the operator T is compact from Λ to itself. So, by the Schäuder fixed-point theorem, there exists a function $u \in \Lambda$ such that

$$u = V(pu^\sigma).$$

It remains to prove that u is a positive continuous solution of problem (1.2). Indeed, since $t \mapsto t^2(1-t)^{\alpha-2}p(t)u^\sigma(t)$ is continuous and integrable on $(0,1)$, then it follows from Proposition 2.2 that the function u is a positive continuous solution of problem (1.2). Finally, let us show that problem (1.2) has a unique positive solution in the cone

$$\digamma = \{u \in C\left([0,1]\right) : \ u \approx \theta\}.$$

So, we assume that u and v are arbitrary solutions of problem (1.2) in $\digamma$. Since $u, v \in \digamma$, then there exists a constant $m \geq 1$ such that

$$\frac{1}{m} \leq \frac{u}{v} \leq m \quad \text{in } (0,1).$$

This implies that the set $J := \left\{m \geq 1 : \frac{1}{m} \leq \frac{u}{v} \leq m\right\}$ is not empty. Now, let $m_0 := \inf J$. It is easy to see that $m_0 \geq 1$. This gives that $u^\sigma \leq m_0^{|\sigma|} v^\sigma$.

On the other hand, put $z := m_0^{|\sigma|} v - u$, and then we have

$$\begin{cases} D^\alpha(z) = p(x)(m_0^{|\sigma|} v^\sigma - u^\sigma) \geq 0 \quad \text{in } (0,1), \\ z(0) = \lim\limits_{x \to 0} D^{\alpha-3} z(x) = 0, \\ z(1) = z'(1) = 0. \end{cases}$$

This implies by Proposition 2.2 that $m_0^{|\sigma|} v - u = V(p(m_0^{|\sigma|} v^\sigma - u^\sigma)) \geq 0$. By symmetry, we obtain that $m_0^{|\sigma|} u \geq v$. Hence, $m_0^{|\sigma|} \in J$. Using the fact that $m_0 := \inf J$ and $|\sigma| < 1$, we get $m_0 = 1$. Then, we conclude that $u = v$.

To illustrate our result proved in Theorem 1.1, we give the following example.

Example 3.1 Let $\sigma \in (-1,1)$ and p be a nonnegative continuous function on $(0,1)$ such that

$$p(x) \approx x^{-3-(\alpha-3)\sigma} \Big(\log\Big(\frac{3}{x}\Big)\Big)^{-2} (1-x)^{-\mu} \Big(\log\Big(\log\Big(\frac{3}{1-x}\Big)\Big)\Big)^{-\beta},$$

where $\mu < \alpha - 1 + \sigma$ and $\beta \in \mathbb{R}$. Then by Theorem 1.1, problem (1.2) has a unique positive solution u in $C([0,1])$ satisfying the following estimates:

(i) If $\mu < \alpha - 2(1-\sigma)$ or $\mu = \alpha - 2(1-\sigma)$ and $\beta > 1$, then for $x \in [0,1]$,

$$u(x) \approx x^{\alpha-3}(1-x)^2 \Big(\log\Big(\frac{3}{x}\Big)\Big)^{\frac{-1}{1-\sigma}}.$$

(ii) If $\mu = \alpha - 2(1-\sigma)$ and $\beta = 1$, then for $x \in [0,1]$,

$$u(x) \approx x^{\alpha-3}(1-x)^2 \Big(\log\Big(\frac{3}{x}\Big)\Big)^{\frac{-1}{1-\sigma}} \Big(\log\Big(\log\Big(\frac{3}{1-x}\Big)\Big)\Big)^{\frac{1}{1-\sigma}}.$$

(iii) If $\mu = \alpha - 2(1-\sigma)$ and $\beta < 1$, then for $x \in [0,1]$,

$$u(x) \approx x^{\alpha-3}(1-x)^2 \Big(\log\Big(\frac{3}{x}\Big)\Big)^{\frac{-1}{1-\sigma}} \Big(\log\Big(\frac{3}{1-x}\Big)\Big)^{\frac{1-\beta}{1-\sigma}}.$$

(iv) If $\alpha - 2(1-\sigma) < \mu < \alpha - 1 + \sigma$, then for $x \in [0,1]$,

$$u(x) \approx x^{\alpha-3}(1-x)^{\frac{\alpha-\mu}{1-\sigma}} \Big(\log\Big(\frac{3}{x}\Big)\Big)^{\frac{-1}{1-\sigma}} \Big(\log\Big(\frac{3}{1-x}\Big)\Big)^{\frac{-\beta}{1-\sigma}}.$$

4 Second Existence Result

In this section, we aim at proving Theorem 1.2 and Corollary 1.1. To this end, we need the following preliminary results. For a nonnegative function q in $\mathcal{K}_\alpha$ such that $\alpha_q < 1$, we define the function $\mathcal{G}(x,t)$ on $[0,1]\times[0,1]$ by

$$\mathcal{G}(x,t) = \sum_{n=0}^{\infty}(-1)^n G_n(x,t), \tag{4.1}$$

where $G_0(x,t) = G(x,t)$ and

$$G_n(x,t) = \int_0^1 G(x,r)G_{n-1}(r,t)q(r)\mathrm{d}r, \quad n \geq 1. \tag{4.2}$$

Next, we establish some inequalities on $G_n(x,t)$. In particular, we deduce that $\mathcal{G}(x,t)$ is well defined.

Lemma 4.1 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q < 1$, and then for each $n \geq 0$ and $(x,t) \in [0,1]\times[0,1]$, we have*

(i) $G_n(x,t) \leq \alpha_q^n G(x,t)$. *In particular, $\mathcal{G}(x,t)$ is well defined in $[0,1]\times[0,1]$.*

(ii)

$$L_n x^{\alpha-2}(1-x)^2 t^2(1-t)^{\alpha-2} \leq G_n(x,t) \leq R_n x^{\alpha-3}(1-x)t^2(1-t)^{\alpha-2}, \tag{4.3}$$

where

$$L_n = \frac{(\alpha-2)^{n+1}}{(\Gamma(\alpha))^{n+1}}\Big(\int_0^1 r^\alpha(1-r)^\alpha q(r)\mathrm{d}r\Big)^n$$

and

$$R_n = \frac{k^{n+1}}{(\Gamma(\alpha))^{n+1}}\Big(\int_0^1 r^{\alpha-1}(1-r)^{\alpha-1} q(r)\mathrm{d}r\Big)^n$$

with

$$k = \max((\alpha-2)^2, \alpha-1).$$

(iii) $G_{n+1}(x,t) = \int_0^1 G_n(x,r)G(r,t)q(r)\mathrm{d}r$.

(iv) $\int_0^1 \mathcal{G}(x,r)G(r,t)q(r)\mathrm{d}r = \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)\mathrm{d}r$.

Proof (i) The assertion is clear for $n = 0$. Assume that inequality in (i) holds for some $n \geq 0$, and then by using (4.2) and (1.11), we obtain

$$G_{n+1}(x,t) \leq \alpha_q^n \int_0^1 G(x,r)G(r,t)q(r)\mathrm{d}r \leq \alpha_q^{n+1}G(x,t).$$

Now, since $0 \leq G_n(x,t) \leq \alpha_q^n G(x,t)$, it follows that $\mathcal{G}(x,t)$ is well defined in $[0,1]\times[0,1]$.

(ii) Using Proposition 2.1(ii) and (4.2), we obtain (4.3) by simple induction.

(iii) The equality is clear for $n = 0$. Assume that for a given integer $n \geq 1$ and $(x,t) \in [0,1]\times[0,1]$, we have

$$G_n(x,t) = \int_0^1 G_{n-1}(x,r)G(r,t)q(r)\mathrm{d}r. \tag{4.4}$$

Using (4.2) and the Fubini-Tonelli's theorem, we obtain

$$\begin{aligned} G_{n+1}(x,t) &= \int_0^1 G(x,r)\Big(\int_0^1 G_{n-1}(r,\xi)G(\xi,t)q(\xi)\mathrm{d}\xi\Big)q(r)\mathrm{d}r \\ &= \int_0^1 \Big(\int_0^1 G(x,r)G_{n-1}(r,\xi)q(r)\mathrm{d}r\Big)G(\xi,t)q(\xi)\mathrm{d}\xi \\ &= \int_0^1 G_n(x,\xi)G(\xi,t)q(\xi)\mathrm{d}\xi. \end{aligned}$$

(iv) Let $n \geq 0$ and $x, r, t \in [0,1]$, By Lemma 4.1(i), we have

$$0 \leq G_n(x,r)G(r,t)q(r) \leq \alpha_q^n G(x,r)G(r,t)q(r).$$

Hence the series $\sum\limits_{n\geq 0} \int_0^1 G_n(x,r)G(r,t)q(r)\mathrm{d}r$ converges. So, we deduce by the dominated convergence theorem and Lemma 4.1(iii) that

$$\begin{aligned} \int_0^1 \mathcal{G}(x,r)G(r,t)q(r)\mathrm{d}r &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 G_n(x,r)G(r,t)q(r)\mathrm{d}r \\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 G(x,r)G_n(r,t)q(r)\mathrm{d}r \\ &= \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)\mathrm{d}r. \end{aligned}$$

Proposition 4.1 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q < 1$. Then the function $(x,t) \to \mathcal{G}(x,t)$ is continuous on $[0,1]\times[0,1]$.*

Proof Firstly, we claim that for $n \geq 0$, the function $(x,t) \to G_n(x,t)$ is continuous on $[0,1]\times[0,1]$. The assertion is clear for $n = 0$. Assume that for a given integer $n \geq 1$, the function $(x,t) \to G_{n-1}(x,t)$ is continuous on $[0,1]\times[0,1]$. So, for each $r \in [0,1]$, the function $(x,t) \to G(x,r)G_{n-1}(r,t)$ is continuous on $[0,1]\times[0,1]$. On the other hand, by Lemma 4.1(i) and Proposition 2.1(ii), we have, for each $(x,t,r) \in [0,1]\times[0,1]\times[0,1]$,

$$\begin{aligned} G(x,r)G_{n-1}(r,t)q(r) &\leq \alpha_q^{n-1}G(x,r)G(r,t)q(r) \\ &\leq \frac{k^2}{(\Gamma(\alpha))^2} r^{\alpha-1}(1-r)^{\alpha-1}q(r), \end{aligned}$$

where $k = \max((\alpha-2)^2, \alpha-1)$. So, we deduce by (4.2) and the dominated convergence theorem that the function $(x,t) \to G_n(x,t)$ is continuous on $[0,1]\times[0,1]$. This proves our claim. Now, by using again Lemma 4.1(i) and Proposition 2.1(ii), we have, for each $x, t \in [0,1]$,

$$G_n(x,t) \leq \alpha_q^n G(x,t) \leq \frac{k}{\Gamma(\alpha)}\alpha_q^n.$$

This implies that the series $\sum\limits_{n\geq 0}(-1)^n G_n(x,t)$ is uniformly convergent on $[0,1]\times[0,1]$ and therefore the function $(x,t) \to \mathcal{G}(x,t)$ is continuous on $[0,1]\times[0,1]$.

Lemma 4.2 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$. Then for $(x,t) \in [0,1] \times [0,1]$, we have*

$$(1-\alpha_q)G(x,t) \leq \mathcal{G}(x,t) \leq G(x,t). \tag{4.5}$$

Proof Since $\alpha_q \leq \frac{1}{2}$, we deduce from Lemma 4.1(i) that

$$|\mathcal{G}(x,t)| \leq \sum_{n=0}^{\infty} (\alpha_q)^n G(x,t) = \frac{1}{1-\alpha_q} G(x,t). \tag{4.6}$$

On the other hand, from the expression of $\mathcal{G}$, we have

$$\mathcal{G}(x,t) = G(x,t) - \sum_{n=0}^{\infty} (-1)^n G_{n+1}(x,t). \tag{4.7}$$

Since the series $\sum\limits_{n\geq 0} \int_0^1 G(x,r)G_n(r,t)q(r)\mathrm{d}r$ is convergent, we deduce by (4.7) and (4.2) that

$$\begin{aligned} \mathcal{G}(x,t) &= G(x,t) - \sum_{n=0}^{\infty} (-1)^n \int_0^1 G(x,r)G_n(r,t)q(r)\mathrm{d}r \\ &= G(x,t) - \int_0^1 G(x,r)\Big(\sum_{n=0}^{\infty} (-1)^n G_n(r,t)\Big) q(r)\mathrm{d}r. \end{aligned}$$

That is

$$\mathcal{G}(x,t) = G(x,t) - V\left(q\mathcal{G}\left(\cdot,t\right)\right)(x). \tag{4.8}$$

Now, from (4.6) and Lemma 4.1(i) (with $n=1$), we obtain

$$V\left(q\mathcal{G}\left(\cdot,t\right)\right)(x) \leq \frac{1}{1-\alpha_q} V\left(qG\left(\cdot,t\right)\right)(x) = \frac{1}{1-\alpha_q} G_1(x,t) \leq \frac{\alpha_q}{1-\alpha_q} G(x,t). \tag{4.9}$$

This implies by (4.8) that

$$\mathcal{G}(x,t) \geq G(x,t) - \frac{\alpha_q}{1-\alpha_q} G(x,t) = \frac{1-2\alpha_q}{1-\alpha_q} G(x,t) \geq 0.$$

So, it follows that $0 \leq \mathcal{G}(x,t) \leq G(x,t)$ and by (4.8) and Lemma 4.1(i) (with $n=1$), we have

$$\mathcal{G}(x,t) \geq G(x,t) - V\left(qG\left(\cdot,t\right)\right)(x) \geq (1-\alpha_q)G(x,t).$$

We recall that for a given nonnegative function $q \in \mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$, the kernels V and V_q are defined on $\mathcal{B}^+((0,1))$ by

$$Vf(x) := \int_0^1 G(x,t)f(t)\mathrm{d}t, \quad V_q f(x) := \int_0^1 \mathcal{G}(x,t)f(t)\mathrm{d}t, \quad x \in [0,1].$$

Also let us introduce the kernel $V(q.)$ defined on $\mathcal{B}^+((0,1))$ by

$$V(q\cdot)f(x) := \int_0^1 G(x,t)q(t)f(t)\mathrm{d}t, \quad x \in [0,1].$$

Using Proposition 4.1, (4.5) and Proposition 2.1(ii), we obtain the following corollary.

Corollary 4.1 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$ and $f \in \mathcal{B}^+((0,1))$, and then the following statements are equivalent:*

(i) *The function $x \to V_q f(x)$ is continuous on $[0,1]$.*

(ii) *The integral $\int_0^1 t^2(1-t)^{\alpha-2} f(t)\mathrm{d}t$ converges.*

Next, we will prove that the kernel V_q satisfies the following resolvent equation.

Lemma 4.3 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$ and $f \in \mathcal{B}^+((0,1))$, and then $V_q f$ satisfies the following resolvent equation:*

$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f). \tag{4.10}$$

In particular, if $V(qf) < \infty$, we have

$$(I - V_q\,(q.))(I + V\,(q\cdot))f = (I + V\,(q\cdot))(I - V_q\,(q\cdot))f = f. \tag{4.11}$$

Proof Let $(x,t) \in [0,1] \times [0,1]$, and then by (4.8), we have

$$G(x,t) = \mathcal{G}(x,t) + V\left(q\mathcal{G}\left(\cdot,t\right)\right)(x),$$

which implies by the Fubini-Tonelli theorem that for $f \in \mathcal{B}^+((0,1))$,

$$\begin{aligned} Vf(x) &= \int_0^1 \left(\mathcal{G}(x,t) + V\left(q\mathcal{G}\left(\cdot,t\right)\right)(x)\right) f(t)\mathrm{d}t \\ &= V_q f(x) + V(qV_q f)(x). \end{aligned}$$

On the other hand, by Lemma 4.1(iii) and the Fubini-Tonelli theorem, we obtain, for $f \in \mathcal{B}^+((0,1))$ and $x \in [0,1]$,

$$\int_0^1\int_0^1 \mathcal{G}(x,r)G(r,t)q(r)f(t)\mathrm{d}r\mathrm{d}t = \int_0^1\int_0^1 G(x,r)\mathcal{G}(r,t)q(r)f(t)\mathrm{d}r\mathrm{d}t,$$

that is,

$$V_q(qVf)(x) = V(qV_q f)(x).$$

So, we obtain

$$Vf = V_q f + V(qV_q f) = V_q f + V_q(qVf)(x).$$

Proposition 4.2 *Let q be a nonnegative function in $\mathcal{K}_\alpha \cap C((0,1))$ such that $\alpha_q \leq \frac{1}{2}$ and $f \in \mathcal{B}^+((0,1))$ such that $t \to t^2(1-t)^{\alpha-2} f(t)$ is continuous and integrable on $(0,1)$. Then $V_q f$ is the unique nonnegative solution in $C([0,1])$ of the perturbed fractional problem (1.12) satisfying*

$$(1-\alpha_q)Vf \leq V_q f \leq Vf. \tag{4.12}$$

Proof Since by Corollary 4.1 the function $x \to V_q f(x)$ is in $C([0,1])$, it follows that the function $x \to q(x)V_q f(x)$ is continuous on $(0,1)$. Using (4.10) and (2.19), there exists a nonnegative constant c such that

$$V_q f(x) \leq V f(x) \leq c x^{\alpha-3}(1-x). \tag{4.13}$$

So, we deduce that

$$\int_0^1 t^2(1-t)^{\alpha-2} q(t) V_q f(t) \mathrm{d}t \leq c \int_0^1 t^{\alpha-1}(1-t)^{\alpha-1} q(t) \mathrm{d}t < \infty.$$

Hence by using Proposition 2.2, the function $u = V_q f = V f - V(qV_q f)$ satisfies the equation

$$\begin{cases} D^\alpha u(x) = f(x) - q(x)u(x), \quad x \in (0,1), \\ u(0) = u(1) = D^{\alpha-3}u(0) = u'(1) = 0. \end{cases}$$

By integrating inequalities (4.5), we obtain (4.12). It remains to prove the uniqueness. Assume that v is another nonnegative solution in $C([0,1])$ of problem (1.12) satisfying (4.12). Since the function $t \to q(t)v(t)$ is continuous on $(0,1)$ and by (4.12)–(4.13), the function $t \to t^2(1-t)^{\alpha-2}q(t)v(t)$ is integrable on $(0,1)$, then it follows by Proposition 2.2 that the function $\widetilde{v} := v + V(qv)$ satisfies

$$\begin{cases} D^\alpha \widetilde{v}(x) = f(x), \quad x \in (0,1), \\ \widetilde{v}(0) = \widetilde{v}(1) = D^{\alpha-3}\widetilde{v}(0) = \widetilde{v}'(1) = 0. \end{cases}$$

From the uniqueness in Proposition 2.2, we deduce that

$$\widetilde{v} := v + V(qv) = Vf.$$

Hence

$$(I + V(q\cdot))(v - u) = 0.$$

Now, by (4.12)–(4.13) and (2.23), we have

$$V(q|v-u|) \leq 2cV(q[\Gamma(\alpha-2)h_1 + h_2]) \leq 2c\alpha_q(\Gamma(\alpha-2)h_1 + h_2) < \infty,$$

we deduce by (4.11) that $u = v$.

Proof of Theorem 1.2 Let $a \geq 0$ and $b \geq 0$ with $a + b > 0$ and recall that

$$\omega(x) := a h_1(x) + b h_2(x).$$

Since the function φ satisfies (H$_2$), there exists a positive function q in $\mathcal{K}_\alpha \cap C((0,1))$ such that $\alpha_q \leq \frac{1}{2}$ and for each $x \in (0,1)$, the map $t \to t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$. Let

$$S := \{u \in \mathcal{B}^+((0,1)) : (1-\alpha_q)\omega \leq u \leq \omega\}$$

and define the operator L on S by

$$Lu = \omega - V_q(q\omega) + V_q((q - \varphi(\cdot, u))u).$$

By (4.10) and (2.25), we have

$$V_q(q\omega) \leq V(q\omega) \leq \alpha_q\omega \leq \omega \tag{4.14}$$

and by (H_2) we obtain

$$0 \leq \varphi(\cdot, u) \leq q \quad \text{ for all } u \in S. \tag{4.15}$$

So, we claim that S is invariant under L. Indeed, using (4.14)–(4.15), we have, for $u \in S$,

$$Lu \leq \omega - V_q(q\omega) + V_q(qu) \leq \omega$$

and

$$Lu \geq \omega - V_q(q\omega) \geq (1 - \alpha_q)\omega.$$

Next, we will prove that the operator L is nondecreasing on S. Indeed, let $u, v \in S$ be such that $u \leq v$. Since for each $x \in (0, 1)$, the function $t \to t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0, 1]$, we obtain

$$Lv - Lu = V_q([v(q - \varphi(\cdot, v)) - u(q - \varphi(\cdot, u))]) \geq 0.$$

Now, we consider the sequence (u_n) defined by $u_0 = (1 - \alpha_q)\omega$ and $u_{n+1} = Lu_n$ for $n \in \mathbb{N}$. Since S is invariant under L, we have $u_1 = Lu_0 \geq u_0$ and by the monotonicity of L, we deduce that

$$(1 - \alpha_q)\omega = u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq \omega.$$

Hence by the dominated convergence theorem and (H_1)–(H_2), we conclude that the sequence (u_n) converges to a function $u \in S$ satisfying $u = (I - V_q(q\cdot))\omega + V_q((q - \varphi(\cdot, u))u)$. That is

$$(I - V_q(q\cdot))u = (I - V_q(q\cdot))\omega - V_q(u\varphi(\cdot, u)).$$

On the other hand, by (4.14), we have $V(qu) \leq V(q\omega) \leq \omega < \infty$, then by applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (4.10)–(4.11), we conclude that u satisfies

$$u = \omega - V(u\varphi(\cdot, u)). \tag{4.16}$$

It remains to prove that u is a solution of problem (1.7). Using (4.15), there exists a constant $c > 0$ such that

$$u(t)\varphi(t, u(t)) \leq q(t)\omega(t) \leq ct^{\alpha-3}(1 - t)q(t). \tag{4.17}$$

This implies by Corollary 2.1 that the function $x \to V(u\varphi(\cdot,u))(x)$ is in $C([0,1])$ and so by (4.16), u is in $C([0,1])$. Now, since by (H_1) and (4.17), the function $t \to t^2(1-t)^{\alpha-2}u(t)\varphi(t,u(t))$ is continuous and integrable on $(0,1)$, we conclude by Proposition 2.2 that u is the required solution. It remains to prove that under condition (H_3), u is the unique solution to problem (1.7) satisfying (1.13). Assume that v is another nonnegative solution in $C([0,1])$ to problem (1.7) satisfying (1.13). Since $v \le \omega$, we deduce by (4.17) that

$$0 \le v(t)\varphi(t,v(t)) \le q(t)\omega(t) \le ct^{\alpha-3}(1-t)q(t).$$

So, the function $t \to t^2(1-t)^{\alpha-2}v(t)\varphi(t,v(t))$ is continuous and integrable on $(0,1)$ and by Proposition 2.2, we conclude that the function $\widetilde{v} := v + V(v\varphi(\cdot,v))$ satisfies

$$\begin{cases} D^{\alpha}\widetilde{v}(x) = 0, \quad x \in (0,1), \\ \widetilde{v}(0) = \widetilde{v}(1) = 0, \quad D^{\alpha-3}\widetilde{v}(0) = a, \quad \widetilde{v}'(1) = -b. \end{cases}$$

From the uniqueness in problem (1.10), we deduce that $\widetilde{v} := v + V(v\varphi(\cdot,v)) = \omega$. That is

$$v = \omega - V(v\varphi(\cdot,v)). \tag{4.18}$$

Now, let h be the function defined on $[0,1]$ by

$$h(x) = \begin{cases} \dfrac{v(x)\varphi(x,v(x)) - u(x)\varphi(x,u(x))}{v(x)-u(x)}, & \text{if } v(x) \neq u(x), \\ 0, & \text{if } v(x) = u(x). \end{cases}$$

Then by (H_3), $h \in \mathrm{B}^+((0,1))$ and by (4.16) and (4.18), we have$(I + V(h\cdot))(v-u) = 0$. On the other hand, by (H_2), we remark that $h \le q$ and by (2.25) we deduce that

$$V(h|v-u|) \le 2V(q\omega) \le 2\alpha_q\omega < \infty.$$

Hence by (4.11), we conclude that $u = v$.

Proof of Corollary 1.1 Let $\varphi(x,t) = \lambda p(x)f(t)$, $\theta(t) = tf(t)$ and $\widetilde{p}(x) := p(x) \cdot \max_{0\le\xi\le\omega(x)} \theta'(\xi)$. It is clear that hypotheses (H_1) and (H_3) are satisfied. Since the function $q(x) := \lambda\widetilde{p}(x)$ belongs to the class $\mathcal{K}_\alpha$, we have $\alpha_q \le \frac{1}{2}$ for $\lambda \in \left[0, \frac{1}{2\alpha_{\widetilde{p}}}\right]$. Moreover, by a simple computation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[t(q(x) - \varphi(x,t\omega(x)))] = q(x) - \lambda p(x)\theta'(t\omega(x)) \ge 0$$

for $t \in [0,1]$ and $x \in (0,1)$. This implies that the function φ satisfies hypothesis (H_2). So, the result follows by Theorem 1.2.

Example 4.1 Let $3 < \alpha \le 4$ and $a,b \ge 0$ with $a+b > 0$. Let $\sigma \ge 0$, and p be a positive continuous function on $(0,1)$ such that

$$\int_0^1 r^{(\alpha-1)+(\alpha-3)\sigma}(1-r)^{\alpha+\sigma-1}p(r)\mathrm{d}r < \infty.$$

Let $\widetilde{p}(x) := (\sigma+1)p(x)(\omega(x))^{\sigma}$. Since $\widetilde{p} \in \mathcal{K}_\alpha$, then for $\lambda \in \left[0, \frac{1}{2\alpha_{\widetilde{p}}}\right]$, the problem

$$\begin{cases} D^\alpha u(x) + \lambda p(x) u^{\sigma+1}(x) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad D^{\alpha-3}u(0) = a, \quad u'(1) = -b \end{cases}$$

has a unique positive solution u in $C([0,1])$ satisfying

$$(1 - \lambda\alpha_{\widetilde{p}})\omega(x) \le u(x) \le \omega(x), \quad x \in [0,1].$$

Example 4.2 Let $3 < \alpha \le 4$ and $a \ge 0$, $b \ge 0$ with $a+b>0$. Let $\sigma \ge 0$, $\gamma > 0$ and p be a positive continuous function on $(0,1)$ such that

$$\int_0^1 r^{(\alpha-1)+(\alpha-3)(\sigma+\gamma)}(1-r)^{\alpha+\sigma+\gamma-1} p(r)\mathrm{d}r < \infty.$$

Let $\theta(s) = s^{\sigma+1}\log(1+s^\gamma)$ and $\widetilde{p}(t) := p(t) \cdot \max_{0\le\xi\le\omega(t)} \theta'(\xi)$. Since $\widetilde{p} \in \mathcal{K}_\alpha$, then for $\lambda \in \left[0, \frac{1}{2\alpha_{\widetilde{p}}}\right]$, the problem

$$\begin{cases} D^\alpha u(x) + \lambda p(x) u^{\sigma+1}(x)\log(1+u^\gamma(x)) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad D^{\alpha-3}u(0) = a, \quad u'(1) = -b \end{cases}$$

has a unique positive solution u in $C([0,1])$ satisfying

$$(1 - \lambda\alpha_{\widetilde{p}})\omega(x) \le u(x) \le \omega(x), \quad x \in [0,1].$$

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