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Degenerate Nonlinear Elliptic Equations Lacking in Compactness

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Abstract In this paper, the authors prove the existence of solutions for degenerate elliptic equations of the form $-\text{div}(a(x)\nabla_p u(x)) = g(\lambda, x, |u|^{p-2}u)$ in $\mathbb{R}^N$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$ and $a(x)$ is a degenerate nonnegative weight. The authors also investigate a related nonlinear eigenvalue problem obtaining an existence result which contains information about the location and multiplicity of eigensolutions. The proofs of the main results are obtained by using the critical point theory in Sobolev weighted spaces combined with a Caffarelli-Kohn-Nirenberg-type inequality and by using a specific minimax method, but without making use of the Palais-Smale condition.

Keywords Degenerate equations, p-Laplacian, Sobolev weighted spaces, Mountain-pass theorem
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1 Introduction

In [16], Motreanu and Rădulescu studied the existence of solutions of the problem

$$-\text{div}(a(x)\nabla u) = f(x,u)u - \lambda u, \quad x \in \Omega \subset \mathbb{R}^N,$$

involving the singular potential $a(x)$ under verifiable conditions for the nonlinear term f when $\lambda > 0$ is sufficiently small. Also, they studied a nonlinear eigenvalue problem for which they proved an existence result containing information about the location and multiplicity of eigensolutions.

The aim of the present paper is to extend the results obtained in [16] to degenerate elliptic equations of the p-Laplacian type. More precisely, here we study the existence of nontrivial weak solutions of the following problem

$$-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = f(x,u)|u|^{p-2}u, \quad x \in \Omega \subset \mathbb{R}^N,$$

where $p > 1$ is a real number, a is a nonnegative weight, λ is a positive real parameter and Ω is a (bounded or unbounded) domain in $\mathbb{R}^N$ $(N \geq 2)$.

The main interest of these equations is due to the presence of the singular potential $a(x)$ in the divergence operator. Problems of this kind arise as models for several physical phenomena

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related to equilibrium of continuous media which may somewhere be "perfect insulators" (see [9, p. 79]). These equations can often be reduced to elliptic equations with the Hardy singular potential (see [17]). For further results and extensions, we refer to [1, 3, 7, 10, 20–22].

The growing attention for the study of the p-Laplacian operator Δ_p in the last few decades is motivated by the fact that it arises in various applications. For instance, in fluid mechanics, the shear stress $\overrightarrow{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\overrightarrow{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. The p-Laplacian also appears in the study of tensorial creep (elastic for $p = 2$, plastic as $p \to \infty$) (see [12]), flow through porous media $\left(p = \frac{3}{2}\right)$ (see [19]) or glacial sliding $\left(p \in \left(1, \frac{4}{3}\right]\right)$ (see [18]). More details on this topic can be found in [13–15].

The proofs of our main results rely on an adequate variational approach where, in view of the presence of a singular potential and a (possibly) unbounded domain, the usual methods fail to apply. Since we are interested in the case of lacking compactness, we suppose $\Omega = \mathbb{R}^N$ and we do not make use of the Palais-Smale condition. The other cases when $\Omega \subset \mathbb{R}^N$ is unbounded can be treated similarly. Moreover, we employ an inequality due to Caldiroli and Musina [6] (see also [5] for the case $a(x) = |x|^\alpha$) which extends the inequalities of Hardy [11] and Caffarelli et al. [4].

This paper is organized as follows. In Section 2 we define the suitable Sobolev weighted spaces and we present our main results. In Section 3 we prove the existence of solutions for our problems, using the mountain-pass theorem and a special version of it involving a suitable hyperplane.

2 Preliminaries and Main Results

Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^N$, with $N \geq 2$, and let $a : \Omega \to [0, \infty)$ be a weight function satisfying $a \in L^1_{\mathrm{loc}}(\Omega)$. We introduce the following assumptions:

(h_α) $\liminf_{x \to z} |x - z|^{-\alpha} a(x) > 0$, $\forall z \in \overline{\Omega}$, with a real number $\alpha \in [0, \infty)$;

(h_α^∞) $\liminf_{|x| \to \infty} |x|^{-\alpha} a(x) > 0$ (if Ω is unbounded).

A model example is $a(x) = |x|^\alpha$. The case $\alpha = 0$ covers the "isotropic" case corresponding to the Laplace operator.

For any $u \in C_c^\infty(\Omega)$, we set

$$\|u\|_a = \left(\int_\Omega a(x)|\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

$$\|u\|_{W^{1,p}(\Omega,a)} := \left(\int_\Omega a(x)|\nabla u|^p \mathrm{d}x + \int_\Omega u^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Let $\mathcal{D}_a^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega, a)$ be the closures of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_a$ and $\|\cdot\|_{W^{1,p}(\Omega,a)}$, respectively. Clearly we have $W_0^{1,p}(\Omega, a) \hookrightarrow \mathcal{D}_a^{1,p}(\Omega)$ with continuous embedding.

For any $\alpha \in (0, p)$, we denote

$$p_\alpha^* := \frac{pN}{N - p + \alpha}.$$

Lemma 2.1 (see [6]) *Assume that the function $a \in L^1_{\rm loc}(\Omega)$ satisfies conditions (h_α) and (h_α^∞), for some $\alpha \in (0,p)$. Then there exists a positive constant C such that*

$$\Big(\int_\Omega |u|^{p_\alpha^*}{\rm d}x\Big)^{\frac{p}{p_\alpha^*}} \le C\Big(\int_\Omega a(x)|\nabla u|^p{\rm d}x\Big)$$

for any $u \in C_c^\infty(\Omega)$.

In order to simplify the arguments, we admit throughout the paper that $a(x) = |x|^\alpha$, for some $\alpha \in (0,p)$, and that $\lambda > 0$. Since we are interested in the case of lacking compactness, we suppose $\Omega = \mathbb{R}^N$.

In the present paper, we deal with the following nonlinear elliptic equation

$$-{\rm div}(|x|^\alpha|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = f(x,u)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

We assume that $p > 1$ is a real number and the nonlinearity $f = f(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ in (2.1) is continuous and satisfies the following hypotheses:

(H1) $f(x,t) \ge 0$ for all $t \ge 0$; $f(x,t) \equiv 0$ for all $t < 0,\ x \in \mathbb{R}^N$;

(H2) $\lim\limits_{t\to 0^+} \frac{f(x,t)}{t^\tau} = 0$ uniformly in $x \in \mathbb{R}^N$, with some constant $\tau > 0$;

(H3) the mapping $(x,t) \mapsto t^{p-1}f(x,t)$ is of class C^1; and there exists the limit $\lim\limits_{t\to\infty} \frac{\rm d}{{\rm d}t}f(x,t)$ for all $x \in \mathbb{R}^N$;

(H4) $\lim\limits_{t\to+\infty} f(x,t) = l > 0$ uniformly in $x \in \mathbb{R}^N$;

(H5) for any $M > 0$, there exists $\theta > 0$ such that

$$(p+\theta)F(x,t) \le f(x,t)t^p \quad \text{for all } t \in (0,M),$$

where $F(x,t) := \int_0^t s^{p-1}f(x,s){\rm d}s$;

(H6) there exists $\eta > 0$ such that

$$\lim_{t\to+\infty} \frac{f(x,t)t^p - pF(x,t)}{t^r} = q(x) \ge \eta > 0 \quad \text{uniformly in } x \in \mathbb{R}^N,$$

with some $r \in \big(\frac{pN}{N+p-\alpha}, p\big)$;

(H7) the function $f(\cdot,t)$ is bounded from above uniformly with respect to t belonging to any bounded subset of $\mathbb{R}_+$.

Remark 2.1 A useful consequence of assumptions (H1)–(H3) is that the derivative with respect to t of the mapping $(x,t) \mapsto t^{p-1}f(x,t)$ vanishes at $t = 0$ uniformly in $x \in \mathbb{R}^N$. Indeed, we have $\frac{\rm d}{{\rm d}t}(t^{p-1}f(x,t))(0) = \lim\limits_{t\to 0^+} \frac{t^{p-1}f(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$.

Moreover, without loss of generality, we may suppose that $0 < \tau < p_\alpha^* - p$.

Remark 2.2 It is easy to see that the assumption (H5) ensures

$$F(x,t) \le \frac{1}{p}f(x,t)t^p, \quad \forall x \in \mathbb{R}^N,\ t \ge 0.$$

Indeed, because $M > 0$ is arbitrary and $\theta F(x,t) \ge 0$ for all $x \in \mathbb{R}^N$ and $t \ge 0$, we easily obtain the above relation.

Remark 2.3 Assumption (H7) can be applied when the function $t \mapsto f(x,t)$ is nondecreasing for all $x \in \mathbb{R}^N$. It is so, because (H4) then implies (H7).

One of the main results of this work is given by the following theorem.

Theorem 2.1 *Assume that conditions* (H1)–(H7) *are fulfilled. Then problem* (2.1) *has a nontrivial weak solution for every* $\lambda \in (0,l)$*, where* $l > 0$ *is the constant in* (H4)*.*

In the sequel, for any $u \in C_c^\infty(\mathbb{R}^N)$, we set

$$\|u\|_\alpha = \Big(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p \mathrm{d}x\Big)^{\frac{1}{p}}.$$

Let $\mathcal{D}_\alpha^{1,p}(\mathbb{R}^N)$ denote the space obtained as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the $\|\cdot\|_\alpha$- norm. Let E be the space defined as the completion of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ with respect to the norm

$$\|u\| := \Big(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^p + \lambda |u|^p \mathrm{d}x\Big)^{\frac{1}{p}}.$$

Definition 2.1 *We say that a function* $u \in E$ *is a weak solution of problem* (2.1) *if*

$$\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} \lambda |u|^{p-2} u v \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u) u^{p-1} v \mathrm{d}x = 0$$

for all $v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$*.*

Remark 2.4 We are working with $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ instead of $C_c^\infty(\mathbb{R}^N)$, because in our approach, it is essential to keep the support of the test functions away from 0 exploiting that every bounded sequence in the space $\mathcal{D}_\alpha^{1,p}(\mathbb{R}^N)$ contains a strongly convergent subsequence in $L^{p_\alpha^*}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$.

Proposition 2.1 $(E, \|\cdot\|)$ *is a reflexive Banach space.*

The proof of this result follows the same ideas as in the case of Sobolev spaces (see, for instance, [8]).

Remark 2.5 We clearly have $E \hookrightarrow \mathcal{D}_\alpha^{1,p}(\mathbb{R}^N)$ with continuous embedding.

Remark 2.6 If Ω is a bounded domain in $\mathbb{R}^N$ and $0 \notin \overline{\Omega}$, then the embedding $\mathcal{D}_\alpha^{1,p}(\Omega) \hookrightarrow L^{p_\alpha^*}(\Omega)$ is compact for $\alpha \in (0,p)$, where $p_\alpha^* := \frac{pN}{N-p+\alpha}$. Moreover, we deduce that $\mathcal{D}_\alpha^{1,p}(\Omega)$ is compactly embedded in $L^i(\Omega)$ for any $i \in [1, p_\alpha^*]$.

In the sequel, we give our second main result, namely, a related nonlinear eigenvalue problem corresponding to the degenerate potential $|x|^\alpha$ with $\alpha \in (0,p)$.

Fix a positive number $\nu > 0$ and let $I : E \to \mathbb{R}$ be a C^1 function satisfying the following conditions

$$I(0) \geq 0, \quad I'(0) \neq 0; \tag{2.2}$$

$$I(u) \leq a_1 + a_2 \|u\|^q, \quad \forall u \in E \tag{2.3}$$

with constants $a_1 \geq 0$, $a_2 \geq 0$, $q \geq p$, $p \geq 2$;

$$\frac{1}{\gamma}\langle I'(u), u\rangle - I(u) \geq -b_1 - b_2\|u\|^p, \quad \forall u \in E \tag{2.4}$$

with constants $\gamma > p$, $b_1 \geq 0$, $b_2 \in \big[0, \nu\big(\frac{1}{p} - \frac{1}{\gamma}\big)\big)$, and

$$I'(v_n) \rightharpoonup I'(v) \text{ in } E^* \quad \text{whenever } v_n \rightharpoonup v \text{ in } E. \tag{2.5}$$

The notation "$\rightharpoonup$" in (2.5) means the weak convergence.

Problem 2.1 Find an eigensolution $(u, \mu) \in (E \setminus \{0\}) \times (0, \infty)$ such that

$$\int_{\mathbb{R}^N} \big(|x|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \lambda|u|^{p-2}u\varphi\big)\,\mathrm{d}x = \mu\langle I'(u), \varphi\rangle, \quad \forall\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \tag{2.6}$$

with fixed constants $\alpha > 0$ and $\lambda > 0$. The concept of solution in (2.6) is clearly compatible with Definition 2.1.

Due to the fact that $I'(0) \neq 0$ in (2.2), a solution $u \in E$ of (2.6) is necessarily nontrivial, that is, $u \in E \setminus \{0\}$. Assume further that

$$\frac{1}{\nu\|u\|^{p-2}} \text{ is not an eigenvalue of (2.6)}, \tag{2.7}$$

that is, (2.6) is not solvable for $\mu = \frac{1}{\nu\|u\|^{p-2}}$. Our main result in studying problem (2.6) is the following.

Theorem 2.2 *Assume that conditions* (2.2)–(2.5) *and* (2.7) *with a given number* $\nu > 0$ *are fulfilled. Then, for every number* $\rho \geq \sqrt[p]{qa_2}$*, there exists an eigensolution* $(u, \mu) \in (E \setminus \{0\}) \times (0, +\infty)$ *of* (2.6) *such that*

$$0 < \mu < \frac{1}{\nu\|u\|^{p-2} + \rho^p\|u\|^{q-2}}. \tag{2.8}$$

If $q = 2$ *in* (2.3)*, then for all* $\rho \geq \sqrt[p]{2a_2}$ *and* $r > \rho$*, there exists an eigensolution* $(u, \mu) \in (E \setminus \{0\}) \times (0, +\infty)$ *of* (2.6) *such that*

$$\frac{1}{\nu\|u\|^{p-2} + r^p} < \mu < \frac{1}{\nu\|u\|^{p-2} + \rho^p}. \tag{2.9}$$

3 Proofs of Main Results

In this section, we give the proofs of our main results which are Theorem 2.1 and Theorem 2.2.

The basic idea in proving Theorem 2.1 is to consider the associate energetic functional of (2.1) and to show that it possesses a nontrivial critical point.

We define the energetic functional associated to problem (2.1) as $J : E \to \mathbb{R}$, where

$$J(u) := \frac{1}{p}\int_{\mathbb{R}^N} (|x|^\alpha|\nabla u|^p + \lambda|u|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x, \quad \forall u \in E.$$

A straightforward argument based on Lemma 2.1, Remark 2.5 and assumptions (H1)–(H4) shows that J is well-defined on the space E and is of class $C^1(E,\mathbb{R})$, with the derivative given by

$$\langle J'(u),v\rangle = \int_{\mathbb{R}^N}(|x|^\alpha|\nabla u|^{p-2}\nabla u\cdot\nabla v\mathrm{d}x+\lambda|u|^{p-2}uv)\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)u^{p-1}v\mathrm{d}x$$

for all $u,v\in E$. Thus, using Definition 2.1, we observe that the weak solutions of (2.1) correspond to the critical points of the functional J.

Lemma 3.1 *Suppose that* (H1)–(H4) *are fulfilled. Let* $(u_n)\subset E$ *be a sequence such that for some* $c\in\mathbb{R}$, *one has* $J(u_n)\to c$ *and* $J'(u_n)\to 0$ *as* $n\to\infty$. *If there exists* $u_0\in E$ *such that* $u_n\rightharpoonup u_0$, *then* $J'(u_n)\rightharpoonup J'(u_0)=0$ *and thus* u_0 *is a weak solution of* (2.1).

Proof Using Remarks 2.5–2.6, we may assume that $u_n\to u_0$ in $L^{p^*_\alpha}(\omega)$, for all bounded domain ω in $\mathbb{R}^N$ with $0\notin\overline{\omega}$.

If we prove that

$$\langle J'(u_n),\varphi\rangle\to\langle J'(u_0),\varphi\rangle,\quad\forall\varphi\in C_c^\infty(\mathbb{R}^N\setminus\{0\}),$$

then by the fact that $J(u_n)\to c$ and $J'(u_n)\to 0$ as $n\to\infty$, u_0 is a weak solution of (2.1).

Therefore, we consider an arbitrary bounded domain ω in $\mathbb{R}^N$ with $0\notin\overline{\omega}$ and an arbitrary function $\varphi\in C_c^\infty(\mathbb{R}^N\setminus\{0\})$ such that $\operatorname{supp}(\varphi)\subset\omega$.

The convergence

$$J'(u_n)\to 0\quad\text{in } E^*$$

implies

$$\langle J'(u_n),\varphi\rangle\to 0\quad\text{as } n\to\infty,$$

that is,

$$\lim_{n\to\infty}\Big(\int_\omega(|x|^\alpha|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi+\lambda|u_n|^{p-2}u_n\varphi)\mathrm{d}x-\int_\omega f(x,u_n)u_n^{p-1}\varphi\mathrm{d}x\Big)=0.\tag{3.1}$$

Since $u_n\rightharpoonup u_0$ in E, it follows that

$$\begin{aligned}&\lim_{n\to\infty}\int_\omega(|x|^\alpha|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi+\lambda|u_n|^{p-2}u_n\varphi)\mathrm{d}x\\&=\int_\omega(|x|^\alpha|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla\varphi+\lambda|u_0|^{p-2}u_0\varphi)\mathrm{d}x.\end{aligned}\tag{3.2}$$

Next, we show that

$$\lim_{n\to\infty}\int_\omega f(x,u_n)u_n^{p-1}\varphi\mathrm{d}x=\int_\omega f(x,u_0)u_0^{p-1}\varphi\mathrm{d}x.\tag{3.3}$$

Indeed, because $u_n\to u_0$ in $L^{p^*_\alpha}(\omega)$, by Remark 2.6, we have

$$u_n\to u_0\quad\text{in } L^i(\omega),\ \forall i\in[1,p^*_\alpha].\tag{3.4}$$

By Remark 2.1, we obtain that

$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}(t^{p-1}f(x,t)) = 0; \tag{3.5}$$

and by (H4), we have

$$0 = \lim_{t\to+\infty} \frac{f(x,t)}{t} = \lim_{t\to+\infty} \frac{t^{p-1}f(x,t)}{t^p} = \lim_{t\to+\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}t}(t^{p-1}f(x,t))}{pt^{p-1}} \tag{3.6}$$

uniformly in $x\in\mathbb{R}^N$, $t\geq 0$, where the last limit exists due to (H3) in conjunction with (H4).

Then for every $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}(t^{p-1}f(x,t))\Big| \leq \varepsilon + C_\varepsilon t^{p-1}, \quad \forall t\geq 0,\ \forall x\in\omega. \tag{3.7}$$

In the above inequality, we essentially used that the derivative $\frac{\mathrm{d}}{\mathrm{d}t}(t^{p-1}f(x,t))$ is continuous, and is thus bounded on any compact set in $\mathbb{R}^N\times\mathbb{R}$.

As known from (3.4), $u_n\to u_0$ in $L^p(\omega)$. Then, passing, if necessary, to a subsequence, there is a function $h\in L^p(\omega)$ such that $|u_n|\leq h$ a.e. in ω. Using (3.7) and the Hölder inequality, we obtain

$$\begin{aligned}
&\Big|\int_\omega (f(x,u_n)u_n^{p-1} - f(x,u_0)u_0^{p-1})\varphi \mathrm{d}x\Big| \\
&\leq \int_\omega |f(x,u_n)u_n^{p-1} - f(x,u_0)u_0^{p-1}||\varphi(x)|\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty(\omega)} \int_\omega |f(x,u_n)u_n^{p-1} - f(x,u_0)u_0^{p-1}|\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty(\omega)} \int_\omega (\varepsilon + C_\varepsilon h(x)^{p-1})|u_n(x)-u_0(x)|\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty(\omega)} \Big[\varepsilon\int_\omega |u_n(x)-u_0(x)|\mathrm{d}x + C_\varepsilon \int_\omega h(x)^{p-1}|u_n(x)-u_0(x)|\mathrm{d}x\Big] \\
&\leq \|\varphi\|_{L^\infty(\omega)}[\epsilon\|u_n-u_0\|_{L^1(\omega)} + C_\varepsilon\|h\|^{p-1}_{L^p(\omega)}\|u_n-u_0\|_{L^p(\omega)}]
\end{aligned}$$

for all $n\in\mathbb{N}$. Thus, we obtain relation (3.3). Therefore, from (3.1)–(3.3), we deduce that

$$\int_\omega (|x|^\alpha|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla\varphi + \lambda|u_0|^{p-2}u_0\varphi)\mathrm{d}x - \int_\omega f(x,u_0)u_0^{p-1}\varphi\mathrm{d}x = 0.$$

Finally, the density of $C_c^\infty(\mathbb{R}^N\setminus\{0\})$ in E ensures that $J'(u_0)=0$, which finishes the proof.

Remark 3.1 Lemma 3.1 holds, assuming in (H1)–(H4) that the convergence is uniform only on the bounded subsets of $\mathbb{R}^N$.

In order to prove that J has a nontrivial critical point, our idea is to show that actually J possesses a mountain-pass geometry. So, we have the following auxiliary results.

Lemma 3.2 *Assume that the conditions* (H1)–(H4) *and* (H7) *hold. Then there exist constants* $\rho>0$ *and* $a>0$ *such that for all* $u\in E$ *with* $\|u\|=\rho$, *one has* $J(u)\geq a$.

Proof By (H1)–(H4), it follows that for any $\sigma > 0$, uniformly with respect to $x \in \mathbb{R}^N$, it is true that

$$\lim_{t\to 0^+} f(x,t) = 0, \quad \lim_{t\to +\infty} \frac{f(x,t)}{t^\sigma} = 0. \tag{3.8}$$

In particular, we have

$$\lim_{t\to 0^+} \frac{F(x,t)}{t^p} = \lim_{t\to 0^+} \frac{t^{p-1} f(x,t)}{pt^{p-1}} = \frac{1}{p} \lim_{t\to 0^+} f(x,t) = 0 \tag{3.9}$$

and for any $\sigma' > p$,

$$\lim_{t\to +\infty} \frac{F(x,t)}{t^{\sigma'}} = \lim_{t\to +\infty} \frac{t^{p-1} f(x,t)}{\sigma' t^{\sigma'-1}} = \frac{1}{\sigma'} \lim_{t\to +\infty} \frac{f(x,t)}{t^{\sigma'-p}} = 0. \tag{3.10}$$

Taking $\sigma' = p_\alpha^*$ implies

$$\lim_{t\to +\infty} \frac{F(x,t)}{t^{p_\alpha^*}} = 0. \tag{3.11}$$

Using (3.9) and (3.11), we obtain that for every $\varepsilon > 0$, there exist constants $0 < \delta_1 < \delta_2$ such that uniformly with respect to $x \in \mathbb{R}^N$, the following estimates hold:

$$\begin{aligned} &0 \le F(x,t) < \varepsilon t^p, \quad \forall t \text{ with } |t| < \delta_1, \\ &0 \le F(x,t) < \varepsilon t^{p_\alpha^*}, \quad \forall t \text{ with } |t| > \delta_2. \end{aligned}$$

Assumption (H7) guarantees that F is bounded on $\mathbb{R}^N \times [\delta_1, \delta_2]$. We deduce that there exists a positive constant C_ε such that

$$0 \le F(x,t) < \varepsilon t^p + C_\varepsilon t^{p_\alpha^*} \quad \text{for all } x \in \mathbb{R}^N. \tag{3.12}$$

Then (3.12) and Lemma 2.1 show that

$$\begin{aligned} J(u) &= \frac{1}{p} \int_{\mathbb{R}^N} (|x|^\alpha |\nabla u|^p + \lambda |u|^p) \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u) \mathrm{d}x \\ &= \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} F(x,u) \mathrm{d}x \\ &\ge \frac{1}{p} \|u\|^p - \varepsilon \int_{\mathbb{R}^N} |u|^p \mathrm{d}x - C_\varepsilon \int_{\mathbb{R}^N} |u|^{p_\alpha^*} \mathrm{d}x \\ &\ge \frac{1}{2p} \|u\|^p + \frac{1}{2p} \|u\|^p - \varepsilon \|u\|^p_{L^p(\mathbb{R}^N)} - C_\varepsilon C^{\frac{p_\alpha^*}{p}} \|u\|^{p_\alpha^*} \\ &\ge \frac{1}{2p} \|u\|^p + \frac{\lambda}{2p} \|u\|^p_{L^p(\mathbb{R}^N)} - \varepsilon \|u\|^p_{L^p(\mathbb{R}^N)} - C_\varepsilon C^{\frac{p_\alpha^*}{p}} \|u\|^{p_\alpha^*} \\ &= \frac{1}{2p} \|u\|^p + \Big(\frac{\lambda}{2p} - \varepsilon\Big) \|u\|^p_{L^p(\mathbb{R}^N)} - C_\varepsilon C^{\frac{p_\alpha^*}{p}} \|u\|^{p_\alpha^*}. \end{aligned}$$

Finally, choosing $\varepsilon \in \big(0, \frac{\lambda}{2p}\big)$ and since $p_\alpha^* > p$, we find $\rho > 0$ and $a > 0$ as required, which finishes the proof.

Remark 3.2 Using the same techniques as in the proof of (3.12), due to assumptions (H1)–(H4) and (H7), we may conclude that for any $\varepsilon > 0$, there exists a positive constant D_ε such that

$$|f(x,t)| \le \varepsilon + D_\varepsilon |t|^\sigma \quad \text{for all } x \in \mathbb{R}^N, \tag{3.13}$$

where $\sigma = r\big(\frac{1}{p_\alpha^*}\big) - 1 > 0$.

In the sequel, we construct an important element of the space E. Let us denote

$$(d(N))^p := \int_{\mathbb{R}^N} \mathrm{e}^{-p|x|^p} \mathrm{d}x, \tag{3.14}$$

and for an arbitrary number $a > 0$,

$$w_a(x) := (d(N))^{-1} a^{\frac{N}{p^2}} \mathrm{e}^{-a|x|^p}, \quad \forall x \in \mathbb{R}^N. \tag{3.15}$$

Proposition 3.1 *The function w_a satisfies $w_a \in E$ whenever $a > 0$.*

Proof Fix $a > 0$ and define

$$h(x) = \mathrm{e}^{-a|x|^p}, \quad \forall x \in \mathbb{R}^N. \tag{3.16}$$

It is enough to show that $h \in E$. We have to establish that for any $\varepsilon > 0$, there is $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that $\|h - \psi\| < \varepsilon$. First we prove that, for every $\varepsilon > 0$, there exists a function $\psi_1 \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|h - \psi_1\| < \varepsilon. \tag{3.17}$$

Given any number $z > 0$, we have

$$\lim_{t \to +\infty} t^{z+p-1} \mathrm{e}^{-t} = 0.$$

We derive that a positive constant $c = c(z)$ can be found with the property that $|t^{z+p-1}\mathrm{e}^{-t}| \le c$ for all $t \in [1, \infty)$, so

$$|t^{z-1}\mathrm{e}^{-t}| \le \frac{c}{t^p}, \quad \forall t \in [1, \infty). \tag{3.18}$$

We check that there exists some constant $\delta > 0$ such that

$$\frac{1}{p^p a^p} \int_{\mathbb{R}^N \setminus B(0,\delta)} |x|^\alpha |\nabla h(x)|^p \mathrm{d}x < \frac{\varepsilon^p}{p^p(p^p a^p + p)}. \tag{3.19}$$

So we have

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B(0,\delta)} |x|^\alpha |\nabla h(x)|^p \mathrm{d}x &= \int_\delta^{+\infty} \Big(\int_{\partial B(0,r)} r^\alpha | - apr^{p-1} \mathrm{e}^{-ar^p}|^p \mathrm{d}\sigma \Big) \mathrm{d}r \\
&= p^p a^p \int_\delta^{+\infty} \Big(r^{\alpha + p(p-1)} \mathrm{e}^{-par^p} \int_{\partial B(0,r)} \mathrm{d}\sigma \Big) \mathrm{d}r \\
&= p^p a^p \omega_N \int_\delta^{+\infty} r^{\alpha + p(p-1) + N - 1} \mathrm{e}^{-par^p} \mathrm{d}r,
\end{aligned}$$

where ω_N is the surface measure of the unit sphere in $\mathbb{R}^N$. To this end, we note that the below equality holds

$$\frac{1}{p^p a^p}\int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^\alpha |\nabla h(x)|^p \mathrm{d}x = \omega_N \int_\delta^{+\infty} r^{\alpha+p(p-1)+N-1}\mathrm{e}^{-par^p}\mathrm{d}r. \tag{3.20}$$

Then using (3.18) for $\delta^p \geq \frac{1}{pa}$, we have

$$\frac{1}{p^p a^p}\int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^\alpha |\nabla h(x)|^p \mathrm{d}x \leq \frac{\widetilde{c}}{\delta^p} \tag{3.21}$$

with a positive constant $\widetilde{c}$. To obtain (3.19), it is enough to choose

$$\delta^p > \max\Big\{\frac{1}{pa}, \frac{p^p(p^p a^p + p)\widetilde{c}}{\varepsilon^p}\Big\}.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfy $\begin{cases}\varphi = 1 \text{ on } B(0,\delta),\\ 0 \leq \varphi \leq 1 \text{ on } \mathbb{R}^N.\end{cases}$

Using (3.19) and taking δ sufficiently large, we find two positive constants C_1 and C_2 satisfying

$$\begin{aligned}
\|h-\varphi h\|_\alpha^p &= \int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^\alpha |\nabla((1-\varphi)h(x))|^p \mathrm{d}x\\
&= \int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^\alpha (1-\varphi)^p |\nabla h(x)|^p \mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^\alpha |\nabla(1-\varphi)(x)|^p h(x)^p \mathrm{d}x\\
&< \frac{a^p\varepsilon^p}{p^p a^p + p} + \int_{\mathbb{R}^N\setminus B(0,\delta)} |x|^\alpha |\nabla(1-\varphi)(x)|^p \mathrm{e}^{-pa|x|^p}\mathrm{d}x\\
&< C_1\varepsilon^p,\\
\lambda\|h-\varphi h\|_{L^p(\mathbb{R}^N)}^p &= \lambda\int_{\mathbb{R}^N\setminus B(0,\delta)} ((1-\varphi)(x))^p\, h(x)^p \mathrm{d}x\\
&= \lambda\int_{\mathbb{R}^N\setminus B(0,\delta)} ((1-\varphi)(x))^p\, \mathrm{e}^{-pa|x|^p}\mathrm{d}x\\
&< C_2\varepsilon^p.
\end{aligned}$$

Setting $\psi_1 := \varphi h \in C_c^\infty(\mathbb{R}^N)$, from the above relations, it follows that

$$\|h-\psi_1\|^p < (C_1+C_2)\varepsilon^p.$$

Since $C_1 + C_2$ is independent of ε, it turns out that

$$\|h-\psi_1\| < \varepsilon.$$

Fix now a function $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that

$$\|\psi_1 - \psi\| < \varepsilon.$$

Then, combining the above two relations, we arrive at the conclusion of Proposition 3.1 and finish the proof.

Lemma 3.3 *If the conditions* (H1)–(H4), (H7) *hold and* $\lambda \in (0, l)$ *with the number* l *in* (H4), *then for the positive number* ρ *given in Lemma* 3.2, *there exists* $e \in E$ *such that* $\|e\| > \rho$ *and* $J(e) < 0$.

Proof Fix the element $w_a \in E$ in Proposition 3.1 for some $a > 0$. From (3.14), we easily get

$$(d(N))^p = \int_{\mathbb{R}^N} \mathrm{e}^{-p|x|^p}\mathrm{d}x = a^{\frac{N}{p}}\omega_N \int_0^\infty \mathrm{e}^{-par^p} r^{N-1}\mathrm{d}r.$$

Using (3.15) and the above relation, we obtain

$$\begin{aligned}\|w_a\|^p_{L^p(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} w_a(x)^p \mathrm{d}x = \int_{\mathbb{R}^N} (d(N))^{-p} a^{\frac{N}{p}} \mathrm{e}^{-pa|x|^p}\mathrm{d}x \\ &= (d(N))^{-p} a^{\frac{N}{p}} \int_0^\infty \Big(\int_{\partial B(0,r)} \mathrm{e}^{-par^p}\mathrm{d}\sigma\Big)\mathrm{d}r \\ &= (d(N))^{-p} a^{\frac{N}{p}} \omega_N \int_0^\infty \mathrm{e}^{-par^p} r^{N-1}\mathrm{d}r \\ &= 1.\end{aligned}$$

Next, using the notation $d(N)$ entering the formula of w_a, we introduce

$$D(N) := p^p (d(N))^{-p} \int_{\mathbb{R}^N} \mathrm{e}^{-p|x|^p} |x|^{p(p-1)+\alpha}\mathrm{d}x. \tag{3.22}$$

We easily get

$$D(N) = p^p (d(N))^{-p} a^{p-1} a^{\frac{\alpha}{p}} a^{\frac{N}{p}} \omega_N \int_0^\infty \mathrm{e}^{-par^p} r^{p(p-1)+\alpha+N-1}\mathrm{d}r.$$

Then, we find

$$\begin{aligned}\|w_a\|^p_\alpha &= \int_{\mathbb{R}^N} |x|^\alpha |\nabla w_a(x)|^p \mathrm{d}x \\ &= \int_0^\infty \Big(\int_{\partial B(0,r)} r^\alpha (d(N))^{-p} a^{\frac{N}{p}} p^p a^p r^{p(p-1)} \mathrm{e}^{-par^p}\mathrm{d}\sigma\Big)\mathrm{d}r \\ &= p^p a^{\frac{N}{p}} a^p (d(N))^{-p} \omega_N \int_0^\infty r^{\alpha+p(p-1)+N-1} \mathrm{e}^{-par^p}\mathrm{d}r \\ &= a^{1-\frac{\alpha}{p}} D(N).\end{aligned} \tag{3.23}$$

Recalling that $0 < \alpha < p$ and making use of the assumption $0 < \lambda < l$, we choose

$$a \in \Big(0, \Big(\frac{l-\lambda}{D(N)}\Big)^{\frac{p}{p-\alpha}}\Big). \tag{3.24}$$

One obtains

$$\|w_a\|^p_\alpha < l - \lambda. \tag{3.25}$$

Since $tw_a(x) \to +\infty$ as $t \to +\infty$ and by (H4),

$$\lim_{u \to +\infty} \frac{F(x,u)}{u^p} = \frac{l}{p}, \tag{3.26}$$

it follows that

$$\lim_{t\to+\infty}\frac{F(x,tw_a(x))}{t^p}=\frac{l}{p}w_a(x)^p\quad\text{for a.e. } x\in\mathbb{R}^N. \tag{3.27}$$

Using Fatou's lemma (see [8, Theorem 1.15–2]), we get

$$\begin{aligned}\limsup_{t\to+\infty}\frac{J(tw_a)}{t^p}&=\frac{1}{p}\|w_a\|_\alpha^p+\frac{\lambda}{p}\|w_a\|^p_{L^p(\mathbb{R}^N)}-\liminf_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,tw_a(x))}{t^p}\mathrm{d}x\\&\le\frac{1}{p}\|w_a\|_\alpha^p+\frac{\lambda}{p}-\int_{\mathbb{R}^N}\lim_{t\to+\infty}\frac{F(x,tw_a(x))}{t^p}\mathrm{d}x\\&=\frac{1}{p}\|w_a\|_\alpha^p+\frac{\lambda}{p}-\frac{l}{p}\\&<\frac{l-\lambda}{p}-\frac{l-\lambda}{p}\\&=0.\end{aligned}$$

In particular, we obtain $J(tw_a)\to-\infty$ as $t\to+\infty$. If $t_0>0$ is large enough and $e=t_0w_a$, we achieve the conclusion of Lemma 3.3 with $e=t_0w_a$ and the proof reaches an end.

Proof of Theorem 2.1 Let us introduce the set

$$\Gamma:=\{\gamma\in C([0,1],E);\gamma(0)=0,\gamma(1)=e\},$$

where E is the space described in Section 2 and $e\in E$ is determined by Lemma 3.3. Moreover, we consider

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t)).$$

Due to Lemma 3.3, we know that $\|e\|>\rho$, so every path $\gamma\in\Gamma$ intersects the sphere $\|u\|=\rho$. Consequently, Lemma 3.2 implies

$$c\ge\inf_{\|u\|=\rho}J(u)\ge a,$$

with the constant $a>0$ in Lemma 3.2, so $c>0$.

By the mountain-pass theorem (see, e.g., [2]), we obtain a sequence $(u_n)\subset E$ so that

$$J(u_n)\to c,\quad J'(u_n)\to 0. \tag{3.28}$$

Firstly, we show that (u_n) is bounded in E. Indeed, from the first convergence in (3.28) we have

$$J(u_n)=\frac{1}{p}\|u_n\|^p-\int_{\mathbb{R}^N}F(x,u_n)\mathrm{d}x=c+o(1). \tag{3.29}$$

We note that

$$\|u_n\|^p-\int_{\mathbb{R}^N}f(x,u_n)u_n^p\mathrm{d}x=\langle J'(u_n),u_n\rangle. \tag{3.30}$$

By (H6), there exist constants $C > 0$ and $M > 0$ such that

$$f(x,t)t^p - pF(x,t) \geq Ct^r, \quad \forall t \geq M,\ x \in \mathbb{R}^N. \tag{3.31}$$

Corresponding to the number $M > 0$ in (3.31), by (H5) there exists some constant $\theta > 0$ such that

$$F(x,t) \leq \frac{1}{p+\theta} f(x,t)t^p, \quad \forall t \in (0, M). \tag{3.32}$$

Thus, by taking into account (3.29)–(3.30), we find

$$\begin{aligned}&\Big(\frac{1}{p} - \frac{1}{p+\theta}\Big)\|u_n\|^p - \int_{\mathbb{R}^N} \Big[F(x,u_n) - \frac{1}{p+\theta} f(x,u_n)u_n^p\Big]\mathrm{d}x \\ &= c - \frac{1}{p+\theta}\langle J'(u_n), u_n\rangle + o(1).\end{aligned} \tag{3.33}$$

Due to (3.33), together with (3.32) and Remark 2.2, we obtain

$$\begin{aligned}\Big(\frac{1}{p} - \frac{1}{p+\theta}\Big)\|u_n\|^p &= c + \int_{\{x;|u_n|\geq M\}} \Big[F(x,u_n) - \frac{1}{p+\theta} f(x,u_n)u_n^p\Big]\mathrm{d}x \\ &\quad + \int_{\{x;|u_n|<M\}} \Big[F(x,u_n) - \frac{1}{p+\theta} f(x,u_n)u_n^p\Big]\mathrm{d}x \\ &\quad - \frac{1}{p+\theta}\langle J'(u_n), u_n\rangle + o(1) \\ &\leq c + \frac{\theta}{p(p+\theta)} \int_{\{x;|u_n|\geq M\}} f(x,u_n)u_n^p \mathrm{d}x \\ &\quad + \frac{1}{p+\theta}|\langle J'(u_n), u_n\rangle| + o(1).\end{aligned} \tag{3.34}$$

On the other hand, using (3.29)–(3.30), it follows that

$$\int_{\mathbb{R}^N} [f(x,u_n)u_n^p - pF(x,u_n)]\mathrm{d}x = pc - \langle J'(u_n), u_n\rangle + o(1). \tag{3.35}$$

Then, from (3.31) and Remark 2.2, we have

$$\begin{aligned}\|J'(u_n)\|\|u_n\| + pc + o(1) &\geq \int_{\{x;|u_n|\geq M\}} [f(x,u_n)u_n^p - pF(x,u_n)]\mathrm{d}x \\ &\geq C\int_{\{x;|u_n|\geq M\}} |u_n|^r \mathrm{d}x.\end{aligned} \tag{3.36}$$

Thus, for a constant $c_0 > 0$, we infer that

$$\int_{\{x;|u_n|\geq M\}} |u_n|^r \mathrm{d}x \leq c_0\left[1 + \|J'(u_n)\|\|u_n\|\right]. \tag{3.37}$$

Relations (3.13) and (3.34) ensure

$$\begin{aligned}\frac{\theta}{p(p+\theta)}\|u_n\|^p &\le c+\frac{\theta}{p(p+\theta)}\int_{\{x;|u_n|\ge M\}}(\varepsilon|u_n|^p+D_\varepsilon|u_n|^{\sigma+p})\mathrm{d}x\\ &\quad+\frac{1}{p+\theta}\|J'(u_n)\|\|u_n\|+o(1)\\ &\le c+\varepsilon\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x+D_\varepsilon\int_{\{x;|u_n|\ge M\}}|u_n|^{\sigma+1}|u_n|^{p-1}\mathrm{d}x\\ &\quad+\frac{1}{p+\theta}\|J'(u_n)\|\|u_n\|+o(1).\end{aligned} \tag{3.38}$$

Taking into account the expressions of p_α^* and σ in (3.13), it follows that

$$\frac{\sigma+1}{r}=1-\frac{p-1}{p_\alpha^*}<1. \tag{3.39}$$

Consequently, using the Hölder inequality, Lemma 2.1 and (3.37), we obtain that there exists a constant $D'_\varepsilon>0$ depending on ε and a constant $C>0$ such that

$$\begin{aligned}\frac{\theta}{p(p+\theta)}\|u_n\|^p &\le c+\varepsilon\|u_n\|^p_{L^p(\mathbb{R}^N)}+D_\varepsilon\Big(\int_{\{x;|u_n|\ge M\}}|u_n|^r\mathrm{d}x\Big)^{\frac{\sigma+1}{r}}\Big(\int_{\mathbb{R}^N}|u_n|^{p_\alpha^*}\mathrm{d}x\Big)^{\frac{p-1}{p_\alpha^*}}\\ &\quad+\frac{1}{p+\theta}\|J'(u_n)\|\|u_n\|+o(1)\\ &\le c+\varepsilon\|u_n\|^p_{L^p(\mathbb{R}^N)}+D'_\varepsilon[1+\|J'(u_n)\|\|u_n\|]^{\frac{\sigma+1}{r}}\|u_n\|^{p-1}\\ &\quad+\frac{1}{p+\theta}\|J'(u_n)\|\|u_n\|+o(1)\\ &\le c+\varepsilon C\|u_n\|^p+D'_\varepsilon[1+\|J'(u_n)\|\|u_n\|]\|u_n\|^{p-1}\\ &\quad+\frac{1}{p+\theta}\|J'(u_n)\|\|u_n\|+o(1).\end{aligned} \tag{3.40}$$

Fix

$$\varepsilon\in\Big(0,\frac{\theta}{pC(p+\theta)}\Big). \tag{3.41}$$

Recalling that $\|J'(u_n)\|\to 0$ (according to the second relation in (3.28)), the above inequality shows that (u_n) is bounded.

By Proposition 2.1, passing, if necessary, to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in E, for some $u_0\in E$. Consequently, Lemma 3.1 and (3.28) imply that u_0 is a weak solution of (2.1).

Finally, we show that u_0 is nontrivial. By (3.28)–(3.30), we obtain that

$$c=\lim_{n\to\infty}J(u_n)=\lim_{n\to\infty}\Big[\frac{1}{p}\langle J'(u_n),u_n\rangle+\int_{\mathbb{R}^N}\Big(\frac{1}{p}f(x,u_n)u_n^p-F(x,u_n)\Big)\mathrm{d}x\Big]. \tag{3.42}$$

Thus, using the boundedness of the sequence (u_n) and the second relation in (3.28), we find

$$c=\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big(\frac{1}{p}f(x,u_n)u_n^p-F(x,u_n)\Big)\mathrm{d}x. \tag{3.43}$$

Let us justify here that Fatou's lemma can be applied. To this end, we notice that assumptions (H1)–(H4) and (H7) ensure the existence of a (sufficiently large) constant $c_0 > 0$ such that

$$|f(x,t)| \leq c_0|t|^\tau, \quad |F(x,t)| \leq c_0|t|^{\tau+p}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.44}$$

By Remark 2.5, it is known that E is continuously embedded in $\mathcal{D}^{1,p}_\alpha(\mathbb{R}^N)$. Applying [6, Proposition 3.4], it follows that E is compactly embedded in $L^{\tau+p}(\mathbb{R}^N)$ because $p < \tau + p < p^*_\alpha$. Consequently, up to a subsequence, we may suppose that $u_n \to u_0$ in $L^{\tau+p}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$, and there is a function $h \in L^{\tau+p}(\mathbb{R}^N)$ such that $|u_n(x)| \leq h(x)$ for almost all $x \in \mathbb{R}^N$. Therefore

$$\frac{1}{p}f(x,u_n)u_n^p - F(x,u_n) \to \frac{1}{p}f(x,u_0)u_0^p - F(x,u_0) \quad \text{for a.e. } x \in \mathbb{R}^N,$$

$$\begin{aligned}\Big|\frac{1}{p}f(x,u_n)u_n^p - F(x,u_n)\Big| &\leq \frac{1}{p}|f(x,u_n)u_n^p| + |F(x,u_n)| \\ &\leq \frac{1}{p}c_0|u_n|^{\tau+p} + c_0|u_n|^{\tau+p} \\ &\leq \frac{p+1}{p}c_0 h(x)^{\tau+p} \quad \text{for a.e. } x \in \mathbb{R}^N.\end{aligned}$$

Thus one may apply Fatou's lemma to the sequence $\big(\frac{1}{p}f(x,u_n)u_n^p - F(x,u_n)\big)$. Using also that u_0 solves the problem (2.1), and then we obtain

$$\begin{aligned} c &\leq \int_{\mathbb{R}^N} \limsup_{n\to\infty} \Big[\frac{1}{p}f(x,u_n)u_n^p - F(x,u_n)\Big]\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \Big[\frac{1}{p}f(x,u_0)u_0^p - F(x,u_0)\Big]\mathrm{d}x \\ &= J(u_0) - \frac{1}{p}\langle J'(u_0), u_0\rangle \\ &= J(u_0).\end{aligned}$$

Taking into account that $c \leq J(u_0)$ and $c > 0$, we conclude that $u_0 \neq 0$ and the proof is complete.

First of all, we recall a minimax-type lemma which will be used in the proof of Theorem 2.2.

Lemma 3.4 (see [16, Lemma 5.1]) *Let E be a real Banach space, let a function $G : E \times \mathbb{R} \to \mathbb{R}$ be of class C^1, and let two positive numbers $\rho < r$ be such that the following condition is fulfilled*

$$\inf_{v\in E} G(v,\rho) > \max\{G(0,0), G(0,r)\}. \tag{3.45}$$

Denoting

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} G(\gamma(t)) \tag{3.46}$$

with

$$\Gamma = \{\gamma \in C([0,1], E \times \mathbb{R}) : \ \gamma(0) = (0,0); \ \gamma(1) = (0,r)\}, \tag{3.47}$$

then there exists a sequence $(u_n) \subset E \times \mathbb{R}$ *such that*

$$G(u_n) \to c, \qquad G'(u_n) \to 0.$$

We are now in the position to prove Theorem 2.2.

Proof of Theorem 2.2 Let us start by choosing positive numbers $\rho < r$ and a function $\beta \in C^1(\mathbb{R})$ such that

$$\beta(0) = \beta(r) = 0, \tag{3.48}$$

$$\rho \geq \sqrt[p]{qa_2}, \quad \beta(\rho) > \frac{qa_1}{p}, \tag{3.49}$$

$$\beta(t) \to +\infty \quad \text{as } |t| \to +\infty, \tag{3.50}$$

$$\beta'(t) < 0 \Leftrightarrow t < 0 \quad \text{or } \rho < t < r. \tag{3.51}$$

Considering the Banach space $E = \overline{C_c^\infty(\mathbb{R}^N \setminus \{0\})}^{\|\cdot\|}$ $\big($where $\|u\|^p = \int_{\mathbb{R}^N}(|x|^\alpha |\nabla u|^p + \lambda |u|^p)\mathrm{d}x\big)$ and the function I in (2.6), we define $G : E \times \mathbb{R} \to \mathbb{R}$ by

$$G(u,t) = t^p \frac{1}{q}\|u\|^q + \frac{p}{q}\beta(t) - I(u) + \frac{\nu}{p}\|u\|^p, \quad \forall (u,t) \in E \times \mathbb{R}. \tag{3.52}$$

Since E is a reflexive Banach space, and $I \in C^1(E, \mathbb{R})$, it is easily seen that $G \in C^1(E \times \mathbb{R})$ and its partial gradients have the expressions

$$G_u(u,t) = t^p\|u\|^{q-2}u - \nabla I(u) + \nu\|u\|^{p-2}u, \quad \forall (u,t) \in E \times \mathbb{R}, \tag{3.53}$$

$$G_t(u,t) = \frac{p}{q}(t^{p-1}\|u\|^q + \beta'(t)), \quad \forall (u,t) \in E \times \mathbb{R}. \tag{3.54}$$

It follows from (3.48), (3.52) and the first relation in (2.2) that

$$G(0,0) = \frac{p}{q}\beta(0) - I(0) \leq 0, \tag{3.55}$$

$$G(0,r) = \frac{p}{q}\beta(r) - I(0) \leq 0. \tag{3.56}$$

Moreover, from (3.52), (2.3) and (3.49), we obtain

$$\begin{aligned} G(u,\rho) &= \frac{1}{q}\rho^p\|u\|^q - I(u) + \frac{p}{q}\beta(\rho) + \frac{\nu}{p}\|u\|^p \\ &\geq \Big(\frac{1}{q}\rho^p - a_2\Big)\|u\|^q + \frac{p}{q}\beta(\rho) - a_1 \\ &\geq \frac{p}{q}\beta(\rho) - a_1 \\ &> 0, \quad \forall u \in E. \end{aligned} \tag{3.57}$$

Therefore, applying Lemma 3.4 we obtain a sequence $((v_n, t_n)) \subset E \times \mathbb{R}$ such that

$$G(v_n,t_n) \to c, \quad G_u(v_n,t_n) \to 0 \quad \text{in } E \quad \text{and} \quad G_t(v_n,t_n) \to 0 \quad \text{in } \mathbb{R}.$$

Taking into account the relations (3.52)–(3.54), these convergences imply

$$\frac{1}{q}t_n^p\|v_n\|^q + \frac{p}{q}\beta(t_n) - I(v_n) + \frac{\nu}{p}\|v_n\|^p \to c, \tag{3.58}$$

$$t_n^p\|v_n\|^{q-2}v_n - \nabla I(v_n) + \nu\|v_n\|^{p-2}v_n \to 0, \tag{3.59}$$

$$t_n^{p-1}\|v_n\|^q + \beta'(t_n) \to 0. \tag{3.60}$$

By (2.3) and (3.58), we see that

$$\begin{aligned} c + o(1) &= \frac{1}{q}t_n^p\|v_n\|^q + \frac{p}{q}\beta(t_n) - I(v_n) + \frac{\nu}{p}\|v_n\|^p \\ &\geq \Big(\frac{1}{q}t_n^p - a_2\Big)\|v_n\|^q + \frac{p}{q}\beta(t_n) - a_1. \end{aligned} \tag{3.61}$$

Then using (3.50), we deduce that the sequence (t_n) is bounded in $\mathbb{R}$. Thus there is $\overline{t} \in \mathbb{R}$ such that along a relabeled subsequence we may suppose

$$t_n \to \overline{t} \quad \text{in } \mathbb{R} \text{ as } n \to \infty. \tag{3.62}$$

Next, we show that

$$(v_n) \text{ is bounded in } E. \tag{3.63}$$

Indeed, in order to prove the above relation, we first consider the case $\overline{t} \neq 0$. Then for n sufficiently large, writing (3.60) in the form

$$\|v_n\|^q = \frac{1}{t_n^{p-1}}(o(1) - \beta'(t_n)), \tag{3.64}$$

and since (t_n) is bounded away from zero, it results that (v_n) is bounded.

Now assume that $\overline{t} = 0$. Then (3.48) and (3.60) ensure that $t_n^{p-1}\|v_n\|^q \to 0$. In view of relation (3.62), we have

$$t_n^p\|v_n\|^q \to 0.$$

It turns out from (3.58) that

$$-I(v_n) + \frac{\nu}{p}\|v_n\|^p \to c. \tag{3.65}$$

On the other hand, from $t_n^{p-1}\|v_n\|^q \to 0$, we deduce

$$t_n^{\frac{(p-1)(q-1)}{q}}\|v_n\|^{q-1} = \left(|t_n|^{p-1}\|v_n\|^q\right)^{\frac{q-1}{q}} \to 0. \tag{3.66}$$

Using (3.62), we obtain

$$t_n^p\|v_n\|^{q-1} \to 0.$$

Combining this fact with (3.59), we infer that

$$-\nabla I(v_n) + \nu\|v_n\|^{p-1} \to 0. \tag{3.67}$$

By (3.65), (3.67) and (2.4), we obtain

$$c + o(1) + \frac{1}{\gamma}\|v_n\|^{p-1} \ge \nu\Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_n\|^p + \frac{1}{\gamma}\langle \nabla I(v_n), v_n\rangle - I(v_n)$$
$$\ge \Big[\nu\Big(\frac{1}{p} - \frac{1}{\gamma}\Big) - b_2\Big]\|v_n\|^p - b_1, \tag{3.68}$$

if n is sufficiently large. Since $b_2 < \nu\big(\frac{1}{p} - \frac{1}{\gamma}\big)$, we obtain that (v_n) is bounded in the situation $\overline{t} = 0$, too.

Because (v_n) is bounded in E (a reflexive Banach space), we know that there is a $u \in E$ such that along a relabeled subsequence, one has $v_n \rightharpoonup u$ in E. According to (3.59), we have

$$(t_n^p\|v_n\|^{q-2} + \nu\|v_n\|^{p-2})v_n - \nabla I(v_n) \to 0.$$

So we obtain

$$\langle \eta_n v_n - \nabla I(v_n), \varphi\rangle \to 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}), \tag{3.69}$$

where

$$\eta_n = t_n^p\|v_n\|^{q-2} + \nu\|v_n\|^{p-2}. \tag{3.70}$$

Passing eventually to a subsequence, from (3.63) we may assume that there exists $\theta := \lim_{n\to\infty}\|v_n\|$ and

$$\|u\| \le \liminf_{n\to\infty}\|v_n\| \le \theta. \tag{3.71}$$

Letting $n \to \infty$ in (3.69)–(3.70), and using (2.5) and (3.62), we obtain

$$\langle u - \mu\nabla I(u), \varphi\rangle = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \tag{3.72}$$

with

$$\mu = \frac{1}{\overline{t}^p\theta^{q-2} + \nu\theta^{p-2}}. \tag{3.73}$$

So from (3.72)–(3.73), we obtain

$$\int_{\mathbb{R}^N} \big(|x|^\alpha|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \lambda|u|^{p-2}u\varphi\big)\,\mathrm{d}x = \mu\langle \nabla I(u), \varphi\rangle, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

In addition, from (3.60), (3.62) and the definition of θ, we obtain the equality

$$\overline{t}^{p-1}\theta^q + \beta'(\overline{t}) = 0. \tag{3.74}$$

This implies that

$$\overline{t}\beta'(\overline{t}) \le 0. \tag{3.75}$$

Notice that $\overline{t} \neq 0$. Indeed, if $\overline{t} = 0$, then (3.73) yields that $\frac{1}{\nu\theta^{p-2}}$ is an eigenvalue of (2.6), which contradicts (2.7). Further, we observe from (3.71), in conjunction with the assumption

$\nabla I(0) \neq 0$ in (2.2) and the relation (3.72), that $\theta \neq 0$. Since $\bar{t} \neq 0$, we deduce from (3.75) and (3.51) that $\rho \leq \bar{t} \leq r$. Knowing by (3.51) that $\beta'(\rho) = \beta'(r) = 0$, it follows from (3.74) that the inequality $\rho \leq \bar{t} \leq r$ can be sharpened to

$$\rho < \bar{t} < r. \tag{3.76}$$

Thus (3.73) and (3.76) imply

$$\frac{1}{r^p\theta^{q-2} + \nu\theta^{p-2}} < \mu < \frac{1}{\rho^p\theta^{q-2} + \nu\theta^{p-2}}. \tag{3.77}$$

For $q = 2$, from (3.77) we obtain

$$\frac{1}{r^p + \nu\theta^{p-2}} < \mu < \frac{1}{\rho^p + \nu\theta^{p-2}}.$$

For $q > 2$, the relations (3.71) and (3.77) involve (2.8), that is,

$$0 < \mu < \frac{1}{\rho^p\|u\|^{q-2} + \nu\|u\|^{p-2}}$$

and the proof is complete.

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