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Null Controllability of Some Reaction-Diffusion Systems with Only One Control Force in Moving Domains*

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Abstract In this article, the authors establish the local null controllability property for semilinear parabolic systems in a domain whose boundary moves in time by a single control force acting on a prescribed subdomain. The proof is based on Kakutani's fixed point theorem combined with observability estimates for the associated linearized system.

Keywords Kakutani's fixed point theorem, Unique continuation, Null controllability
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1 Introduction and Main Results

In this article we investigate the question of local null controllability of a semilinear coupled parabolic system in the case of time dependent domains with a single control force. To make notations clear, let Ω be a bounded connected open set of $\mathbb{R}^n$ with the boundary C^2. For $T > 0$, we represent by Q_T the cylinder $\Omega \times (0, T)$ of $\mathbb{R}^{n+1}$ with a lateral boundary Σ_T defined by $\Gamma \times (0, T)$. Let us consider a family of functions $\{\tau_t\}_{0\le t\le T}$, where for each t, τ_t is a deformation of Ω into an open bounded set Ω_t of $\mathbb{R}^n$ defined by

$$\Omega_t = \{x \in \mathbb{R}^n;\ x = x_\tau(y) \ \text{ for } y \in \Omega\}.$$

For $t = 0$, we identify Ω_0 with Ω and τ_0 with the identity mapping. For convenience of notation, for vector $y \in \Omega$, we will write $y = (y_1, y_2, \cdots, y_n)$ and the points in the deformed domain Ω_t, $0 < t < T$, will be denoted by $x = (x_1, x_2, \cdots, x_n)$. The smooth boundary of Ω_t is represented by Γ_t. The non-cylindrical domain $\widehat{Q}_T$ and its lateral boundary $\widehat{\Sigma}_T$ are defined by

$$\widehat{Q}_T = \bigcup_{0\le t\le T} \{\Omega_t \times \{t\}\} \quad \text{and} \quad \widehat{\Sigma}_T = \bigcup_{0\le t\le T} \{\Gamma_t \times \{t\}\},$$

respectively.

We assume the following regularity on the functions τ_t for $0 \le t \le T$:

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(A1) τ_t is a C^2 diffeomorphism from Ω to Ω_t.

(A2) τ_t lies in $C^1([0,T];C^0(\overline{\Omega},\mathbb{R}^n))\cap C^0([0,T];C^2(\overline{\Omega},\mathbb{R}^n))$.

Thus we have a natural diffeomorphism $\tau_t : Q_T \to \widehat{Q}_T$ defined by

$$(y,t)\in Q_T \to (x,t)\in \widehat{Q}_T, \quad \text{where } x=\tau_t(y).$$

To simplify the presentation, the reference domain Ω is assumed to be bounded and of class C^2. Nevertheless, we remark that most of the results we present here still hold when Ω is Lipschitz continuous and unbounded. The regularity assumptions on the diffeomorphism τ_t may also be weakened. However, the minimal assumptions on the reference domain Ω and the transformation τ_t will depend very much on the notion of solution and the type of control problem under consideration.

Concerning the class of domains $\widehat{Q}_T$ which we are considering, it is important to point out that the assumptions above are not very restrictive. For instance, the condition (A2) that τ_t depends in a C^1 way on time (that, in practice, can often be replaced by a Lipschitz dependence) indicates that the domain does not evolve in time too roughly but allows all kinds of deformations on its shape. But, the conditions that Ω_t can be mapped into the reference domain Ω at every t by means of a C^2 diffeomorphism impose that the topology of Ω_t does not change as time evolves. This is the main restriction that we impose on the geometry of the space-time domain $\widehat{Q}_T$ under consideration. In particular, we do not address here the problems in which holes appear or disappear in Ω_t as time increases. This type of situation requires a separate analysis since solutions may develop singularities at those points where the topology of Ω_t changes.

Our main goal is to establish the null controllability for the following general reaction-diffusion system which arises in mathematical biology:

$$\begin{cases} \widehat{\psi}' - \Delta\widehat{\psi} + f_1(\widehat{\psi},\widehat{w}) = 0 & \text{in } \widehat{Q}_T,\\ \widehat{w}' - \Delta\widehat{w} + f_2(\widehat{\psi},\widehat{w}) = \chi_{\widehat{\omega}}\widehat{g} & \text{in } \widehat{Q}_T,\\ \widehat{\psi} = \widehat{w} = 0 & \text{on } \widehat{\Sigma}_T,\\ \widehat{\psi}(x,0)=\widehat{\psi}_0(x), \quad \widehat{w}(x,0)=\widehat{w}_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the control force $\widehat{g}$ acts on a unique equation of the system through an arbitrarily small open set $\widehat{\omega}$, where $\widehat{\omega}$ is the image by τ_t of a non-empty open subset ω of Ω.

In (1.1) we have $\widehat{\psi}=\widehat{\psi}(x,t)$, $\widehat{w}=\widehat{w}(x,t)$, $\widehat{\psi}'=\frac{\partial\widehat{\psi}}{\partial t}$, $\widehat{w}'=\frac{\partial\widehat{w}}{\partial t}$, $\chi_{\widehat{\omega}}$ is the characteristic function of $\widehat{\omega}$, and $\widehat{\psi}_0(x)$ and $\widehat{w}_0(x)$ are the initial states.

Throughout this paper we assume that the nonlinear functions $f_i:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are globally Lipschitz and $f_i(0,0)=0$. By this, we mean that there exist $M_1, M_2>0$ such that

$$\left| \begin{array}{l} |f_i(\psi,w)-f_i(\overline{\psi},\overline{w})| \le M_i(|\psi-\overline{\psi}|+|w-\overline{w}|), \quad i=1,2,\\ f_i(0,0)=0. \end{array}\right. \tag{1.2}$$

Assume also that there exists a positive constant $C_0>0$ such that

$$\frac{\partial f_1}{\partial w}(\psi,w)\ge C_0, \quad \forall(\psi,w)\in\mathbb{R}\times\mathbb{R}. \tag{1.3}$$

Under natural hypotheses on the $\widehat{\psi}_0, \widehat{w}_0$, we see that for each $\widehat{g}$, there exists exactly one solution $\widehat{\psi}, \widehat{w}$ to (1.1), with

$$\widehat{\psi}, \widehat{w} \in C^0\big([0,T]; L^2(\Omega)\big).$$

The main aim of this paper is to analyze the controllability properties of (1.1) when the control acts on a single equation of system.

The system (1.1) is to be said null controllable at time T if the following holds: For any given $\widehat{\psi}_0, \widehat{w}_0 \in L^2(\Omega)$, there exist controls $\widehat{g} \in L^2(\widehat{\omega} \times (0,T))$ such that the corresponding solutions to (1.1) satisfy

$$\widehat{\psi}(x,T) = 0, \quad \widehat{w}(x,T) = 0 \quad \text{in } \Omega, \tag{1.4}$$

with an estimate of the form

$$|\widehat{g}|^2_{L^2(\widehat{\omega}\times(0,T))} \le C(|\widehat{\psi}_0|^2_{L^2(\Omega)} + |\widehat{w}_0|^2_{L^2(\Omega)}). \tag{1.5}$$

The system (1.1) is to be said locally null controllable at time T if the previous property holds for any $\widehat{\psi}_0, \widehat{w}_0$ in a ball $B(0;\delta) \subset L^2(\Omega)$, with δ depending on T.

Recently, important progress has been made in the controllability analysis of semilinear parabolic equations. We refer to the works [7–8, 10–11, 14] and the references therein in the context of bounded domains, and the works [3, 6] in the context of more general domains. It is natural, from both the theoretical and applied viewpoints, to try to extend the known results to systems of kind (1.1). It is particulary important to highlight that system (1.1) has only one control, which is in accordance with the theoretical philosophy of trying to control a system with the least possible controls; to this direction we cite [1, 12, 16–17].

Our main result is the following.

Theorem 1.1 *Assume that the non-cylindrical domain* $\widehat{Q}_T$ *and functions* $f_i, i = 1,2$, *satisfy the geometric conditions* (A1)–(A2) *and the conditions* (1.2)–(1.3) *respectively. Then the nonlinear system* (1.1) *is locally null controllable at any time* $T > 0$.

The methodology in the present paper consists in turning the non-cylindrical state equation (1.1) into a cylindrical one (see (1.8) below) by the diffeomorphism τ_t.

To carry on this methodology, we first denote by $\varphi_t(x)$ the inverse map of τ_t, that is, $\varphi_t = \tau_t^{-1}$. According to the assumption (A1), φ_t is a C^2-map from Ω_t to Ω, for all $0 \le t \le T$. We shall use the notation $\varphi(x,t) = \varphi_t(x)$. Thus the state on Q_T is defined by

$$\begin{aligned} \psi(y,t) &= \widehat{\psi}(\tau_t(y),t) = \widehat{\psi}(\tau(y,t),t) \quad \text{for all } y \in \Omega, \\ w(y,t) &= \widehat{w}(\tau_t(y),t) = \widehat{w}(\tau(y,t),t) \quad \text{for all } y \in \Omega. \end{aligned} \tag{1.6}$$

Equivalently in $\widehat{Q}_T$ we have

$$\begin{aligned} \widehat{\psi}(x,t) &= \psi(\varphi_t(x),t) \quad \text{for all } x \in \Omega_t, \\ \widehat{w}(x,t) &= w(\varphi_t(x),t) \quad \text{for all } x \in \Omega_t. \end{aligned} \tag{1.7}$$

Therefore, the initial-boundary value problem (1.1) is equivalent to:

$$\begin{cases} \psi_t + A(t)\psi + f_1(\psi, w) = 0 & \text{in } Q_T, \\ w_t + A(t)w + f_2(\psi, w) = \chi_w g & \text{in } Q_T, \\ \psi = w = 0 & \text{on } \Sigma_T, \\ \psi(x,0) = \psi_0(x), \quad w(x,0) = w_0(x) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where

$$\left| \begin{array}{l} A(t)\psi(y,t) = \widetilde{A}(t)\psi(y,t) + \vec{b}(y,t)\cdot\nabla\psi, \\ \widetilde{A}(t)\psi(y,t) = -\displaystyle\sum_{k,j=1}^{n} \frac{\partial}{\partial y_k}\Big(\alpha_{kj}(y,t)\frac{\partial\psi}{\partial y_j}\Big), \\ \alpha_{kj}(y,t) = \dfrac{\partial\varphi_k}{\partial x_i}(\tau_t(y),t)\,\dfrac{\partial\varphi_j}{\partial x_i}(\tau_t(y),t), \\ \vec{b}(y,t) = (b_j(y,t))_{1\le j\le n}, \\ b_j(y,t) = \dfrac{\partial\varphi_j}{\partial t}(\tau_t(y),t) + \displaystyle\sum_{k=1}^{n}\frac{\partial\alpha_{kj}}{\partial y_k}(y,t) - \Delta_x\varphi_j(\tau_t(y),t), \\ g(y,t) = \widehat{g}(\tau_t(y),t), \\ \psi_0(y) = \widehat{\psi}_0(\tau_0(y)), \quad w_0(y) = \widehat{w}_0(\tau_0(y)). \end{array} \right.$$

The system (1.8) is a variable coefficient parabolic equation in the cylindrical domain Q_T. From the technical point of view, a new problem arises because the state equation (1.8) contains a uniformly coercive operator $\widetilde{A}(t)$. More precisely, for $\psi, w \in H_0^1(\Omega)\times H_0^1(\Omega)$ and by Gaussian lemma, we obtain the bilinear form $\alpha(t,\psi,w)$ defined by

$$\alpha(t,\psi,w) = (\widetilde{A}(t)\psi, w) = \sum_{k,j=1}^{n}\int_\Omega \alpha_{kj}(y,t)\frac{\partial\psi}{\partial y_j}\frac{\partial w}{\partial y_k}\,\mathrm{d}y.$$

This bilinear form is bounded because $\varphi(x,t) = {\tau_t}^{-1}(x)$ is a C^2 diffeomorphism between Ω_t and Ω (see (A1)). Then its matrix $M = \big(\frac{\partial\varphi_j}{\partial x_i}\big)_{1\le i,j\le n}$ is inversible and for all $\eta\in\mathbb{R}^n$ we have $\|M^{-1}\eta\|_{\mathbb{R}^n} \le \frac{1}{\alpha_0}\|\eta\|_{\mathbb{R}^n}$, $\alpha_0 > 0$. From the last inequality, we have the estimate:

$$\alpha(t,\psi,\psi) \ge \alpha_0^2\|\psi\|^2_{H_0^1(\Omega)}, \tag{1.9}$$

proving the coercivity of α in $H_0^1(\Omega)\times H_0^1(\Omega)$. Note that according to assumptions (A1) and (A2), the boundary value problem (1.8) is a classical problem studied by Lions in [18]. If we take $\psi_0, w_0 \in H_0^1(\Omega)$ and $g \in L^2(0,T;L^2(\Omega))$, then (1.8) has a strong solution $\psi, w \in C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Otherwise, if $\psi_0, w_0 \in L^2(\Omega)$ and $g \in L^2(0,T;L^2(\Omega))$, then (1.8) has a weak solution $\psi, w \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. In both cases we have uniqueness.

By using the diffeomorphism $(y,t)\to(x,t)$, from Q_T to $\widehat{Q}_T$, we obtain a unique solution $\widehat{\psi}, \widehat{w}$ to the problem (1.1) with the regularity, namely:

(1) If $\widehat{\psi}_0, \widehat{w}_0 \in H_0^1(\Omega)$, $\widehat{g} \in L^2(0,T;L^2(\Omega_t))$, then $\widehat{\psi}, \widehat{w} \in C([0,T];H_0^1(\Omega_t)) \cap L^2(0,T;H^2(\Omega_t)) \cap H^1(0,T;L^2(\Omega_t))$.

(2) If $\widehat{\psi}_0, \widehat{w}_0 \in L^2(\Omega)$, $\widehat{g} \in L^2(0,T;L^2(\Omega_t))$, then $\widehat{\psi}, \widehat{w} \in C([0,T];L^2(\Omega_t)) \cap L^2(0,T;H_0^1(\Omega_t))$.

At this point we underline that, under assumptions (A1)–(A2), the transformation $y \to x$ does indeed map the space of functions $C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ into $C([0,T];H_0^1(\Omega_t)) \cap L^2(0,T;H^2(\Omega_t)) \cap H^1(0,T;L^2(\Omega_t))$.

For a detailed discussion about the existence, uniqueness and regularity of solutions to the system (1.1), we refer to [23]. There, they have used the energy method combined with Poincaré's inequality (satisfied uniformly in the domains Ω_t for all $0 \le t \le T$), the uniform (with respect to t) elliptic regularity, and the classical trace results and interpolation. They did not use the diffeomorphism $(y,t) \to (x,t)$ from Q_T to $\widehat{Q}_T$, and they worked directly on the system (1.1) to see how the structure of the non-cylindrical domain affects the estimates that are used in the energy method.

This paper is organized as follows. Section 2 is devoted to proving the null controllability of a linearized system, which is similar to (1.8). In Section 3, we prove Theorem 1.1 by a fixed point method.

We close this section by mentioning some basic references on the analysis of partial differential equations in non-cylindrical domains. Among many references we mention the following: Lions [19], Cooper and Bardos [5], Medeiros [20], Inoue [15], Nakao and Narazaki [23] for wave equations; Bernardi, Bonfanti and Lutteroti [2], Miranda and Medeiros [22] for Schrödinger equations; He and Hsiano [13] for Euler equations; Miranda and Limaco [21] for Navier-Stokes equations; Chen and Frid [4] for hyperbolic systems of the conservation law.

2 Analysis of the Controllability of the Linearized System

The main result of this article will be proved in Section 3 by means of a fixed point argument. For this, we observe that for any $\psi, w \in L^2(Q_T)$, the following identity holds:

$$\begin{aligned} & f_1(\psi,w) - f_1(0,0) \\ = & \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma} f_1(\sigma\psi,\sigma w)\,\mathrm{d}\sigma \\ = & \int_0^1 \frac{\partial}{\partial z} f_1(\sigma\psi,\sigma w)\,\mathrm{d}\sigma\ \psi + \int_0^1 \frac{\partial}{\partial \eta} f_1(\sigma\psi,\sigma w)\,\mathrm{d}\sigma\ w \\ = & a(\psi,w)\psi + b(\psi,w)w, \end{aligned} \tag{2.1}$$

where $\frac{\partial f_1}{\partial z}$ and $\frac{\partial f_1}{\partial \eta}$ denote the partial derivatives of f_1 with respect to the variables ψ and w, respectively, and the functions $a(\psi,w)$ and $b(\psi,w)$ are defined as

$$\left| \begin{aligned} & a(\psi,w) = \int_0^1 \frac{\partial}{\partial z} f_1(\sigma\psi,\sigma w)\,\mathrm{d}\sigma, \\ & b(\psi,w) = \int_0^1 \frac{\partial}{\partial \eta} f_1(\sigma\psi,\sigma w)\,\mathrm{d}\sigma. \end{aligned} \right. \tag{2.2}$$

Similarly, we define the functions $c = c(\psi,w)$ and $d = d(\psi,w)$ as

$$\left| \begin{aligned} & c(\psi,w) = \int_0^1 \frac{\partial}{\partial z} f_2(\sigma\psi,\sigma w)\,\mathrm{d}\sigma, \\ & d(\psi,w) = \int_0^1 \frac{\partial}{\partial \eta} f_2(\sigma\psi,\sigma w)\,\mathrm{d}\sigma. \end{aligned} \right. \tag{2.3}$$

Note that

$$f_2(\psi,w)-f_2(0,0)=c(\psi,w)\psi+d(\psi,w)w. \tag{2.4}$$

Moreover, we assume the bounds:

$$\left|\begin{array}{l}|a(\psi,w)|_{L^\infty(Q_T)}\le M_1,\quad |b(\psi,w)|_{L^\infty(Q_T)}\le M_1,\\ |c(\psi,w)|_{L^\infty(Q_T)}\le M_2,\quad |d(\psi,w)|_{L^\infty(Q_T)}\le M_2,\end{array}\right. \tag{2.5}$$

where M_1, M_2 are the positive constants given in (1.2).

With this notation, the system (1.8) can be rewritten in the form

$$\begin{cases}\psi_t+A(t)\psi+a(\psi,w)\psi+b(\psi,w)w=0 & \text{in } Q_T,\\ w_t+A(t)w+c(\psi,w)\psi+d(\psi,w)w=\chi_w g & \text{in } Q_T,\\ \psi=w=0 & \text{on } \Sigma_T,\\ \psi(0)=\psi_0,\quad w(0)=w_0 & \text{in } \Omega.\end{cases} \tag{2.6}$$

Given $\overline{\psi},\overline{w}\in L^2(Q_T)$ we now consider the linearized system

$$\begin{cases}\psi_t+A(t)\psi+a(\overline{\psi},\overline{w})\psi+b(\overline{\psi},\overline{w})w=0 & \text{in } Q_T,\\ w_t+A(t)w+c(\overline{\psi},\overline{w})\psi+d(\overline{\psi},\overline{w})w=\chi_w g & \text{in } Q_T,\\ \psi=w=0 & \text{on } \Sigma_T,\\ \psi(0)=\psi_0,\quad w(0)=w_0 & \text{in } \Omega.\end{cases} \tag{2.7}$$

Observe that the system (2.7) is linear on the states ψ and w, and has potentials $a,b,c,d\in L^\infty(Q_T)$ satisfying the uniform bound given in (2.5).

Again, with this notation we rewrite the system (2.7) as

$$\begin{cases}\psi_t+A(t)\psi+a(y,t)\psi+b(y,t)w=0 & \text{in } Q_T,\\ w_t+A(t)w+c(y,t)\psi+d(y,t)w=\chi_w g & \text{in } Q_T,\\ \psi=w=0 & \text{on } \Sigma_T,\\ \psi(0)=\psi_0,\quad w(0)=w_0 & \text{in } \Omega.\end{cases} \tag{2.8}$$

As usual, the controllability of (2.8) is closely related to the properties of the solutions to the associated adjoint states. In this case, the adjoint systems are

$$\begin{cases}-u_t+A^*(t)u+a(y,t)u+c(y,t)v=0 & \text{in } Q_T,\\ -v_t+A^*(t)v+b(y,t)u+d(y,t)v=0 & \text{in } Q_T,\\ u=v=0 & \text{on } \Sigma_T,\\ u(T)=u_T,\quad v(T)=v_T & \text{in } \Omega,\end{cases} \tag{2.9}$$

where $A^*(t)$ is the formal adjoint of the operator $A(t)$, $u_T, v_T\in L^2(\Omega)$.

Next we sketch the points used in the proof of the null controllability of the system (2.8) using the observability estimate. First, we use a global-Carleman inequality satisfied by the solutions to (2.9). Second, this inequality allows us to establish an observability estimate. Third, we prove the null controllability of (2.8) by using the observability estimate.

In this approach, the following technical result, due to Fursikov and Imanuvilov [11], is fundamental.

Lemma 2.1 *Let $\omega \Subset \Omega$ be a non-empty open set. There exists a function $\mu \in C^2(\overline{\Omega})$ satisfying*

$$\begin{aligned} &\mu(y) > 0, \quad \forall y \in \Omega, \\ &\mu = 0, \quad \forall y \in \partial\Omega, \\ &|\nabla \mu(y)| \geq k > 0, \quad \forall y \in \Omega \setminus \omega. \end{aligned}$$

Let us introduce the functions

$$\phi(y,t) = \frac{\mathrm{e}^{\lambda\mu(y)}}{\beta(t)}, \quad \alpha(y,t) = \frac{\mathrm{e}^{\lambda\mu(y)} - \mathrm{e}^{2\lambda\Psi}}{\beta(t)} < 0, \tag{2.10}$$

where $\Psi = \|\mu\|_{L^\infty}$, $\beta(t) = t(T-t)$ for $0 \leq t \leq T$ and $\lambda > 0$.

We will use the following Carleman inequality.

Theorem 2.1 *There exist positive constants λ_0, s_0, C_0 and C_1 such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$ and any solution to* (2.9) (*corresponding to some $u_T, v_T \in L^2(\Omega)$*), *one has*

$$\begin{aligned} &\iint_{Q_T} \mathrm{e}^{2s\alpha}[(s\phi)^{-1}(|u_t|^2 + |\Delta u|^2) + \lambda^2 s\phi|\nabla u|^2 + \lambda^4(s\phi)^3|u|^2]\,\mathrm{d}y\mathrm{d}t \\ &\leq C_0\Big(\iint_{Q_T} \mathrm{e}^{2s\alpha}|v|^2\,\mathrm{d}y\mathrm{d}t + \iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3|u|^2\,\mathrm{d}y\mathrm{d}t\Big), \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\iint_{Q_T} \mathrm{e}^{2s\alpha}[(s\phi)^{-1}(|v_t|^2 + |\Delta v|^2) + \lambda^2 s\phi|\nabla v|^2 + \lambda^4(s\phi)^3|v|^2]\,\mathrm{d}y\mathrm{d}t \\ &\leq C_1\Big(\iint_{Q_T} \mathrm{e}^{2s\alpha}|u|^2\,\mathrm{d}y\mathrm{d}t + \iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3|v|^2\,\mathrm{d}y\mathrm{d}t\Big). \end{aligned} \tag{2.12}$$

Furthermore, C_0, C_1 and λ_0 depend only on Ω and ω, and s_0 can be chosen of the form

$$s_0 = C(\Omega,\omega)(T + T^2). \tag{2.13}$$

This result was essentially proved in [11] (in fact, similar Carleman inequalities were established there for much more general linear parabolic equations); see also [9]. In fact, the coefficients of the principal part $A^*(t)$, according to the assumptions (A1) and (A2), are of class C^1 and a, b, c and d are uniformly bounded. Under these conditions, the Carleman inequalities presented in [11] or [9] guarantee (2.11) and (2.12).

We remark that the explicit dependence on time of the constants is not given in [11]. We refer to [9] where the above formula for s_0 was obtained.

Remark 2.1 The Carleman inequalities presented in [11] and [9] were used there to derive the null controllability. More precisely, they were applied to the adjoint equation after the coordinate transformation $x \to y$, which requires C^1 or the Lipschitz condition on the coefficients in the principal part. This means that the geometric assumptions (A1)–(A2) are almost necessary to establish the existence results.

Since, as far as we know, there is no negative result for the null controllability in the case of parabolic equations with bounded and coercive coefficients, we expect that the null controllability operates under much weaker assumptions than (A1)–(A2).

Let us introduce the following notation: For given λ and s as in Theorem 2.1, we set

$$I(u) = \iint_{Q_T} \mathrm{e}^{2s\alpha}[(s\phi)^{-1}(|u_t|^2 + |\Delta u|^2) + \lambda^2 s\phi|\nabla u|^2 + \lambda^4 (s\phi)^3 |u|^2]\, \mathrm{d}y\mathrm{d}t \tag{2.14}$$

and

$$I(v) = \iint_{Q_T} \mathrm{e}^{2s\alpha}[(s\phi)^{-1}(|v_t|^2 + |\Delta v|^2) + \lambda^2 s\phi|\nabla v|^2 + \lambda^4 (s\phi)^3 |v|^2]\, \mathrm{d}y\mathrm{d}t. \tag{2.15}$$

As consequence of Theorem 2.1, we have the following lemma.

Lemma 2.2 *Consider the same notations as in Theorem* 2.1. *For $\lambda, s \gg 0$, the solutions u, v of* (2.9) *satisfiy the estimate:*

$$I(u) + I(v) \le C\Big(\iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4 (s\phi)^3 (|u|^2 + |v|^2)\, \mathrm{d}y\mathrm{d}t\Big). \tag{2.16}$$

This lemma already implies the null controllability of (2.8) by two control forces. This corresponds to the case where we plug a second force $f\chi_\omega$ into the right-hand side of the first equation of (2.8). So, now, the problem is to get rid of the term $\iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3|u|^2\mathrm{d}y\,\mathrm{d}t$ in the right-hand side of (2.16). The main plan to do this is to use the second equation in (2.9) to estimate this last integral in terms of $\iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3|v|^2\mathrm{d}y\,\mathrm{d}t$. The construction of the functional (2.21) below turns around this idea. This paper is based on the following crucial result.

Theorem 2.2 *Consider the hypotheses of Lemma* 2.2 *and assume, moreover, that there exists a constant $b_0 > 0$ and a domain ω_b such that*

$$\omega_b \Subset \omega \tag{2.17}$$

and

$$|b| \ge b_0 \quad \text{in } \omega_b \times (0, T_0) \tag{2.18}$$

for some $T_0 > 0$. Then for all $r \in [0, 2)$ there exists a constant $C = C(r, T, \lambda)$ such that

$$\iint_{\omega'\times(0,T)} \mathrm{e}^{2\alpha}(u^2 + v^2)\, \mathrm{d}y\mathrm{d}t \le C \iint_{\omega\times(0,T)} \mathrm{e}^{r\alpha} v^2 \mathrm{d}y\mathrm{d}t \tag{2.19}$$

for all ω' satisfying $\omega' \Subset \omega_b \Subset \omega \Subset \Omega$.

Proof The main idea is to estimate $\iint_{\omega'\times(0,T)} \mathrm{e}^{2\alpha}u^2\mathrm{d}y\mathrm{d}t$ by $\iint_{\omega\times(0,T)} \mathrm{e}^{r\alpha}v^2\mathrm{d}y\mathrm{d}t$ for some $r \in [0, 2)$ using the second equation of (2.9). To do this, let $\xi \in C^\infty(\mathbb{R}^n)$ be a truncation function satisfying

$$\left|\begin{array}{ll} \xi(y) = 1, & \forall y \in \omega', \\ 0 < \xi(y) \le 1, & \forall y \in \omega'', \\ \xi(y) = 0, & \forall y \in \mathbb{R}^n \setminus \omega'', \end{array}\right. \tag{2.20}$$

where $\omega' \Subset \omega'' \Subset \omega_b \Subset \omega \Subset \Omega$.

Assume for example that $b \geq b_0 > 0$ in $\omega_b \times (0,T)$ and introduce the function $\eta := \xi^6$. For real numbers β_0, β_1, p and $q > 0$, which will be chosen, set

$$\Gamma(t) = \int_\Omega (\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}u^2 + \beta_0\mathrm{e}^{2\alpha}\eta uv + \beta_1\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}v^2)\,\mathrm{d}y. \tag{2.21}$$

In the other case $-b \geq b_0 > 0$, we modify the term $(\beta_0\mathrm{e}^{2\alpha}\eta uv)$ to $(-\beta_0\mathrm{e}^{2\alpha}\eta uv)$ in the expression for Γ.

Differentiating Γ with respect to t and replacing u_t and v_t by their expressions given by (2.9), we obtain

$$\begin{aligned}\Gamma'(t) = &\int_\Omega (p\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}u^2 + 2\beta_0\mathrm{e}^{2\alpha}\eta uv + \beta_1 q\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}v^2)\alpha_t\mathrm{d}y\\
&+\int_\Omega (2\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}uu_t + \beta_0\mathrm{e}^{2\alpha}\eta u_t v + \beta_0\mathrm{e}^{2\alpha}\eta uv_t + 2\beta_1\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}vv_t)\,\mathrm{d}y\\
=&\int_\Omega (p\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}u^2 + 2\beta_0\mathrm{e}^{2\alpha}\eta uv + \beta_1 q\mathrm{e}^{q\alpha}n^{\frac{2}{3}}v^2)\alpha_t\,\mathrm{d}y\\
&+2\int_\Omega \mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}u(A^*u + au + cv)\,\mathrm{d}y\\
&+\beta_0\int_\Omega \mathrm{e}^{2\alpha}\eta[(A^*u + au + cv)v + u(A^*v + bu + dv)]\,\mathrm{d}y\\
&+2\beta_1\int_\Omega \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}v(A^*v + bu + dv)\,\mathrm{d}y.\end{aligned} \tag{2.22}$$

Hence, integration from 0 to $t \leq T$ and using $\Gamma(0) = \Gamma(T) = 0$ (because $\alpha(0) = 0$, $\alpha(T) = -\infty$; see (2.10)) yield

$$\begin{aligned}&\iint_{Q_T} \beta_0\mathrm{e}^{2\alpha}\eta bu^2\mathrm{d}y\mathrm{d}t\\
=&\iint_{Q_T} -\{(p\alpha_t + 2a)\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}u^2 + (\beta_1(q\alpha_t + 2d)\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}} + \beta_0\mathrm{e}^{2\alpha}\eta c)v^2\\
&+(\beta_0(2\alpha_t + a + d)\mathrm{e}^{2\alpha}\eta + 2\beta_1\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}b + 2\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}c)uv\}\,\mathrm{d}y\mathrm{d}t - 2\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}uA^*u\,\mathrm{d}y\mathrm{d}t\\
&-\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta(uA^*v + vA^*u)\,\mathrm{d}y\mathrm{d}t - 2\beta_1\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}vA^*v\,\mathrm{d}y\mathrm{d}t\\
=&\,\mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4.\end{aligned} \tag{2.23}$$

Next we estimate the above four integrals separately.

Estimate for J_1 In fact, since $\alpha_t \notin L^\infty(Q_T)$, we introduce $r \in [0,2)$ and write $\mathrm{e}^{2\alpha} = \mathrm{e}^{(2-r)\alpha}\mathrm{e}^{r\alpha}$. Assuming that

$$p > 2, \quad q > 1 + \frac{r}{2}, \quad r < 2, \tag{2.24}$$

and that $\beta_0, \beta_1 \geq 1$, by using the Cauchy-Schwarz inequality, we get

$$\mathrm{J}_1 \leq C\Big(\Big[1 + |a|_\infty + \frac{1}{T^4}\Big]\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t$$

$$+ |\beta|^2 \Big[1 + |a|^2_\infty + |b|^2_\infty + |c|^2_\infty + |d|^2_\infty + \frac{1}{T^8}\Big] \iint_{Q_T} \eta^{\frac{1}{3}} \mathrm{e}^{r\alpha} v^2 \mathrm{d}y\mathrm{d}t\Big), \tag{2.25}$$

where $C = C(p, q, |\eta|_\infty)$ and $|\beta|^2 = \beta_0^2 + \beta_1^2$.

Estimate for $\mathbf{J_2}$ We have

$$\begin{aligned}
\mathrm{J}_2 &= -2 \iint_{Q_T} \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} u A^* u \,\mathrm{d}y\mathrm{d}t \\
&= 2 \iint_{Q_T} \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} u \Big(\sum_{i,j=1}^{n} \frac{\partial}{\partial y_j} (\alpha_{ij} \frac{\partial u}{\partial y_i}) + \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} \,\mathrm{div}(\vec{b}u) \Big) \,\mathrm{d}y\mathrm{d}t \\
&= -2 \iint_{Q_T} \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} \frac{\partial u}{\partial y_j} \alpha_{ij} \frac{\partial u}{\partial y_i} \,\mathrm{d}y\mathrm{d}t \\
&\quad - 2 \iint_{Q_T} \Big[\frac{\partial}{\partial y_j} (\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}}) \alpha_{ij} u \frac{\partial u}{\partial y_i} + \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} \frac{\partial b_i}{\partial y_i} u^2 + \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} b_i u \frac{\partial u}{\partial y_i} \Big] \,\mathrm{d}y\mathrm{d}t \\
&\leq -2\alpha_0 \iint_{Q_T} \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} |\nabla u|^2 + \iint_{Q_T} \frac{\partial}{\partial y_i} \Big(\alpha_{ij} \frac{\partial}{\partial y_j} (\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}}) \Big) u^2 \mathrm{d}y\mathrm{d}t \\
&\quad + \iint_{Q_T} \Big[\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} \Big| \frac{\partial b_i}{\partial y_i} \Big| u^2 - \frac{\partial}{\partial y_i} (\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} b_i) u^2 \Big] \,\mathrm{d}y\mathrm{d}t,
\end{aligned} \tag{2.26}$$

where we have used the inequality (1.9).

Set

$$\left| \begin{aligned}
N_1 &= \iint_{Q_T} \frac{\partial}{\partial y_i} \Big(\alpha_{ij} \frac{\partial}{\partial y_j} (\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}}) \Big) u^2 \mathrm{d}y\mathrm{d}t, \\
N_2 &= \iint_{Q_T} \mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} \Big| \frac{\partial b_i}{\partial y_i} \Big| u^2 \mathrm{d}y\mathrm{d}t, \\
N_3 &= - \iint_{Q_T} \frac{\partial}{\partial y_i} (\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}} b_i) u^2 \mathrm{d}y\mathrm{d}t.
\end{aligned} \right. \tag{2.27}$$

Derivate the integral N_1 with respect to y_i, $\frac{\partial}{\partial y_i}$, we find 8 terms, namely:

$$\begin{aligned}
&\mathrm{e}^{-2\alpha} \eta^{-1} \frac{\partial}{\partial y_i} \Big(\alpha_{ij} \frac{\partial}{\partial y_j} (\mathrm{e}^{p\alpha} \eta^{\frac{4}{3}}) \Big) \\
&= \mathrm{e}^{(p-2)\alpha} \frac{\partial \alpha_{ij}}{\partial y_i} \frac{\partial \alpha}{\partial y_j} \eta^{\frac{1}{3}} + \mathrm{e}^{(p-2)\alpha} p^2 \alpha_{ij} \frac{\partial \alpha}{\partial y_i} \frac{\partial \alpha}{\partial y_j} \eta^{\frac{1}{3}} \\
&\quad + \mathrm{e}^{(p-2)\alpha} p \alpha_{ij} \frac{\partial^2 \alpha}{\partial y_i \partial y_j} \eta^{\frac{1}{3}} + \mathrm{e}^{(p-2)\alpha} p \alpha_{ij} \frac{\partial \alpha}{\partial y_j} \frac{4}{3} \eta^{-\frac{2}{3}} \frac{\partial \eta}{\partial y_i} \\
&\quad + \mathrm{e}^{(p-2)\alpha} \frac{\partial \alpha_{ij}}{\partial y_i} \frac{4}{3} \eta^{-\frac{2}{3}} \frac{\partial \eta}{\partial y_j} + \mathrm{e}^{(p-2)\alpha} \alpha_{ij} p \eta^{-\frac{2}{3}} \frac{4}{3} \frac{\partial \alpha}{\partial y_i} \frac{\partial \eta}{\partial y_j} \\
&\quad + \mathrm{e}^{(p-2)\alpha} \alpha_{ij} \frac{4}{3} (\frac{4}{3} - 1) \eta^{\frac{4}{3}-3} \frac{\partial \eta}{\partial y_i} \frac{\partial \eta}{\partial y_j} + \mathrm{e}^{(p-2)\alpha} \alpha_{ij} \frac{4}{3} \eta^{\frac{4}{3}-2} \frac{\partial^2 \eta}{\partial y_i \partial y_j} \\
&= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8.
\end{aligned} \tag{2.28}$$

In the following, we estimate each of these terms.

From the definition of $\eta = \xi^6$, we obtain

$$\Big| \eta^{-\frac{2}{3}} \frac{\partial \eta}{\partial y_i} \Big| = \Big| 6\xi \frac{\partial \xi}{\partial y_i} \Big| \leq C, \quad \Big| \eta^{-\frac{5}{6}} \frac{\partial \eta}{\partial y_i} \Big| = \Big| 6 \frac{\partial \xi}{\partial y_i} \Big| \leq C,$$

$$\Big|\eta^{-\frac{2}{3}}\frac{\partial^2\eta}{\partial y_i\partial y_j}\Big| = \Big|30\frac{\partial\xi}{\partial y_j}\frac{\partial\xi}{\partial y_i} + 6\xi\frac{\partial^2\xi}{\partial y_i\partial y_j}\Big| \le C.$$

It follows from this last computation and the assumptions (A1)–(A2) that

$$\left|\begin{aligned}&|a_1|_\infty \le C\Big(1+\frac{1}{T^4}\Big), \quad |a_2|_\infty \le C\Big(1+\frac{1}{T^4}\Big), \quad |a_3|_\infty \le \Big(1+\frac{1}{T^4}\Big),\\ &|a_4|_\infty \le C\Big(1+\frac{1}{T^4}\Big), \quad |a_5|_\infty \le C, \quad |a_6|_\infty \le C\Big(1+\frac{1}{T^4}\Big),\\ &|a_7|_\infty \le C, \quad |a_8|_\infty \le C,\end{aligned}\right. \tag{2.29}$$

where $C = C(p, |\eta|_\infty)$.

Plugging these estimates into N_1, we get

$$\begin{aligned}N_1 &= \iint_{Q_T} \mathrm{e}^{-2\alpha}\eta^{-1}\frac{\partial}{\partial y_i}\Big(\alpha_{ij}\frac{\partial}{\partial y_j}(\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}})\Big)\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t\\ &\le \Big|\mathrm{e}^{-2\alpha}\eta^{-1}\Big(\frac{\partial}{\partial y_i}\Big(\alpha_{ij}\frac{\partial}{\partial y_j}(\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}})\Big)\Big)\Big|_\infty \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t\\ &\le C\Big(1+\frac{1}{T^4}\Big)\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t. \end{aligned}\tag{2.30}$$

Proceeding as previously, we obtain

$$N_2 \le \Big|\mathrm{e}^{(p-2)\alpha}\eta^{\frac{1}{3}}\frac{\partial b_i}{\partial y_i}\Big|_\infty \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t \le C\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t, \tag{2.31}$$

because $\big|\mathrm{e}^{(p-2)\alpha}\eta^{\frac{1}{3}}\frac{\partial b_i}{\partial y_i}\big|_\infty \le C$.

Also,

$$N_3 = -\iint_{Q_T} \mathrm{e}^{-2\alpha}\eta^{-1}\Big(\frac{\partial}{\partial y_i}(\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}b_i)\Big)\mathrm{e}^{2\alpha}\eta u^2. \tag{2.32}$$

On the other hand,

$$\frac{\partial}{\partial y_i}(\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}b_i) = p\mathrm{e}^{p\alpha}\frac{\partial\alpha}{\partial y_i}\eta^{\frac{4}{3}}b_i + \mathrm{e}^{p\alpha}\frac{4}{3}\eta^{\frac{1}{3}}\frac{\partial\eta}{\partial y_i}b_i + \mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}\frac{\partial b_i}{\partial y_i},$$

and thus

$$N_3 \le \Big|\mathrm{e}^{-2\alpha}\eta^{-1}\Big(\frac{\partial}{\partial y_i}(\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}b_i)\Big)\Big|_\infty \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t \le C\Big(1+\frac{1}{T^4}\Big)\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t. \tag{2.33}$$

Substituting (2.30)–(2.31) and (2.33) into (2.26) we have

$$\mathrm{J}_2 \le -2\alpha_0\iint_{Q_T} \mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2 + C\Big(1+\frac{1}{T^4}\Big)\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t. \tag{2.34}$$

Estimate for $\mathbf{J_3}$ We have

$$\begin{aligned}\mathrm{J}_3 &= \beta_0\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u\Big(-\frac{\partial}{\partial y_i}\Big(\alpha_{ij}\frac{\partial v}{\partial y_j}\Big) - \frac{\partial}{\partial y_i}(b_i v)\Big)\,\mathrm{d}y\mathrm{d}t\\ &\quad + \beta_0\iint_{Q_T} \mathrm{e}^{2\alpha}\eta v\Big(-\frac{\partial}{\partial y_i}\Big(\alpha_{ij}\frac{\partial u}{\partial y_j}\Big) - \frac{\partial}{\partial y_i}(b_i u)\Big)\,\mathrm{d}y\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \beta_0 \iint_{Q_T} \mathrm{e}^{2\alpha}\eta \frac{\partial u}{\partial y_i}\alpha_{ij}\frac{\partial v}{\partial y_j}\ \mathrm{d}y\mathrm{d}t + \beta_0 \iint_{Q_T} \frac{\partial}{\partial y_i}(\mathrm{e}^{2\alpha}\eta)\alpha_{ij}u\frac{\partial v}{\partial y_j}\ \mathrm{d}y\mathrm{d}t \\
&\quad - \beta_0 \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u\frac{\partial (b_i v)}{\partial y_i}\ \mathrm{d}y\mathrm{d}t + \beta_0 \iint_{Q_T} \mathrm{e}^{2\alpha}\eta \frac{\partial v}{\partial y_i}\alpha_{ij}\frac{\partial u}{\partial y_j}\ \mathrm{d}y\mathrm{d}t \\
&\quad + \beta_0 \iint_{Q_T} \frac{\partial}{\partial y_i}(\mathrm{e}^{2\alpha}\eta)\alpha_{ij}v\frac{\partial u}{\partial y_j}\ \mathrm{d}y\mathrm{d}t - \beta_0 \int_{Q_T} \mathrm{e}^{2\alpha}\eta v\frac{\partial (b_i u)}{\partial y_i}\ \mathrm{d}y\mathrm{d}t. \qquad (2.35)
\end{aligned}
$$

Next, we bound from above the fifth and the second terms on the right-hand side of (2.35), respectively, as follows:

$$
\begin{aligned}
&\beta_0 \iint_{Q_T} \frac{\partial}{\partial y_i}(\mathrm{e}^{2\alpha}\eta)\alpha_{ij}v\frac{\partial u}{\partial y_j}\ \mathrm{d}y\mathrm{d}t \\
&= \beta_0 \iint_{Q_T} \mathrm{e}^{\frac{r\alpha}{2}} v\eta^{\frac{1}{6}}\mathrm{e}^{2\alpha-\frac{r\alpha}{2}}\Big(2\alpha_{ij}\frac{\partial \alpha}{\partial y_i}\frac{\partial u}{\partial y_j}\eta^{\frac{5}{6}} + \alpha_{ij}\frac{\partial \eta}{\partial y_i}\frac{\partial u}{\partial y_j}\eta^{-\frac{1}{6}}\Big)\ \mathrm{d}y\mathrm{d}t \\
&\le \frac{\beta_0^2}{2\epsilon}\iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}v^2 + \epsilon\int_{Q_T} \mathrm{e}^{4\alpha-r\alpha}2\Big[4\alpha_{ij}^2\Big(\frac{\partial \alpha}{\partial y_i}\Big)^2\eta^{\frac{5}{3}} + \alpha_{ij}^2\Big(\frac{\partial \eta}{\partial y_i}\Big)^2\eta^{-\frac{1}{3}}\Big]\Big(\frac{\partial u}{\partial y_j}\Big)^2\mathrm{d}y\mathrm{d}t \\
&\le \frac{\beta_0^2}{2\varepsilon}\iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}y\mathrm{d}t \\
&\quad + 2\epsilon C(\alpha_{ij})\iint_{Q_T} \mathrm{e}^{(q-r)\alpha}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}(4|\nabla\alpha|^2\eta^{\frac{1}{3}} + |\nabla\eta|^2\eta^{-\frac{5}{3}})\Big|\frac{\partial u}{\partial y_j}\Big|^2\mathrm{d}y\mathrm{d}t \\
&\le \frac{\beta_0^2}{2\epsilon}\iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}y\mathrm{d}t + 2\epsilon C\Big(1+\frac{1}{T^8}\Big)\iint_{Q_T} \mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t, \qquad (2.36)
\end{aligned}
$$

where we have used in (2.36) that

$$
\left|\begin{aligned}
&\Big|\frac{\partial \alpha_i}{\partial y_j}\Big| \le |\nabla\alpha|, \quad \frac{\nabla\eta}{\eta^{\frac{5}{6}}} = 6\nabla\xi \in L^\infty(\Omega) \Rightarrow \frac{|\nabla\eta|^2}{\eta^{\frac{5}{3}}} = 36|\nabla\xi|^2 \in L^\infty(\Omega), \\
&\nabla\alpha = \frac{\lambda \mathrm{e}^{\lambda\psi}\nabla\psi}{t(T-t)}, \quad p+q\le\alpha \Rightarrow \mathrm{e}^{4\alpha} \le \mathrm{e}^{(p+q)\alpha}, \quad q-r = q-\frac{r}{2}-\frac{r}{2} > 0.
\end{aligned}\right. \qquad (2.37)
$$

Similarly,

$$
\begin{aligned}
&\beta_0 \iint_{Q_T} \frac{\partial}{\partial y_i}(\mathrm{e}^{2\alpha}\eta)\alpha_{ij}u\frac{\partial v}{\partial y_j}\ \mathrm{d}y\mathrm{d}t \\
&= \iint_{Q_T} \mathrm{e}^{\alpha}\eta^{\frac{1}{2}}\alpha_{ij}u\Big(2\beta_0\mathrm{e}^{\alpha}\frac{\partial \alpha}{\partial y_i}\eta^{\frac{1}{2}} + \beta_0\mathrm{e}^{\alpha}\eta^{-\frac{1}{2}}\frac{\partial \eta}{\partial y_i}\Big)\frac{\partial v}{\partial y_j}\ \mathrm{d}y\mathrm{d}t \\
&\le \frac{1}{2}\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t + \beta_0^2\iint_{Q_T} 2\mathrm{e}^{2\alpha}\Big[\Big|\frac{\partial \alpha}{\partial y_i}\Big|^2\eta + \eta^{-1}\Big|\frac{\partial \eta}{\partial y_i}\Big|^2\Big]\alpha_{ij}^2\Big|\frac{\partial v}{\partial y_j}\Big|^2\mathrm{d}y\mathrm{d}t \\
&\le \frac{1}{2}\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t + \beta_0^2\iint_{Q_T} \mathrm{e}^{(2-q)\alpha}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}(|\nabla\alpha|^2\eta^{\frac{1}{3}} + \eta^{-\frac{5}{3}}|\nabla\eta|^2)\alpha_{ij}^2|\nabla v|^2\mathrm{d}y\mathrm{d}t \\
&\le \frac{1}{2}\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t + \beta_0^2|\mathrm{e}^{(2-q)\alpha}(|\nabla\alpha|^2\eta^{\frac{1}{3}} + \eta^{-\frac{5}{3}}|\nabla\eta|^2)\alpha_{ij}^2|_\infty\iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t \\
&\le \frac{1}{2}\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t + \beta_0^2 C\Big(1+\frac{1}{T^8}\Big)\iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t. \qquad (2.38)
\end{aligned}
$$

Substituting (2.36), (2.38) into (2.35), we obtain

$$
\mathrm{J}_3 \le 2\beta_0\iint_{Q_T} \mathrm{e}^{2\alpha}\eta\alpha_{ij}\frac{\partial u}{\partial y_i}\frac{\partial v}{\partial y_j}\ \mathrm{d}y\mathrm{d}t + \frac{{\beta_0}^2}{2\epsilon}\iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t + \frac{1}{2}\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t
$$

$$+ \epsilon C\Big(1+\frac{1}{T^8}\Big)\iint_{Q_T} \mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2 \mathrm{d}y\mathrm{d}t + C{\beta_0}^2\Big(1+\frac{1}{T^8}\Big)\iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2 \mathrm{d}y\mathrm{d}t$$
$$- \beta_0 \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u \frac{\partial}{\partial i}(b_i v)\,\mathrm{d}y\mathrm{d}t + \beta_0 \iint_{Q_T} \mathrm{e}^{2\alpha}\eta v \frac{\partial}{\partial y_i}(b_i v)\,\mathrm{d}y\mathrm{d}t. \tag{2.39}$$

Estimate for $\mathbf{J_4}$ We have

$$\begin{aligned}
\mathrm{J}_4 &= -2\beta_1 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}} v\Big(-\frac{\partial}{\partial y_j}\Big(\alpha_{ij}\frac{\partial v}{\partial y_j}\Big) - \frac{\partial}{\partial y_i}(b_i v)\Big)\,\mathrm{d}y\mathrm{d}t\\
&= -2\beta_1 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}\frac{\partial v}{\partial y_j}\alpha_{ij}\frac{\partial v}{\partial y_j}\,\mathrm{d}y\mathrm{d}t - 2\beta_1\iint_{Q_T}\frac{\partial}{\partial y_j}(\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}})v\alpha_{ij}\frac{\partial v}{\partial y_j}\,\mathrm{d}y\mathrm{d}t\\
&\quad + 2\beta_1 \iint_{Q_T} \mathrm{e}^{2\alpha}\eta^{\frac{2}{3}} v\frac{\partial}{\partial y_i}(b_i v)\,\mathrm{d}y\mathrm{d}t\\
&\le -2\beta_1\alpha_0 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t\\
&\quad + \beta_1 \iint_{Q_T}\frac{\partial}{\partial y_j}\Big(\frac{\partial}{\partial y_j}(\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}})\alpha_{ij}\Big)v^2\mathrm{d}y\mathrm{d}t + 2\beta_1\iint_{Q_T}\mathrm{e}^{2\alpha}\eta^{\frac{2}{3}}v\frac{\partial}{\partial y_i}(b_i v)\,\mathrm{d}y\mathrm{d}t\\
&\le -2\beta_1\alpha_0 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla|^2 + C\Big(1+\frac{1}{T^4}\Big)\beta_1\iint_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t\\
&\quad + 2\beta_1 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}} v\frac{\partial}{\partial y_i}(b_i v)\,\mathrm{d}y\mathrm{d}t.
\end{aligned} \tag{2.40}$$

From (2.25), (2.34), (2.39) and (2.40) applied to (2.23), we obtain

$$\begin{aligned}
&b_0\beta_0 \iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t\\
&\le C_1\Big(1+|a|_\infty+\frac{1}{T^4}\Big)\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t\\
&\quad+\Big(\beta_0+\beta_1+|a,b,c,d|_\infty+\frac{1}{T^4}\Big)\iint_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}y\mathrm{d}t - 2\alpha_0\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t\\
&\quad+C\Big(1+\frac{1}{T^4}\Big)\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t + 2\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta\alpha_{ij}\frac{\partial u}{\partial y_j}\frac{\partial v}{\partial y_j}\mathrm{d}y\mathrm{d}t\\
&\quad+\frac{{\beta_0}^2}{2\epsilon}\int_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t + \frac{1}{2}\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t + \epsilon C\Big(1+\frac{1}{T^8}\Big)\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\\
&\quad+C{\beta_0}^2\Big(1+\frac{1}{T^8}\Big)\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t - \beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u\frac{\partial}{\partial y_i}(b_i v)\,\mathrm{d}y\mathrm{d}t\\
&\quad-\beta_0\int_{Q_T}\mathrm{e}^{2\alpha}\eta v\frac{\partial}{\partial y_i}(b_i u) - 2\beta_1\alpha_0\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t\\
&\quad+C\Big(1+\frac{1}{T^4}\Big)\beta_1\iint_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2 + 2\beta_1\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}v\frac{\partial}{\partial y_i}(b_i v)\,\mathrm{d}y\mathrm{d}t.
\end{aligned} \tag{2.41}$$

But fixing now

$$\frac{b_0\beta_0}{2} \ge C_1\Big(1+|a|_\infty+\frac{1}{T^4}\Big) + C\Big(1+\frac{1}{T^4}\Big)+\frac{3}{2}, \tag{2.42}$$

we obtain

$$\frac{\beta_0 b_0}{2}\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t$$

$$
\begin{aligned}
&\leq \Big(\beta_0^2+\beta_1^2+|a,b,c,d|_\infty+\frac{\beta_0^2}{2\varepsilon}+C\Big(1+\frac{1}{T^4}\Big)\beta_1\Big)\iint_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}y\mathrm{d}t\\
&\quad+\Big(-\alpha_0+\epsilon C\Big(1+\frac{1}{T^8}\Big)\Big)\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t-\alpha_0\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t\\
&\quad-\Big(2\beta_1\alpha_0-C\beta_0^2\Big(1+\frac{1}{T^8}\Big)\Big)\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t+\alpha_0\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t\\
&\quad+\frac{\beta_0^2C}{\alpha_0}\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t-\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u\frac{\partial}{\partial y_i}(b_iv)\,\mathrm{d}y\mathrm{d}t\\
&\quad-\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta v\frac{\partial}{\partial y_i}(b_iu)\,\mathrm{d}y\mathrm{d}t+2\beta_1\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}v\frac{\partial}{\partial y_i}(b_iv)\,\mathrm{d}y\mathrm{d}t. \qquad (2.43)
\end{aligned}
$$

Besides, we also have used in (2.43) that

$$
\begin{aligned}
&\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta\alpha_{ij}\frac{\partial u}{\partial y_i}\frac{\partial v}{\partial y_j}\,\mathrm{d}y\mathrm{d}t\\
&\leq\beta_0\iint_{Q_T}\Big(\mathrm{e}^{\frac{p\alpha}{2}}\eta^{\frac{2}{3}}\alpha_0^{\frac{1}{2}}\frac{\partial u}{\partial y_i}\Big)\Big(\mathrm{e}^{q\alpha}\eta^{\frac{1}{3}}\alpha_0^{-\frac{1}{2}}\frac{\partial v}{\partial y_j}\Big)\alpha_{ij}\,\mathrm{d}y\mathrm{d}t\\
&\leq\alpha_0\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t+\frac{\beta_0^2C}{\alpha_0}\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t. \qquad (2.44)
\end{aligned}
$$

Analysis of the Terms in (2.43) We have

$$
\begin{aligned}
&\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u\frac{\partial}{\partial y_i}(b_iv)\,\mathrm{d}y\mathrm{d}t\\
&=\iint_{Q_T}\mathrm{e}^{\alpha}\eta^{\frac{1}{2}}u\Big(\beta_0\eta^{\frac{1}{2}}\Big(\frac{\partial b_i}{\partial y_i}v+b_i\frac{\partial v}{\partial y_i}\Big)\Big)\mathrm{e}^{\alpha}\,\mathrm{d}y\mathrm{d}t\\
&\leq\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t+C\beta_0^2\iint_{Q_T}\mathrm{e}^{2\alpha}\eta(|v|^2+|\nabla v|^2)\,\mathrm{d}y\mathrm{d}t\\
&\leq\iint_{Q_T}\mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t+C\beta_0^2\iint_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t+C\beta_0^2\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t. \qquad (2.45)
\end{aligned}
$$

Also

$$
\begin{aligned}
&\beta_0\iint_{Q_T}\mathrm{e}^{2\alpha}\eta v\frac{\partial}{\partial y_i}(b_iu)\,\mathrm{d}y\mathrm{d}t\\
&\leq\beta_0\iint_{Q_T}\mathrm{e}^{\frac{p\alpha}{2}}\mathrm{e}^{\frac{q\alpha}{2}}\eta v\Big(\frac{\partial b_i}{\partial y_i}u+b_i\frac{\partial u}{\partial y_i}\Big)\,\mathrm{d}y\mathrm{d}t\\
&\leq\frac{\beta_0^2}{\epsilon}\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}y\mathrm{d}t+4\epsilon\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{5}{3}}(|u|^2+|\nabla u|^2)\,\mathrm{d}y\mathrm{d}t\\
&\leq\frac{\beta_0^2}{\epsilon}\iint_{Q_T}\mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}y\mathrm{d}t+\epsilon C\iint_{Q_T}\mathrm{e}^{p\alpha}\eta|u|^2\mathrm{d}y\mathrm{d}t+\epsilon C\iint_{Q_T}\mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t, \qquad (2.46)
\end{aligned}
$$

because $q>1+\frac{r}{2}>\frac{r}{2}+\frac{r}{2}=r$.

Also, and finally, we have

$$
2\beta_1\iint_{Q_T}\mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}v\frac{\partial}{\partial y_i}(b_iv)\mathrm{d}y\mathrm{d}t
$$

$$
\begin{aligned}
&= 2\beta_1 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}} v^2 \frac{\partial b_i}{\partial y_i}\,\mathrm{d}y\mathrm{d}t + 2\beta_1 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}} v b_i \frac{\partial v}{\partial y_i}\,\mathrm{d}y\mathrm{d}t \\
&\le C\beta_1 \iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t + \beta_1 \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}} b_i \frac{\partial}{\partial y_i}(v^2)\mathrm{d}y\mathrm{d}t \\
&\le C\beta_1 \iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t \\
&\quad + \beta_1 \iint_{Q_T} \Big(q\mathrm{e}^{q\alpha}\frac{\partial \alpha}{\partial y_i}\eta^{\frac{2}{3}} b_i + \mathrm{e}^{q\alpha}\frac{2}{3}\eta^{-\frac{1}{3}}\frac{\partial \eta}{\partial y_i} b_i + \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}\frac{\partial b_i}{\partial y_i}\Big) v^2\mathrm{d}y\mathrm{d}t \\
&\le C\beta_1 \iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t + \beta_1 C\Big(1+\frac{1}{T^4}\Big) \iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}} v^2\mathrm{d}y\mathrm{d}t. \qquad (2.47)
\end{aligned}
$$

Combining (2.45)–(2.47) with (2.43) and again with (2.42) yields

$$
\begin{aligned}
&\frac{\beta b_0}{2} \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t \\
\le& \Big(\beta_0^2 + \beta_1^2 + |a,b,c,d|_\infty + \frac{\beta_0^2}{2\varepsilon} + C\Big(1+\frac{1}{T^4}\Big)\beta_1 + C\beta_0^2 \\
&+ \frac{\beta_0^2}{\varepsilon} + C\beta_1 + \beta_1\Big(1+\frac{1}{T^4}\Big)\Big) \iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}}|v|^2\mathrm{d}y\mathrm{d}t \\
&+ \Big(-\alpha_0 + \varepsilon C\Big(1+\frac{1}{T^4}\Big) + 2\varepsilon C\Big) \iint_{Q_T} \mathrm{e}^{p\alpha}\eta^{\frac{4}{3}}|\nabla u|^2\mathrm{d}y\mathrm{d}t \\
&- \Big(2\beta_1\alpha_0 - C\beta_0^2\Big(1+\frac{1}{T^8}\Big) - \frac{\beta_0^2}{\alpha_0}C - C\beta_0^2\Big) \iint_{Q_T} \mathrm{e}^{q\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}y\mathrm{d}t. \qquad (2.48)
\end{aligned}
$$

Select

$$
\left|
\begin{aligned}
&\beta_1 > \frac{1}{2\alpha_0}\Big(C\beta_0^2\Big(1+\frac{1}{T^8}\Big) + \frac{\beta_0^2}{\alpha_0}C + C\beta_0\Big), \\
&\beta_0 \quad \text{as in (2.42)} \quad \text{and} \epsilon \text{ small enough satisfying:} \\
&\quad \epsilon\Big(C\Big(1+\frac{1}{T^8}\Big) + 2C\Big) < \alpha_0.
\end{aligned}
\right. \qquad (2.49)
$$

By using (2.24) and (2.49) in the last inequality (2.48) we obtain

$$
\iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t \le C\int_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}} v^2\mathrm{d}y\mathrm{d}t. \qquad (2.50)
$$

From the definition of $\xi(y)$ given in (2.20), we obtain

$$
\begin{aligned}
&\iint_{w'\times(0,T)} \mathrm{e}^{2\alpha}u^2\mathrm{d}y\mathrm{d}t \le \iint_{Q_T} \mathrm{e}^{2\alpha}\eta u^2\mathrm{d}y\mathrm{d}t \\
\le& C\iint_{Q_T} \mathrm{e}^{r\alpha}\eta^{\frac{1}{3}} v^2\mathrm{d}y\mathrm{d}t \le C\iint_{w\times(0,T)} \mathrm{e}^{r\alpha} v^2\mathrm{d}y\mathrm{d}t. \qquad (2.51)
\end{aligned}
$$

Since $r<2$, $\alpha<0$, then

$$
\iint_{w'\times(0,T)} \mathrm{e}^{2\alpha}v^2\mathrm{d}y\mathrm{d}t \le \iint_{w\times(0,T)} \mathrm{e}^{2\alpha}v^2\mathrm{d}y\mathrm{d}t \le \iint_{w\times(0,T)} \mathrm{e}^{r\alpha}v^2\mathrm{d}y\mathrm{d}t. \qquad (2.52)
$$

This completes the proof of Theorem 2.2.

Note that if we modify the expression of functional $\Gamma(t)$ defined in (2.21) by taking $s\alpha$ instead of α, $s>s_0>0$, we have the following result.

Corollary 2.1 *As an immediate consequence of Theorem* 2.2*, it follows that for all* $r \in [0,2)$*, there exists a constant* $C = C(r,T)$ *such that*

$$\iint_{\omega' \times (0,T)} \mathrm{e}^{2s\alpha}(u^2+v^2)\,\mathrm{d}y\mathrm{d}t \le C \iint_{\omega\times(0,T)} \mathrm{e}^{rs\alpha} v^2 \mathrm{d}y\mathrm{d}t. \tag{2.53}$$

Now, we will prove the observability inequality for weak solutions of the adjoint system (2.9). Observe that it is a consequence of the Carleman inequality proved in Theorem 2.1 and Corollary 2.1.

Theorem 2.3 *Let the assumptions of Theorem* 2.2 *be satisfied. Then there exists a positive constant* C *depending on* T*,* s *and* λ*, such that every pair of solutions* $u = u(y,t),\ v = v(y,t)$ *to* (2.9) *satisfies*

$$|u(0)|^2_{L^2(\Omega)} + |v(0)|^2_{L^2(\Omega)} \le C \iint_{\omega\times(0,T)} \mathrm{e}^{\frac{rs\alpha}{2}} v^2 \mathrm{d}y\mathrm{d}t, \tag{2.54}$$

where s and λ are taken as in Theorem 2.1.

Proof By the Carleman inequality (Theorem 2.1), we have

$$\begin{aligned}
&\iint_{Q_T} \mathrm{e}^{2s\alpha}\{(s\phi)^{-1}(|u_t|^2+|v_t|^2+|\Delta u|^2+|\Delta v|^2)+\lambda^2 s\phi(|\nabla u|^2+|\nabla v|^2)\}\,\mathrm{d}y\mathrm{d}t\\
&+\iint_{Q_T} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t\\
&\le C\iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t
\end{aligned} \tag{2.55}$$

for all $\omega \Subset \Omega,\ \lambda \ge \lambda_0$, and $s \ge s_0$. If we set $\omega = \omega'$ (with ω' as in Theorem (2.2)) in (2.55), we have

$$\iint_{Q_T} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t \le C\iint_{\omega'\times(0,T)} \mathrm{e}^{2s\alpha}\lambda^4(s\phi)^3(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t. \tag{2.56}$$

We have

$$\phi^3 \le \frac{C}{t^3(T-t)^3}, \tag{2.57}$$

because $\phi(x,t) \le \frac{\mathrm{e}^{\lambda||\psi||}}{\beta(t)} \le \frac{C}{t(T-t)}$.

By using Corollary 2.1 and (2.56)–(2.57) we obtain

$$\begin{aligned}
&\iint_{Q_T} \mathrm{e}^{2s\alpha}\phi^3(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t \le C\iint_{\omega'\times(0,T)} \mathrm{e}^{2s\alpha}\phi^3(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t\\
&\le C_1\Big(\frac{1}{T^6}\Big)\iint_{\omega'\times(0,T)} \mathrm{e}^{s\alpha}(|u|^2+|v|^2)\,\mathrm{d}y\mathrm{d}t \le C_2\iint_{\omega\times(0,T)} \mathrm{e}^{\frac{rs\alpha}{2}}|v|^2\mathrm{d}y\mathrm{d}t.
\end{aligned} \tag{2.58}$$

On the other hand, we also have

$$\phi^3 \ge \frac{1}{t^3(T-t)^3}, \tag{2.59}$$

because $\phi(x,t) = \frac{e^{\lambda\psi(x)}}{\beta(t)} \geq \frac{1}{t(T-t)}$.

Combining (2.58)–(2.59) yields

$$C\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_{\Omega}(|u|^2+|v|^2)\,dydt \leq C\iint_{\omega\times(0,T)} e^{\frac{rs\alpha}{2}}|v|^2 dydt. \tag{2.60}$$

Multiplying both sides of the first equation of (2.9) by u and integrating on Ω, and multiplying both sides of the second equation of (2.9) by v and integrating on Ω, we obtain

$$\left|\begin{aligned} &-\frac{1}{2}\frac{d}{dt}|u|^2 + \int_{\Omega}\alpha_{ij}\frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_j} \leq \int_{\Omega}\frac{\partial b_i}{\partial y_i}u^2 + b_i\frac{\partial u}{\partial y_i}u + au^2 + cvu,\\ &-\frac{1}{2}\frac{d}{dt}|v|^2 + \int_{\Omega}\alpha_{ij}\frac{\partial v}{\partial y_i}\frac{\partial v}{\partial y_j} \leq \int_{\Omega}\frac{\partial b_i}{\partial y_i}v^2 + b_i\frac{\partial v}{\partial y_i}v + cv^2 + duv.\end{aligned}\right. \tag{2.61}$$

Recalling the assumptions (A1)–(A2) and using (2.5) and (1.9), we rewrite (2.61) and obtain

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}(|u|^2_{L^2(\Omega)} + |v|^2_{L^2(\Omega)}) + \alpha_0(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)})\\ &\leq C(|u|^2_{L^2(\Omega)} + |v|^2_{L^2(\Omega)}) + \frac{\alpha_0}{2}(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}).\end{aligned} \tag{2.62}$$

Thus

$$|u(0)|^2_{L^2(\Omega)} + |v(0)|^2_{L^2(\Omega)} \leq C(|u|^2_{L^2(\Omega)} + |v|^2_{L^2(\Omega)}). \tag{2.63}$$

Employing (2.60) and (2.63) we finally obtain

$$\begin{aligned} &|u(0)|^2_{L^2(\Omega)} + |v(0)|^2_{L^2(\Omega)}\\ &= \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}}(|u(0)|^2_{L^2(\Omega)} + |v(0)|^2_{L^2(\Omega)})\,dt\\ &\leq C\int_{\frac{T}{4}}^{\frac{3T}{4}}(|u|^2_{L^2(\Omega)} + |v|^2_{L^2(\Omega)})\,dt\\ &\leq C\iint_{\omega\times(0,T)} e^{\frac{rs\alpha}{2}}v^2 dydt.\end{aligned} \tag{2.64}$$

Theorem 2.4 *Assume that b satisfies the same assumptions* (2.17)–(2.18) *as in Theorem* 2.2. *For each $\psi_0, w_0 \in L^2(\Omega)$, there exists a control $g \in L^2(\omega\times(0,T))$ such that the weak solution $\psi = \psi(y,t), w = w(y,t)$ of the state equation* (2.8) *satisfies*

$$\psi(y,T) = 0, \quad w(y,T) = 0 \quad \textit{in } \Omega, \tag{2.65}$$

with an estimate for the control of the form

$$|g|^2_{L^2(\omega\times(0,T))} \leq C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)}). \tag{2.66}$$

Proof We prove this theorem by using a variational method and the observability inequality (see (2.54)). For $g \in L^2(Q_T)$, $r \in [0,2)$ and $\epsilon > 0$ given, let us introduce the functional J_ϵ by

$$J_\epsilon(g) = \iint_{Q_T} e^{\frac{-rs\alpha}{2}}g^2 dydt + \frac{1}{\epsilon}\,|\psi(T)|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\,|w(T)|^2_{L^2(\Omega)}. \tag{2.67}$$

Here, the pair ψ, w is the solution of (2.8) associated to the initial data $\psi(T), w(T)$. It is not difficult to check that J_ϵ is continuous, strictly convex and coercive in $L^2(Q_T)$, so it possesses a unique minimum $g_\epsilon \in L^2(Q_T)$, whose associated solution is denoted by $\psi_\epsilon, w_\epsilon$.

We find $g_\epsilon \in L^2(Q_T)$, and by means of the state equation (2.8), we find the weak solution $\psi_\epsilon, w_\epsilon$. The next step consists in proving the convergence of $g_\epsilon, \psi_\epsilon$ and w_ϵ, that is,

$$\lim_{\epsilon\to 0} g_\epsilon = g, \quad \lim_{\epsilon\to 0} \psi_\epsilon = \psi, \quad \lim_{\epsilon\to 0} w_\epsilon = w. \tag{2.68}$$

And the further step consists in proving that the pair ψ, w is the weak solution of (2.8) corresponding to the control g and that

$$\psi(y,T) = 0, \quad w(y,T) = 0 \quad \text{in } \Omega. \tag{2.69}$$

Initially, we observe that by the maximum principle (or see, for instance, [9]) we obtain that

$$g_\epsilon = \mathrm{e}^{\frac{rs\alpha}{2}} \chi_\omega v_\epsilon \quad \text{a.e. in } Q_T, \tag{2.70}$$

where the pair u_ϵ, v_ϵ is the weak solution of the parabolic problem:

$$\begin{cases} -u_{\epsilon,t} + A^*(t)u_\epsilon + a(y,t)u_\epsilon + c(y,t)v_\epsilon = 0 & \text{in } Q_T, \\ -v_{\epsilon,t} + A^*(t)v_\epsilon + b(y,t)u_\epsilon + d(y,t)v_\epsilon = 0 & \text{in } Q_T, \\ u_\epsilon = v_\epsilon = 0 & \text{on } \Sigma_T, \\ u_\epsilon(T) = -\dfrac{1}{\epsilon}\psi_\epsilon(T), \quad v_\epsilon(T) = -\dfrac{1}{\epsilon} w_\epsilon(T) & \text{in } \Omega, \end{cases} \tag{2.71}$$

with $\psi_\epsilon, w_\epsilon$ being the solution of

$$\begin{cases} \psi_{\epsilon,t} + A(t)\psi_\epsilon + a(\psi,w)\psi_\epsilon + b(\psi,w)w_\epsilon = 0 & \text{in } Q_T, \\ w_{\epsilon,t} + A(t)w_\epsilon + c(\psi,w)\psi_\epsilon + d(\psi,w)w_\epsilon = \chi_w g_\epsilon & \text{in } Q_T, \\ \psi_\epsilon = w_\epsilon = 0 & \text{on } \Sigma_T, \\ \psi_\epsilon(0) = \psi_0, \quad w_\epsilon(0) = w_0 & \text{in } \Omega. \end{cases} \tag{2.72}$$

Recall that our objective is to show that $\psi_\epsilon(y,T) = 0$ and, $w_\epsilon(y,T) = 0$ in Ω. For this, we need to estimate the functions g_ϵ and the pair $\psi_\epsilon, \omega_\epsilon$ in order to assure the convergence of g_ϵ to g and $\psi_\epsilon, \omega_\epsilon$ to ψ, ω as ϵ goes to zero. In the following, we describe how to obtain such estimates. As the first step, multiply both sides of the first equation of (2.71) by ψ_ϵ and both sides of the second equation of (2.71) by w_ϵ, and integrate on Q_T. As the second step, multiply both sides of the first equation of (2.71) by u_ϵ and both side of the second equation of (2.71) by v_ϵ, and integrate on Q_T. Adding the results of these steps, we obtain

$$\begin{aligned} &\iint_{Q_T} \mathrm{e}^{\frac{rs\alpha}{2}} |v_\epsilon|^2 \chi_w \, \mathrm{d}y\mathrm{d}t - \int_\Omega w_\epsilon(T)v_\epsilon(T)\, \mathrm{d}y - \int_\Omega \psi_\epsilon(T)u_\epsilon(T)\, \mathrm{d}y \\ = &- \int_\Omega w_\epsilon(0)v_\epsilon(0)\, \mathrm{d}y - \int_\Omega \psi_\epsilon(0)u_\epsilon(0)\, \mathrm{d}y. \end{aligned} \tag{2.73}$$

By the inequality of observability for (2.71) (see Theorem 2.3), we obtain from (2.71) and (2.73):

$$\iint_{Q_T} \mathrm{e}^{\frac{rs\alpha}{2}} |v_\epsilon|^2 \mathrm{d}y\mathrm{d}t + \frac{1}{\epsilon}|w_\epsilon(T)|^2_{L^2(\Omega)} + \frac{1}{\epsilon}|\psi_\epsilon(T)|^2_{L^2(\Omega)}$$

$$\begin{aligned}&\le |w_0|_{L^2(\Omega)}|v_\epsilon(0)|_{L^2(\Omega)} + |\psi_0|_{L^2(\Omega)}|u_\epsilon(0)|_{L^2(\Omega)}\\ &\le C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)}) + \frac{1}{2}\iint_{Q_T} \mathrm{e}^{\frac{rs\alpha}{2}}|v_\epsilon|^2\mathrm{d}y\mathrm{d}t. \end{aligned}\tag{2.74}$$

Thus from (2.74) we obtain

$$\iint_{Q_T}|g_\epsilon|^2\mathrm{d}y\mathrm{d}t = \iint_{Q_T}\mathrm{e}^{\frac{rs\alpha}{2}}\mathrm{e}^{\frac{rs\alpha}{2}}|v_\epsilon|^2\mathrm{d}y\mathrm{d}t \le C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)}),\tag{2.75}$$

from which it follows that

$$g_\epsilon \rightharpoonup g \quad \text{in } L^2(\omega\times(0,T)) \quad \text{ with } |g|_{L^2(\omega\times(0,T))} \le C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)}).\tag{2.76}$$

Also from (2.74) we have

$$\psi_\epsilon(y,T)\to 0,\quad w_\epsilon(y,T)\to 0\quad \text{strongly } L^2(\Omega) \text{ as } \epsilon\to 0.\tag{2.77}$$

From (2.72) we obtain

$$\begin{aligned}&\psi_\epsilon \rightharpoonup \psi \quad \text{weakly } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)),\\ &w_\epsilon \rightharpoonup w \quad \text{weakly } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)),\\ &\psi_{\varepsilon,\,t} \rightharpoonup \psi_t \quad \text{weakly } L^2(0,T;H^{-1}(\Omega)),\\ &w_{\varepsilon,\,t} \rightharpoonup w_t \quad \text{weakly } L^2(0,T;H^{-1}(\Omega)).\end{aligned}\tag{2.78}$$

Applying a compactness result (see, for example, Lions [18]), we can extract a subsequence of (ψ_ϵ), (w_ϵ), which shall still be represented by (ψ_ϵ), (w_ϵ), such that

$$\left|\begin{aligned}&\psi_\varepsilon\to\psi \quad \text{strongly } L^2(Q_T),\\ &w_\varepsilon\to w \quad \text{strongly } L^2(Q_T).\end{aligned}\right.\tag{2.79}$$

It is easy to see that the limit g is such that the solution ψ, w of the system

$$\begin{cases}\psi_t + A(t)\psi + a(\psi,w)\psi + b(\psi,w)w = 0 & \text{in } Q_T,\\ w_t + A(t)w + c(\psi,w)\psi + d(\psi,w)w = \chi_w g & \text{in } Q_T,\\ \psi = w = 0 & \text{on } \Sigma_T,\\ \psi(0)=\psi_0,\quad w(0)=w_0 & \text{in } \Omega\end{cases}\tag{2.80}$$

satisfies (2.65).

Moreover, by the lower semi-continuity of the norm with respect to the weak topology and in view of (2.76)–(2.79), we deduce that (2.66) holds. This completes the proof of Theorem 2.4.

3 Null Controllability of the Nonlinear Problem

This section is devoted to proving the main result in this paper, namely, Theorem 1.1. By the inverse mapping τ^{-1}, we prove that Theorem 3.1 below implies Theorem 1.1. For this reason, we only need to prove Theorem 3.1. It will be a consequence of Theorem 2.4 and Kakutani's fixed-point theorem.

Remark 3.1 The system (1.1) is to be said locally null controllable at time T if the previous property holds for any ψ_0, w_0 in a ball $B(0;\delta)\subset L^2(\Omega)$, with δ depending on T.

Theorem 3.1 *Assume that the conditions of Theorem* 1.1 *hold. Then for any* ψ_0, w_0 *in a ball* $B(0;\delta) \subset L^2(\Omega)$*, with* δ *depending on* T*, the nonlinear system* (1.8) *is locally null controllable at time* T*.*

More precisely, for any ψ_0, w_0 *in a ball* $B(0;\delta) \subset L^2(\Omega)$*, with* δ *depending on* T *and* $T>0$*, there exists a control* $g \in L^2(\omega \times (0,T))$ *such that the solution* ψ, w *of* (1.8) *satisfies*

$$\psi(y,T)=0, \quad w(y,T)=0 \quad \text{in } \Omega. \tag{3.1}$$

Proof We apply the fixed point method, as is usually done. As we will work with the multi-valued function, we need an infinite dimensional version of Shizuo Kakutani's fixed point theorem. In order to do this, we introduce the following Hilbert space:

$$W = W(0,T,H_0^1(\Omega),H^{-1}(\Omega)) = \{\xi \in L^2(0,T;H_0^1(\Omega))\colon \xi_t \in L^2(0,T;H^{-1}(\Omega))\},$$

which is equipped with the norm

$$|\xi|_W^2 = |\xi|^2_{L^2(0,T;H_0^1(\Omega))} + |\xi_t|^2_{L^2(0,T;H^{-1}(\Omega))}$$

(see, for instance [18]). We observe that

$$\left| \begin{array}{ll} W(0,T,H_0^1(\Omega),H^{-1}(\Omega)) \subset L^2(Q_T) & \text{with compact imbedding,} \\ W(0,T,H_0^1(\Omega),H^{-1}(\Omega)) \subset C([0,T];L^2(\Omega)) & \text{with continuous imbedding.} \end{array} \right. \tag{3.2}$$

Let us fix $R>0$ and denote by $B=B(0,R)$ the closed ball in $W \times W$ of center 0 and radius R. Hence, B is a convex and compact subset of $X := L^2(Q_T) \times L^2(Q_T)$.

For each $(\overline{\psi}, \overline{w}) \in B$ and $g \in L^2(\Omega)$, we consider the null controllability problem for

$$\begin{cases} \psi_t + A(t)\psi + a(\overline{\psi},\overline{w})\psi + b(\overline{\psi},\overline{w})w = 0 & \text{in } \ Q_T, \\ w_t + A(t)w + c(\overline{\psi},\overline{w})\psi + d(\overline{\psi},\overline{w})w = \chi_w g & \text{in } Q_T, \\ \psi = w = 0 & \text{on } \ \Sigma_T, \\ \psi(0)=\psi_0, \quad w(0)=w_0 & \text{in } \ \Omega, \end{cases} \tag{3.3}$$

where a, b, c and d are given in (2.2)–(2.3).

In view of (1.3) and Theorem 2.4, there exists a control $g \in L^2(\omega \times (0,T))$ such that the associated state ψ, w satisfies (2.65)–(2.66).

We define the mapping $\Phi : B \to 2^X$ as follows: For $(\overline{\psi},\overline{w}) \in B$, we set, by definition

$$\begin{aligned} \Phi(\overline{\psi},\overline{w}) = \{&(\psi,w) \in W \times W, \text{ weak solution of (3.3) for } g \in L^2(Q_T), \text{ with} \\ &|g|^2_{L^2(Q_T)} \le C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)}), \text{ such that } \psi(y,T)=0, \ w(y,T)=0 \text{ in}\, \Omega\}, \end{aligned}$$

Then, the goal is to prove that the multi-valued mapping Φ satisfies the hypotheses of Kakutani's fixed-point theorem.

We consider $(\overline{\psi},\overline{w}) \in B$. Then $\Phi(\overline{\psi},\overline{w})$ is non-empty and convex (a consequence of Theorem 2.4). Let us now prove that $\Phi : B \to 2^B$, that is, $\Phi(B) \subset B$. In fact, for all $(\overline{\psi},\overline{w}) \in B$, if $(\psi,w) \in \Phi(\overline{\psi},\overline{w})$, by the definition of $\Phi(\overline{\psi},\overline{w})$, (ψ,w) is a weak solution of (3.3).

By the same argument to obtain (2.62), which is applied to (3.3), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\psi|^2_{L^2(\Omega)} + |w|^2_{L^2(\Omega)}) + \alpha_0(\|\psi\|^2_{H^1_0(\Omega)} + \|w\|^2_{H^1_0(\Omega)})$$
$$\le C(|\psi|^2_{L^2(\Omega)} + |w|^2_{L^2(\Omega)} + |g|^2_{L^2(\omega)}) + \frac{\alpha_0}{2}(\|\psi\|^2_{H^1_0(\Omega)} + \|w\|^2_{H^1_0(\Omega)}). \tag{3.4}$$

Thus

$$|\psi|^2_{L^2(\Omega)} + |w|^2_{L^2(\Omega)} + 2\alpha_0\int_0^T (\|\psi\|^2_{H^1_0(\Omega)} + \|w\|^2_{H^1_0(\Omega)})\mathrm{d}t$$
$$\le (|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)} + |g|^2_{L^2(\omega\times(0,T))})\mathrm{e}^{2CT} = C_1. \tag{3.5}$$

Hence

$$\int_0^T (|\psi|^2_{L^2(\Omega)} + |w|^2_{L^2(\Omega)})\ \mathrm{d}t \le C_2, \tag{3.6}$$

where $C_2 = C_2(|\psi_0|_{L^2(\Omega)}, |w_0|_{L^2(\Omega)}, |g|_{L^2(\omega\times(0,T))}, T)$.

Fix any $z \in H^1_0(\Omega)$ with $\|z\|_{H^1_0(\Omega)} \le 1$. From the first equation of (3.3) and by using Poincaré's inequality, assumptions (A1)–(A2) and the estimate (2.5), we obtain

$$|\langle \psi_t, z\rangle|_{H^{-1}(\Omega)\times H^1_0(\Omega)} \le C_3(\|\psi\|_{H^1_0(\Omega)} + \|w\|_{H^1_0(\Omega)})\|z\|_{H^1_0(\Omega)}, \tag{3.7}$$

and thus

$$|\psi_t|^2_{H^{-1}(\Omega)} \le 2C_3(\|\psi\|^2_{H^1_0(\Omega)} + \|w\|^2_{H^1_0(\Omega)}). \tag{3.8}$$

Again, using (3.5) yields

$$\int_0^T |\psi_t|^2_{H^{-1}(\Omega)}\mathrm{d}t \le C_4. \tag{3.9}$$

By a similar argument we obtain finally

$$\int_0^T (|\psi_t|^2_{H^{-1}(\Omega)} + |w_t|^2_{H^{-1}(\Omega)})\mathrm{d}t \le C_5. \tag{3.10}$$

We observe that C_2 and C_5 depend on $|\psi_0|_{L^2(\Omega)}, |w_0|_{L^2(\Omega)}, |g|_{L^2(\omega\times(0,T))}$ and T.

Thus, if ψ_0, w_0 are sufficiently small, i.e., if

$$\max\{C_2, C_5\} < \frac{R^2}{2}, \tag{3.11}$$

then $\Phi(B) \subset B$.

We claim now that $\Phi(\overline{\psi}, \overline{w})$ is closed in X. Indeed, let $(\overline{\psi}, \overline{w})$ be fixed in B, and $(\psi_n, w_n) \in \Phi(\overline{\psi}, \overline{w})$ such that: $\psi_n \to \psi$, $w_n \to w$ strongly in $L^2(Q_T)$ for all n. By the definition of $\Phi(\overline{\psi}, \overline{w})$ we have

$$\begin{cases} \psi_{n,t} + A(t)\psi_n + a(\overline{\psi}, \overline{w})\psi_n + b(\overline{\psi}, \overline{w})w_n = 0 & \text{in } \ Q_T, \\ w_{n,t} + A(t)w_n + c(\overline{\psi}, \overline{w})\psi_n + d(\overline{\psi}, \overline{w})w_n = \chi_w g_n & \text{in } Q_T, \\ \psi_n = w_n = 0 & \text{on } \ \Sigma_T, \\ \psi_n(0) = \psi_0, \quad w_n(0) = w_0 & \text{in } \ \Omega, \end{cases} \tag{3.12}$$

with $|g_n|^2_{L^2(Q_T)} \leq C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)})$.

We extract a subsequence $(g_n)_{n\in\mathbb{N}}$ such that

$$g_n \rightharpoonup g \quad \text{weakly in } \; L^2(Q_T). \tag{3.13}$$

By the same argument to obtain (3.10) from (3.3), we get

$$|\psi_{n,t}|^2_{L^2(0,T;H^{-1}(\Omega))} + |w_{n,t}|^2_{L^2(0,T;H^{-1}(\Omega))} + |\psi_n|^2_{L^2(0,T;H^1_0(\Omega))} + |w_n|^2_{L^2(0,T;H^1_0(\Omega))} \leq R^2 \tag{3.14}$$

or

$$|(\psi_n, w_n)|^2_{W\times W} \leq R^2. \tag{3.15}$$

From (3.14) we extract a subsequence $(\psi_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ such that

$$\left|\begin{array}{ll} \psi_n \rightharpoonup \psi & \text{weakly } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)), \\ w_n \rightharpoonup w & \text{weakly } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)), \\ \psi_{n,\,t} \rightharpoonup \psi_t & \text{weakly } L^2(0,T;H^{-1}(\Omega)), \\ w_{n,\,t} \rightharpoonup w_t & \text{weakly } L^2(0,T;H^{-1}(\Omega)), \\ \psi_n \longrightarrow \psi & \text{strongly } L^2(Q_T), \\ w_n \longrightarrow w & \text{strongly } L^2(Q_T). \end{array}\right. \tag{3.16}$$

We have assured the last two convergence by Aubin-Lions compactness result; see, for example, Lions [18] (or equivalently, as a consequence of the compactness of the embedding of $W(0,T,H^1_0(\Omega),H^{-1}(\Omega))$ into $L^2(Q_T)$, cf. (3.2)).

From (3.13) and (3.16), we pass to the limits in (3.12) as $n \to \infty$ and obtain

$$\begin{cases} \psi_t + A(t)\psi + a(\overline{\psi},\overline{w})\psi + b(\overline{\psi},\overline{w})w = 0 & \text{in } \; Q_T, \\ w_t + A(t)w + c(\overline{\psi},\overline{w})\psi + d(\overline{\psi},\overline{w})w = \chi_w g & \text{in } Q_T, \\ \psi = w = 0 & \text{on } \; \Sigma_T, \\ \psi(0) = \psi_0, \quad w(0) = w_0 & \text{in } \; \Omega, \end{cases} \tag{3.17}$$

and $|g|^2_{L^2(Q_T)} \leq C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)})$. Thus, $(\psi, w) \in \Phi(\overline{\psi},\overline{w})$ and $\Phi(\overline{\psi},\overline{w})$ is closed in X.

Thus, since B is a compact of X and $\Phi(\overline{\psi},\overline{w}) \subset B$ is closed, it implies that $\Phi(\overline{\psi},\overline{w})$ is a compact of X.

We now intend to show that Φ has the closed graph in $X \times X$. This is not difficult to check: Assume that $(\overline{\psi_n},\overline{w_n}) \to (\overline{\psi},\overline{w})$ strongly in X and $(\psi_n, w_n) \to (\psi, w)$ strongly in X, with $(\psi_n, w_n) \in \Phi(\overline{\psi_n},\overline{w_n})$ for all n. It remains to show that $(\psi, w) \in \Phi(\overline{\psi},\overline{w})$. In fact, from $(\psi_n, w_n) \in \Phi(\overline{\psi_n},\overline{w_n})$, it follows that (ψ_n, w_n) is a weak solution of the following problem:

$$\begin{cases} \psi_{n,t} + A(t)\psi_n + a(\overline{\psi_n},\overline{w_n})\psi_n + b(\overline{\psi_n},\overline{w_n})w_n = 0 & \text{in } \; Q_T, \\ w_{n,t} + A(t)w_n + c(\overline{\psi_n},\overline{w_n})\psi_n + d(\overline{\psi_n},\overline{w_n})w_n = \chi_w g_n & \text{in } Q_T, \\ \psi_n = w_n = 0 & \text{on } \; \Sigma_T, \\ \psi_n(0) = \psi_0, \quad w_n(0) = w_0 & \text{in } \; \Omega \end{cases} \tag{3.18}$$

with $\iint_{Q_T} |g_n|^2 \mathrm{d}y\mathrm{d}t \leq C(|\psi_0|^2_{L^2(\Omega)} + |w_0|^2_{L^2(\Omega)})$.

Recall (see, e.g., (3.15) or equivalently (3.14)) that the following energy inequality holds for (3.18):

$$|(\psi_n, w_n)|^2_{W\times W} \leq R^2. \tag{3.19}$$

By Aubin-Lions compactness theorem (see (3.3)) and using that $(\psi_n, w_n) \to (\psi, w)$ strongly in $X \times X$, we can derive the estimates similar to (3.16) for the sequence $(\psi_n, w_n)_{n\in\mathbb{N}}$ of a weak solution of (3.18). They are as follows:

$$\left| \begin{array}{ll} \psi_n \rightharpoonup \psi & \text{weakly}\, L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \\ w_n \rightharpoonup w & \text{weakly}\, L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \\ \psi_{n,\,t} \rightharpoonup \psi_t & \text{weakly}\, L^2(0,T;H^{-1}(\Omega)), \\ w_{n,\,t} \rightharpoonup w_t & \text{weakly}\, L^2(0,T;H^{-1}(\Omega)), \\ \psi_n \to \psi & \text{strongly}\, L^2(Q_T) \ \text{ and a.e. in } \ Q_T, \\ w_n \to w & \text{strongly}\, L^2(Q_T) \ \text{ and a.e. in } \ Q_T. \end{array} \right. \tag{3.20}$$

Notice that by hypothesis we have

$$\left| \begin{array}{ll} \overline{\psi_n} \to \overline{\psi} & \text{strongly}\, L^2(Q_T)\, \text{and a.e. in}\, Q_T, \\ \overline{w_n} \to \overline{w} & \text{strongly}\, L^2(Q_T)\, \text{and a.e. in}\, Q_T. \end{array} \right. \tag{3.21}$$

From the convergences above, passing to the limits in (3.18) as $n \to \infty$, it is then easy to see that $(\psi, w) \in \Phi(\overline{\psi}, \overline{w})$.

Therefore, the multi-valued mapping $\Phi : B \to 2^X$ satisfies the conditions of Kakutani's fixed-point theorem, which are: B is a non-empty convex compact set, $\Psi(B) \subset B$, and Ψ has a closed graph in $X \times X$. Hence it has a fixed point. The proof of Theorem 3.1 is complete.

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