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Continuous-Time Independent Edge-Markovian Random Graph Process*

Ruijie DU[1] Hanxing WANG[2] Yunbin FU[2]

Abstract In this paper, the continuous-time independent edge-Markovian random graph process model is constructed. The authors also define the interval isolated nodes of the random graph process, study the distribution sequence of the number of isolated nodes and the probability of having no isolated nodes when the initial distribution of the random graph process is stationary distribution, derive the lower limit of the probability in which two arbitrary nodes are connected and the random graph is also connected, and prove that the random graph is almost everywhere connected when the number of nodes is sufficiently large.

Keywords Complex networks, Random graph, Random graph process, Stationary distribution, Independent edge-Markovian random graph process
2000 MR Subject Classification 05C80

1 Introduction

In the study of complex networks (see [1–2]), network evolution is usually described by the evolution of graphs (see [3–5]). Graph evolution includes adding vertices, adding edges, deleting vertices, deleting edges, and reconnecting. Reconnection is to consider the connection of edges on the assumption that the number of vertices remains unchanged. Researchers often use the Markov process to describe the edge reconnection, and discuss the random graph process model based on the classic theory of Markov chains; this assumption is also consistent with the evolution characteristic of many actual networks.

Using the Markov process to describe the dynamic evolution of the network is an important research content of the random graph process, which has gained a lot of achievements. Han [6] assumed that the random graph process was a homogeneous Markov process; its state space was a simple directed graph set which had the same set of vertices. By adding or deleting an edge (probabilities of adding or deleting edges related to edge numbers of the graph), transfer of the state was achieved. Under this assumption, stationary distribution of the random graph sequences was studied; Chen Avin et al [7] proposed a general Markov

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[1]Department of Mathematics, Shanghai University, Shanghai 200444, China; School of Mathematics and Information, Shanghai Lixin University of Commerce, Shanghai 201620, China.
E-mail: duruijie@lixin.edu.cn
[2]School of Mathematics and Information, Shanghai Lixin University of Commerce, Shanghai 201620, China. E-mail: whxlgg@163.com fuyunbin@lixin.edu.cn
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dynamic graph model: The state space was a set composed of graphs which had the same set of vertices; they also discussed the encountering time and continuing time between the nodes of the model. Clementi et al [8] proposed a discrete Markov dynamic graph model based on edge-independent evolution, namely, probability of every state which disconnects (connected) in time t and connected (disconnects) in time $t+1$ was $p(n)$ $(q(n))$, where n was the number of nodes. Under the assumptions of this model, Clementi discussed the flooding time of such dynamic graphs. As a special case of [7], the edge-independent evolution Markov dynamic graph model was easier to handle mathematically. As the supplement of [8], Herv et al [9] proposed that the evolution graph defined on the time sequences was mapped to a weighted random graph, which eliminated corrections of dynamic network evolution, and they also proved that the weighted graph had the same topological properties as the time evolution graph. In 2011, according to the highly dynamic topology and characteristics that are evaluated with time in the opportunity network, Cai et al [10] proposed the time evolution graph model based on edge-independent evolution, and the model assumed that the edge evolution between any node pair was independent of the other node pairs. They used the discrete Markov chain and the birth and death process to characterize the evolution of time correlation and the Laplace successor rule to estimate the probability of the edge births and deaths, and then analyzed the convergence characteristics of the dynamic evolution network. Du [11] constructed a probability space and a random graph process: $\{G_t\}_{t\geq 0}$, G_0 is an m-complete graph, and $v_0, v_1, \cdots, v_{m-1}$ are m vertices of it. Du proved that this mathematical model has the same marginal distributions and boundary conditions as the BA model, and $\{G_t\}_{t\geq 0}$ can be considered as graph value Markov chains constructed on its probability space.

Unlike other researches on the discrete Markov dynamic graph model with independent edge-Markovian evolution, this article discusses the topology properties of the continuous-time Markov dynamic graph model with edge-independent evolution, and by using the independence of the chain process and Markov assumptions, we have explored the statistical properties of the distribution of the number of isolated nodes and the connection probability.

2 Distribution of the Number of Isolated Nodes

Definition 2.1 (see [1]) *Suppose that (Ω, F, P) is a probability space, and $\{G_t(\omega) : t \geq 0\}$ is an undirected random simple graph process defined on (Ω, F, P), where $\forall V(G_t) = \{1, 2, \cdots, N\}$, $N \geq 2,\ t \geq 0$, the corresponding adjacency matrix process is*

$$A(G_t(\omega)) = (\xi_{ij}(t, \omega))_{ij}.$$

If the chain processes $\{\xi_{ij}(t, \omega) : t \geq 0\}$ $(i < j)$ satisfy the following conditions:

(1) *C_N^2 chain processes are independent of each other, namely, $\{\xi_{ij}(t, \omega) : t \geq 0\}$ $(i < j)$ independent of each other;*

(2) *Every chain process $\{\xi_{ij}(t, \omega) : t \geq 0\}$ is a continuous-time Markov chain which has two states 0 and 1, and its density matrix is*

$$Q = \begin{pmatrix} -\beta_{ij} & \beta_{ij} \\ \alpha_{ij} & -\alpha_{ij} \end{pmatrix},$$

where $P'_{ij}(0; 0, 1) = \beta_{ij},\ P'_{ij}(0; 1, 0) = \alpha_{ij}$, then $\{G_t(\omega) : t \geq 0\}$ is defined as an independent edge-Markovian random graph process.

The transfer function of the chain processes $\{\xi_{ij}(t,\omega) : t \geq 0\}$ $(i < j)$ is denoted as $P_{(ij)}(t; x, y),\ t \geq 0,\ x, y \in \{0, 1\}$. Obviously, the independent edge-Markovian random graph process is a Markov random graph process having stationary distribution $\pi(\cdot),\ \forall(x_{kl})$, which is a symmetric matrix with $\{0, 1\}$ element. Then

$$\pi((x_{kl})) = \prod_{i<j} \pi_{ij}(x_{ij}),$$

where $\pi_{ij}(\cdot)$ is the distribution on $0, 1$, $\pi_{ij}(1) = \frac{\beta_{ij}}{\alpha_{ij}+\beta_{ij}}$, $\pi_{ij}(0) = \frac{\alpha_{ij}}{\alpha_{ij}+\beta_{ij}}$.

In the article below, unless specially described, we always assume that the chain processes have the same distributions, i.e., each chain process has the same density matrix

$$\theta = \begin{pmatrix} -\beta & \beta \\ \alpha & -\alpha \end{pmatrix}.$$

Therefore, it has the same stationary distribution, namely, $\forall i \neq j$,

$$\pi_{ij}(1) = p = \frac{\beta}{\alpha+\beta}, \quad \pi_{ij}(0) = \frac{\alpha}{\alpha+\beta}.$$

If we choose the stationary distribution as its initial distribution, then the independent edge-Markovian random graph process is a stationary random graph process, namely, $\forall t \geq 0,\ i \neq j$,

$$P(\xi_{ij}(t) = 1) = \pi_{ij}(1) = p,$$
$$P(\xi_{ij}(t) = 0) = \pi_{ij}(0) = q.$$

That is to say, $\forall t \geq 0$, G_t is such a random graph that $|V(G_t)| = N \geq 2$, and for any two nodes i and j, their connection probability is p, disconnected probability is q, and whether it is connected or not every node pair is mutually independent.

Suppose that $\{G_t : t \geq 0\}$ is a graph family, $V(G_t) = V,\ t \geq 0$. If node $i \in V$ is an isolated node of all $G_s,\ t_0 \leq s \leq t$, then i is defined as the interval isolated node of the graph family $\{G_t : t \geq 0\}$ on $[t_0, t]$.

The interval isolated node is a concept associated with the evolution of the graph. If node i is the interval isolated node of the graph family $\{G_t : t \geq 0\}$ on $[t_0, t]$, that is in the evolution process of the graph, node i always maintains the state of isolated nodes in the period $[t_0, t]$.

Definition 2.2 (see [2]) *Suppose that $\{G_t(\cdot) : t \geq 0\}$ is the independent edge-Markovian random graph process, where $V(G_t) = \{1, 2, \cdots, N\},\ N \geq 2,\ t \geq 0$, and also suppose that the chain process is identically distributed, and its initial distribution is the stationary distribution, $\forall i \in V$. Suppose further $t \geq 0$,*

$$i(\omega, t) = \begin{cases} 1, & i \text{ is an isolated node of } G_t(\omega), \\ 0, & \text{otherwise}, \end{cases}$$
$$\eta_t(\omega) = \sum_{i \in V} i(\omega, t).$$

Then the random variable $\eta_t(\omega)$ is the number of isolated nodes of the random graph process $\{G_t(\cdot) : t \geq 0\}$ at time t.

$\forall 0 \le t_0 < t,\ i \in V$, suppose

$$i(\omega, t_0, t) = \begin{cases} 1, & i \text{ is an isolated node of } G_s(\omega),\ t_0 \le s \le t, \\ 0, & \text{otherwise,} \end{cases}$$
$$\xi(\omega; t_0, t) = \sum_{i \in V} i(\omega; t_0, t).$$

Then the random variable $\xi(\omega; t_0, t)$ is the number of interval isolated nodes of the random graph process $\{G_t(\cdot) : t \ge 0\}$ on $[t_0, t]$.

Note that the random events $\{i(\omega, t) = 1\}$ and $\{j(\omega, t) = 1\},\ i \ne j$ are not independent of each other, because $P\{i(\omega, t) = 1\} = P\{j(\omega, t) = 1\} = q^{N-1}$, $P\{i(\omega, t) = 1, j(\omega, t) = 1\} = q^{2N-3}$. So $\eta_t(\omega)$ does not obey binomial distribution. Similarly, the number of interval isolated nodes $\xi(\omega; t_0, t)$ does not obey binomial distribution either.

Lemma 2.1 (see [3]) *Suppose that $G(\omega)$ is a random graph, and $V(G(\omega)) = \{1, 2, \cdots, n\}$, $n \ge 2$; the probability of its arbitrary two different nodes connected is p, and the probability of disconnected is $q = 1 - p$; and all node pairs, whether the nodes are connected or not, are independent of each other. Then the probability of $G(\omega)$ having no isolated nodes is*

$$P_n(0) = 1 - [nq^{n-1} - C_n^2 q^{2n-3} + \cdots + (-1)^{k+1} C_n^k q^{\frac{[2n-(k+1)]k}{2}} + \cdots + (-1)^{n+1} q^{C_n^2}].$$

Proof Suppose that A_i is a random event, and i is an isolated node, $i = 1, 2, \cdots, n$. Then in addition to the multi-less complement principle

$$\begin{aligned} P\Big(\bigcup_{i=1}^n A_i\Big) &= \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i A_j) + \cdots + (-1)^{k+1} \sum_{i_1, \cdots, i_k} P\Big(\bigcap_{j=1}^k A_{i_j}\Big) + \cdots \\ &\quad + (-1)^{n+1} P\Big(\bigcap_{i=1}^n A_i\Big) \\ &= C_n^1 q^{n-1} - C_n^2 q^{n-1} q^{n-2} + \cdots + (-1)^{k+1} C_n^k q^{n-1} q^{n-2} \cdots q^{n-k} + \cdots + (-1)^{n+1} q^{C_n^2} \\ &= nq^{n-1} - C_n^2 q^{2n-3} + \cdots + (-1)^{k+1} C_n^k q^{\frac{2n-(k+1)}{2} \cdot k} + \cdots + (-1)^{n+1} q^{C_n^2}, \end{aligned}$$

we can get

$$\begin{aligned} P_n(0) &= 1 - P\Big(\bigcup_{i=1}^n A_i\Big) \\ &= 1 - [nq^{n-1} - C_n^2 q^{2n-3} + \cdots + (-1)^{k+1} C_n^k q^{\frac{[2n-(k+1)]k}{2}} + \cdots + (-1)^{n+1} q^{C_n^2}]. \end{aligned}$$

This completes the proof.

Theorem 2.1 *Suppose that $\{G_t(\cdot) : t \ge 0\}$ is an independent identically distribution edge-Markovian random graph process which has stationary distribution as its initial distribution, $V(G_t) = \{1, 2, \cdots, N\},\ N \ge 2,\ t \ge 0$, and the density matrix of the chain process is*

$$\theta = \begin{pmatrix} -\beta & \beta \\ \alpha & -\alpha \end{pmatrix}.$$

Then $\eta_t(\omega)$ is the number of isolated nodes of the random graph process $\{G_t(\cdot):t\geq 0\}$ at time $t\geq 0$, and its distribution is

$$P(\eta_t(\omega)=0)=1-[Nq^{N-1}-C_N^2q^{2N-3}+\cdots+(-1)^{k+1}C_N^kq^{\frac{[2N-(k+1)]k}{2}}+\cdots+(-1)^{N+1}q^{C_N^2}],$$
$$P(\eta_t(\omega)=n)=C_N^nq^{\frac{[2N-(n+1)]n}{2}}P_{N-n}(0),\quad n=1,2,\cdots,N-1,$$

where

$$P_{N-n}(0)=1-[(N-n)q^{N-n-1}-C_{N-n}^2q^{2N-2n-3}+\cdots+(-1)^{N-n+1}q^{C_{N-n}^2}],\quad q=\frac{\alpha}{\alpha+\beta}.$$

Remark 2.1 Obviously, $(\eta_t(\omega)=N-1)=(\eta_t(\omega)=N)$. Therefore,

$$\sum_{n=0}^{N-1}P(\eta_t(\omega)=n)=1.$$

That is to say, the value of $\eta_t(\omega)$ is $0,1,2,\cdots,N-1$, or $0,1,2,\cdots,N-2,N$.

Proof of Theorem 2.1 According to the assumptions, $\forall t\geq 0$, $G_t(\omega)$ have the independent chains, and the probability of any two nodes connected is $p=\frac{\beta}{\alpha+\beta}$, while the probability of disconnected is $q=\frac{\alpha}{\alpha+\beta}$. According to Lemma 2.1,

$$P(\eta_t(\omega)=0)=P_N(0)=1-[Nq^{N-1}-C_N^2q^{2N-3}+\cdots+(-1)^{N+1}q^{C_N^2}].$$

Suppose that A_i is a random event, i is an isolated node of $G_t(\omega)$, $i=1,2,\cdots,N$. Then from Lemma 2.1 and the total probability formula for $V=V(G_t)$, $\forall n\geq 1$,

$$\begin{aligned}P(\eta_t(\omega)=n)&=\sum_{i_1,\cdots,i_n}P(A_{i_1}A_{i_2}\cdots A_{i_n})P(\eta_t(\omega)=n\mid A_{i_1}A_{i_2}\cdots,A_{i_n})\\&=\sum_{i_1,\cdots,i_n}q^{\frac{[2N-(n+1)]n}{2}}P(V-(i_1,i_2,\cdots,i_n)\text{ has no isolated node}\mid A_{i_1}A_{i_2}\cdots A_{i_n})\\&=\sum_{i_1,\cdots,i_n}q^{\frac{[2N-(n+1)]n}{2}}P_{N-n}(0)=C_N^nq^{\frac{[2N-(n+1)]n}{2}}P_{N-n}(0),\end{aligned}$$

where

$$P_{N-n}(0)=1-[(N-n)q^{N-n-1}-C_{N-n}^2q^{2N-2n-3}+\cdots+(-1)^{N-n+1}q^{C_{N-n}^2}].$$

This completes the proof.

Lemma 2.2 *Suppose that $\{G_t(\cdot):t\geq 0\}$ is the stationary Markov random graph process in Theorem 2.1. Then $\forall t\geq 0$, the probability of $\{G_t(\cdot):t\geq 0\}$ having no interval isolated nodes on $[0,t]$ is*

$$P_N^{(t)}(0)=1-[Nq_t^{N-1}-C_N^2q_t^{2N-3}+\cdots+(-1)^{k+1}C_N^kq_t^{\frac{[2N-(k+1)]k}{2}}+\cdots+(-1)^{N+1}q_t^{C_N^2}],$$

where $q_t=qe^{-\beta t}$.

Proof Suppose that the random adjacency matrix process according to $\{G_t(\cdot):t\geq 0\}$ is

$$\{A_t(\cdot):t\geq 0\}=\{(\xi_{ij}(t))_{ij}:t\geq 0\}.$$

Suppose further that A_i is a random event, i is an isolated node on $[0,t]$, $i=1,2,\cdots,N$. Then

$$\begin{aligned}P(A_i) &= P(i(\omega;0,t)=1) = P(i \text{ is an isolated node on } [0,t]) \\ &= P(\xi_{ij}(s)=0, 0\le s\le t, i\ne j) = \prod_{i\ne j} P(\xi_{ij}(s)=0, 0\le s\le t) \\ &= \prod_{i\ne j} P(\xi_{ij}(0)=0) P(\xi_{ij}(s)=0, 0\le s\le t \mid \xi_{ij}(0)=0) \\ &= \prod_{i\ne j} q\cdot e^{-\beta t} = (q\cdot e^{-\beta t})^{N-1} = q_t^{N-1}.\end{aligned}$$

$\forall A_i, A_j,\ i\ne j$,

$$\begin{aligned}P(A_iA_j) &= P(\xi_{ik}(s)=0, 0\le s\le t, k\ne i;\ \xi_{jr}(s)=0, 0\le s\le t, r\ne j) \\ &= P(\xi_{ij}(s)=0, 0\le s\le t) \prod_{\substack{k\ne i\\ k\ne j}} P(\xi_{ij}(s)=0, 0\le s\le t) \prod_{\substack{r\ne i\\ r\ne j}} P(\xi_{jr}(s)=0, 0\le s\le t) \\ &= q_t\cdot q_t^{2N-4} = q_t^{2N-3}.\end{aligned}$$

Generally, for any k different events $A_{i_1}A_{i_2}\cdots A_{i_k}, 1\le k\le N$, similarly, we can get

$$P(A_{i_1}A_{i_2}\cdots A_{i_n}) = q_t^{N-1}\cdot q_t^{N-2}\cdots q_t^{N-k} = q_t^{\frac{[2N-(k+1)]k}{2}}.$$

According to the multi-less complement principle,

$$\begin{aligned}P(\xi(\omega;0,t)) &= P\Big(\bigcup_{i=1}^{N} A_i\Big) \\ &= \sum_{i=1}^{N} P(A_i) - \sum_{i<j} P(A_iA_j) + (-1)^{k+1} \sum_{i_1,\cdots,i_k} P(A_{i_1}A_{i_2}\cdots A_{i_k}) \\ &\quad + (-1)^{N+1} P(A_1A_2\cdots A_N) \\ &= Nq_t^{N-1} - C_N^2 q_t^{2N-3} + \cdots + (-1)^{N+1} C_N^k q_t^{\frac{[2N-(k+1)]k}{2}} + \cdots + (-1)^{N+1} q_t^{C_N^2}.\end{aligned}$$

So

$$\begin{aligned}P_N^{(t)}(0) &= P(\xi(\omega;0,t)=0) = 1 - P(\xi(\omega;0,t)\ge 1) \\ &= 1 - [Nq_t^{N-1} - C_N^2 q_t^{2N-3} + \cdots + (-1)^{N+1} q_t^{C_N^2}].\end{aligned}$$

This completes the proof.

Theorem 2.2 *Suppose that $\{G_t(\cdot): t\ge 0\}$ is the stationary Markov random graph process in Theorem 2.1. Then $\forall t\ge 0$, $\xi(\omega;0,t)$ is the number of interval isolated nodes of $\{G_t(\cdot): t\ge 0\}$ on $[0,t]$, and its distribution is*

$$P(\xi(\omega;0,t)=0) = 1 - [Nq_t^{N-1} - C_N^2 q_t^{2N-3} + \cdots + (-1)^{N+1} C_N^k q_t^{\frac{[2N-(k+1)]k}{2}} + \cdots + (-1)^{N+1} q_t^{C_N^2}],$$
$$P(\xi(\omega;0,t)=n) = C_N^n q_t^{\frac{[2N-(n+1)]n}{2}} P_{N-n}^{(t)}(0), \quad n=1,2,\cdots,N-1,$$

where

$$P_{N-n}^{(t)}(0) = 1 - [(N-n)q_t^{N-n-1} - C_{N-n}^2 q_t^{2N-2n-3} + \cdots + (-1)^{N-n+1} q_t^{C_{N-n}^2}].$$

Proof According to Lemma 2.2,

$$\begin{aligned}&P(\xi(\omega;0,t)=0)\\=&1-[Nq_t^{N-1}-C_N^2q_t^{2N-3}+\cdots+(-1)^{N+1}C_N^kq_t^{\frac{[2N-(k+1)]k}{2}}+\cdots+(-1)^{N+1}q_t^{C_N^2}].\end{aligned}$$

Suppose that A_i is a random event, i is an interval isolated node on $[0,t]$, $i=1,2,\cdots,N$. According to the total probability formula and Lemma 2.2, for $V=V(G_t)$, $\forall 1\le n\le N-1$,

$$\begin{aligned}&P(\xi(\omega;0,t)=n)\\=&\sum_{i_1,\cdots,i_n}P(A_{i_1}A_{i_2}\cdots A_{i_n})P(\xi(\omega;0,t)=n\mid A_{i_1}A_{i_2}\cdots A_{i_n})\\=&\sum_{i_1,\cdots,i_n}q_t^{\frac{[2N-(n+1)]n}{2}}P(V-(i_1,i_2,\cdots,i_n)\text{ has no interval isolated node on }[0,t]\mid A_{i_1}A_{i_2}\cdots A_{i_n})\\=&\sum_{i_1,\cdots,i_n}q_t^{\frac{[2N-(n+1)]n}{2}}P_{N-n}^{(t)}(0)=C_N^nq_t^{\frac{[2N-(n+1)]n}{2}}P_{N-n}^{(t)}(0),\end{aligned}$$

where

$$P_{N-n}^{(t)}(0)=1-[(N-n)q_t^{N-n-1}-C_{N-n}^2q_t^{2N-2n-3}+\cdots+(-1)^{N-n+1}q_t^{C_{N-n}^2}].$$

This completes the proof.

Remark 2.2 Because $\{G_t(\cdot):t\ge 0\}$ is the stationary Markov random graph process, according to its stationarity, $\forall t_0\ge 0,\ t\ge 0,\ \xi(\omega;t_0,t_0+t)$ is the number of interval isolated nodes on $[t_0,t_0+t]$, and it has the same distribution as $\xi(\omega;0,t)$, that is to say, the distribution of $\xi(\omega;t_0,t_0+t)$ has nothing to do with t_0, but only has something to do with the interval length t.

Theorem 2.3 *Suppose that $\{G_t(\cdot):t\ge 0\}$ is the stationary Markov random graph process in Theorem 2.1. Then when $N\to\infty$, $\{G_t(\cdot):t\ge 0\}$ has no isolated node at any time $t\ge 0$ and no interval isolated node on any interval.*

Proof For $V=V(G_t(\cdot)),\ t\ge 0,\ \forall i\in V,\ A_i$ are random events, i is the isolated node. $\forall i\ne j$, A_{ij} denotes that i and j are disconnected. Then

$$P(A_i)=P\Big(\bigcap_{\substack{j\in V\\ i\ne j}}A_{ij}\Big)=\prod_{\substack{j\in V\\ i\ne j}}P(A_{ij})=\prod_{\substack{j\in V\\ i\ne j}}q=q^{N-1}.$$

Therefore,

$$\lim_{N\to\infty}P(A_i)=\lim_{N\to\infty}q^{N-1}=0.$$

Then

$$P\Big(\bigcup_{i\in V}A_i\Big)\le\sum_{i\in V}P(A_i)=0.$$

Namely, $\{G_t(\cdot):t\ge 0\}$ has no isolated node at any time $t\ge 0$. Similarly, we can also prove that it has no interval isolated node on any interval.

This completes the proof.

Corollary 2.1 *Suppose* $0 < q < 1$. *Then*

$$\lim_{N\to\infty}[Nq^{N-1} - C_N^2 q^{2N-3} + \cdots + (-1)^{N+1} C_N^k q^{\frac{[2N-(k+1)]k}{2}} + \cdots + (-1)^{N+1} q^{C_N^2}] = 0.$$

The proof of Corollary 2.1 can be obtained directly from Theorems 2.1 and 2.3. However, it is worth noting that to prove the corollary by the method usually used to derive limit does not seem obvious. By applying the random graph method, Corollary 2.1 gives an interesting limit of the sum formula.

3 Connection Probability

Suppose that G is a graph. A pathway of G refers to a finite non-empty sequence $W = V_0 e_1 V_1 e_2 \cdots e_k V_k$. W is a pathway from V_0 to V_k, and integer k is called the length of the pathway W. A pathway with length k is defined as k-pathway.

In the sample graph, pathway $V_0 e_1 V_1 e_2 \cdots e_k V_k$ is determined by its vertices sequence $V_0 V_1 \cdots V_k$, so the pathway of the sample graph is also denoted by its vertices sequence. Even in a non-simple graph, in the case of no ambiguity, we also use the vertices sequence to represent a pathway.

Suppose that $G(\omega)$ is an undirected random graph. For $\omega_0 \in \Omega$ and $u, v \in V(G(\omega_0))$, if there is a pathway from u to V in $G(\omega)$, then u and v are defined as connected in $G(\omega)$. Suppose $\Omega_0 \subseteq \Omega$. If $\forall \omega \in \omega_0$, u and v are connected in $G(\omega)$, then we call u and v are connected in $G(\omega)$ on Ω_0. Specially, if $P(\Omega_0) = 1$, then u and v are defined as almost everywhere connected on Ω. If any two nodes in $G(\omega)$ are almost everywhere connected, then $G(\omega)$ is defined as an almost everywhere connected random graph, and G is abbreviatedly called a connected random graph.

Lemma 3.1 (see [3]) *Suppose that* $\{G_t(\cdot) : t \geq 0\}$ *is an independent identically distribution edge-Markovian random graph process which has stationary distribution as its initial distribution in Theorem* 2.1, *where* $V(G_t(\omega)) = \{1, 2, \cdots, N\}$, $t \geq 0$. *Then* $\forall t \geq 0$, *the probability of* $G_t(\cdot)$ *existing a* k*-pathway* $(k \geq 1)$ *in which* i *is the start point is*

$$P(\textit{existing a } k\textit{-pathway in which } i \textit{ is the start point}) = (1 - q^{N-1})^k, \quad k \geq 1.$$

Proof Assume $i = 1$, and suppose that A_{1j} is a random event, nodes 1 and j are connected, $2 \leq j \leq N$. Suppose that $A_j(k)$ denotes a random event: There is a k-pathway in which j is the start point, $1 \leq j \leq N$, $k \geq 1$. Next, we prove the lemma by the mathematical induction method.

When $k = 1$, the conclusion is obviously established. In fact,

$$P(A_1(1)) = P\Big(\bigcup_{j=2}^{N} A_{1j}\Big) = 1 - P\Big(\bigcap_{j=2}^{N} \overline{A}_{ij}\Big) = 1 - q^{N-1}.$$

Suppose that the conclusion is established when $k - 1$, namely,

$$P(A_j(k-1)) = (1 - q^{N-1})^{k-1}, \quad 1 \leq j \leq N.$$

Obviously, $A_1(k) \subseteq A_1(1),\ k \geq 1,\ A_1(1) = \bigcup\limits_{j=2}^{N} A_{1j}$. From the above and the total probability formula, we get

$$\begin{aligned}
P(A_1(k)) &= P(A_1(1) \cap A_1(k)) = P\Big(\bigcap_{j=2}^{N} A_{1j} \cap A_1(k)\Big) = \sum_{j=2}^{N} P(\overline{A}_{12}\overline{A}_{13}\cdots\overline{A}_{1j-1}A_{1j}A_1(k)) \\
&= \sum_{j=2}^{N} P(\overline{A}_{12}\overline{A}_{13}\cdots\overline{A}_{1j-1}A_{1j}) \cdot P(A_1(k) \mid \overline{A}_{12}\overline{A}_{13}\cdots\overline{A}_{1j-1}A_{1j}) \\
&= \sum_{j=2}^{N} pq^{j-2} P(A_j(k-1)) = \sum_{j=2}^{N} pq^{j-2}(1-q^{N-1})^{k-1} \\
&= p(1-q^{N-1})^{k-1}\frac{1-q^{N-1}}{1-q} = (1-q^{N-1})^k.
\end{aligned}$$

So the conclusion is established according to the principle of induction.

This completes the proof.

Lemma 3.2 (see [3]) *Suppose that $\{G_t(\cdot) : t \geq 0\}$ is an independent identically distribution edge-Markovian random graph process which has stationary distribution as its initial distribution in Theorem* 2.1*, where $V(G_t(\omega)) = \{1, 2, \cdots, N\},\ t \geq 0$. For $i, j \in V(G_t(\omega))$, suppose that $A_{ij}(k)$ denotes a random event: There is a k-pathway in which i is the start point, and j is the end point. Then*

$$P(A_{ij}(k)) = pq^{k-1}(1-q^{N-1})^{k-1}.$$

Proof Suppose that $i = 1,\ j = N$, and $A_1(k-1)$ denotes the event: There is a $(k-1)$-pathway in which 1 is the start point. All $(k-1)$-pathways in which 1 is the start point are marked, denoted as $B_1, B_2, \cdots$. Then $A_1(k-1) = \bigcup\limits_{j\geq 1} B_j,\ A_{1N}(k) \subseteq A_1(k-1)$. Apply the assumption and the total probability formula, we can get

$$\begin{aligned}
P(A_{1N}(k)) &= P(A_{1N}(k) \cap A_1(k-1)) = P\Big(A_{1N}(k) \cap \Big(\bigcup_{j\geq 1} B_j\Big)\Big) \\
&= P\Big(A_{1N}(k) \cap \Big(\bigcup_{j\geq 1}\Big(B_j - \bigcup_{i=1}^{j-1} B_i\Big)\Big)\Big) = \sum_{j\geq 1} P\Big(A_{1N}(k) \cap \Big(B_j - \bigcup_{i=1}^{j-1} B_i\Big)\Big) \\
&= \sum_{j\geq 1} P\Big(\Big(B_j - \bigcup_{i=1}^{j-1} B_i\Big)\Big) P\Big(A_{1N}(k) \Big| \Big(B_j - \bigcup_{i=1}^{j-1} B_i\Big)\Big) = \sum_{j\geq 1} P\Big(B_j - \bigcup_{i=1}^{j-1} B_i\Big) \cdot q^{k-1}p \\
&= pq^{k-1}P(A_1(k-1)) = pq^{k-1}(1-q^{N-1})^{k-1}.
\end{aligned}$$

This completes the proof.

Theorem 3.1 *Suppose that $\{G_t(\cdot) : t \geq 0\}$ is an independent identically distribution edge-Markovian random graph process which has stationary distribution as its initial distribution in Theorem* 2.1*, where $V(G_t(\omega)) = \{1, 2, \cdots, N\},\ t \geq 0$. $\forall i, j \in V(G_t(\omega))$, C_{ij} denotes the event: Nodes i and j are connected. Then*

$$P(C_{ij}) = \frac{p}{p+q^N}.$$

Proof Suppose that ς_{ij} denotes the length of the pathway ($+\infty$ can be obtained) in which i is the start point, and j is the end point. Then ς_{ij} is a random variable which can obtain the positive integer value or $+\infty$. From Lemma 3.2, we have

$$P(C_{ij}) = P(\varsigma_{ij} < +\infty) = \sum_{k=1}^{\infty} P(\varsigma_{ij} = k) = \sum_{k=1}^{\infty} pq^{k-1}(1-q^{N-1})^{k-1}$$
$$= \sum_{k=1}^{\infty} p[q(1-q^{N-1})]^{k-1} = \frac{p}{1-q(1-q^{N-1})} = \frac{p}{p+q^N}.$$

This completes the proof.

Theorem 3.2 *Suppose that $\{G_t(\cdot) : t \geq 0\}$ is a random graph process in Theorem 2.1, and C denotes the random event: $G_t(\omega)$ is connected. Then*

(1) $P(C) \geq 1 - \frac{N(N-1)q^N}{2(p+q^N)}$.

(2) *When $N \to +\infty$, $G_t(\omega)$ is almost everywhere connected.*

Proof

$$P(C) = P\Big(\bigcap_{i<j} C_{ij}\Big) = 1 - P\Big(\bigcup_{i<j} \overline{C}_{ij}\Big) \geq 1 - \sum_{i<j} P(\overline{C}_{ij})$$
$$= 1 - \sum_{i<j}\Big(1 - \frac{p}{p+q^N}\Big) = 1 - \sum_{i<j} \frac{q^N}{p+q^N} = 1 - \frac{N(N-1)q^N}{2(p+q^N)}.$$

Obviously,

$$P(C) = \lim_{N \to +\infty} \Big(1 - \frac{N(N-1)q^N}{2(p+q^N)}\Big) = 1.$$

This completes the proof.

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