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A Note on Model (Co)slice Categories*

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Abstract There are various adjunctions between model (co)slice categories. The author gives a proposition to characterize when these adjunctions are Quillen equivalences. As an application, a triangle equivalence between the stable category of a Frobenius category and the homotopy category of a non-pointed model category is given.

Keywords Model slice categories, Homotopy categories, Quillen equivalences
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1 Introduction

Given a category $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$, one can construct adjunctions between coslice categories $(X \downarrow \mathcal{C})$ and $(Y \downarrow \mathcal{C})$, and between slice categories $(\mathcal{C} \downarrow X)$ and $(\mathcal{C} \downarrow Y)$. If $\mathcal{C}$ is a closed model category, all these (co)slice categories inherit a model structure from $\mathcal{C}$. In this case, the adjunctions between these (co)slice categories are Quillen adjunctions. Meanwhile, if we start from a Quillen adjunction between two closed model categories $\mathcal{C}$ and $\mathcal{D}$, we can also construct Quillen adjunctions between some (co)slice categories. This note is aimed to characterize when these adjunctions are Quillen equivalences.

As an application of these characterizations, we construct a triangle equivalence between the stable category of a weakly idempotent complete Frobenius category and the homotopy category of its coslice category (see Corollary 3.3). As a byproduct, we get a non-pointed model category whose homotopy category is a triangulated category (see Theorem 3.1). This shows that the pointed condition of Quillen's theorem (see [8, Chapter I, Section 2, Theorem 2]) is not always necessary.

The contents of the note are as follows. In Subsection 2.1, we recall the standard material of Quillen equivalences. In Subsections 2.2–2.3, we recall the notion of model (co)slice categories and construct six Quillen adjunctions between model (co)slice categories (see Proposition 2.2). In Section 3, we give a proposition to distinguish when these Quillen adjunctions are Quillen equivalences (see Proposition 3.1). We end up this note by proving Theorem 3.1.

2 Preliminaries of (Co)slice Categories and Quillen Equivalences

In this section we recall the notions of Quillen equivalences and model (co)slice categories. Our main references about model categories are [8, 1, 5, 4].

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2.1 Quillen equivalences

Recall that a category $\mathcal{C}$ is said to be bicomplete if it contains small limits and colimits. A closed model category $\mathcal{C}$ is a bicomplete category in which there are three classes of morphisms, called cofibrations, fibrations and weak equivalences and they satisfy some axioms. We will denote the three classes of morphisms by $\mathrm{Cof}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{We}(\mathcal{C})$, respectively. A morphism which is both a (co)fibration and a weak equivalence is called an acyclic (co)fibration. An object $X \in \mathcal{C}$ is called cofibrant if $\emptyset \to A \in \mathrm{Cof}(\mathcal{C})$ and fibrant if $X \to * \in \mathrm{Fib}(\mathcal{C})$, where $\emptyset$ is the initial object of $\mathcal{C}$ and $*$ the terminal object of $\mathcal{C}$. We use $\mathcal{C}_c$ and $\mathcal{C}_f$ to denote the classes of cofibrant and fibrant objects, respectively. An object in $\mathcal{C}_{cf} := \mathcal{C}_c \cap \mathcal{C}_f$ is called bifibrant.

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are closed model categories. An adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is called a Quillen adjunction if F preserves cofibrations and acyclic cofibrations, or equivalently, G preserves fibrations and acyclic fibrations (see [5, Definition 1.3.1, Lemma 1.3.4]). Sometimes we will call F a left Quillen functor and G a right Quillen functor.

Definition 2.1 (see [5, Definition 1.3.12]) *A Quillen adjunction $(F, G; \varphi) : \mathcal{C} \to \mathcal{D}$ is called a Quillen equivalence if for all $X \in \mathcal{C}_c$ and $Y \in \mathcal{D}_f$, a morphism $f : F(X) \to Y \in \mathrm{We}(\mathcal{D})$ if and only if $\varphi(f) : X \to G(Y) \in \mathrm{We}(\mathcal{C})$.*

If $(F, G) : \mathcal{C} \to \mathcal{D}$ is a Quillen equivalence, then the left derived functor $\mathbb{L}F$ and the right derived functor $\mathbb{R}G$ exist (see [8, Chapter I, Section 4]). Furthermore, they induce an equivalent adjunction $(\mathbb{L}F, \mathbb{R}L) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ between the homotopy categories (see [8, Chapter I, Theorem 3]).

In a model category $\mathcal{C}$, we use $p_X : Q(X) \to X$ to denote the cofibrant approximation of an object X (it is an acyclic fibration) and $r_X : X \to R(X)$ the fibrant approximation of X (it is an acyclic cofibration), respectively (see [8, Chapter I, Section 1] or [1, Section 5]). The following is the most useful criterion for checking when a given Quillen adjunction is a Quillen equivalence.

Proposition 2.1 (see [5, Definition 1.3.13, Corollary 1.3.16]) *Suppose that $(F, G, \varphi; \eta, \varepsilon)$: $\mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

(i) *(F, G, φ) is a Quillen equivalence.*

(ii) *$(\mathbb{L}F, \mathbb{R}G) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is an adjoint equivalence of categories.*

(iii) *If $F(f)$ is a weak equivalence for a morphism f in $\mathcal{C}_c$, so is f. And the morphism $F(Q(G(Y))) \xrightarrow{F(p_{G(Y)})} FG(Y) \xrightarrow{\varepsilon_Y} Y$ is a weak equivalence for every $Y \in \mathcal{D}_f$.*

(iv) *If $G(g)$ is a weak equivalence for a morphism g in $\mathcal{D}_f$, so is g. And the morphism $X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(r_{F(X)})} G(R(F(X)))$ is a weak equivalence for every $X \in \mathcal{C}_c$.*

2.2 The model (co)slice categories

Definition 2.2 *Let $\mathcal{C}$ be a category. For an object X in $\mathcal{C}$, the coslice category $(X \downarrow \mathcal{C})$ is the category in which an object is a morphism $X \xrightarrow{u} C$ in $\mathcal{C}$, and a morphism from $X \xrightarrow{u} C$ to $X \xrightarrow{u'} C'$ is a morphism $\alpha : C \to C'$ in $\mathcal{C}$ such that $u' = \alpha \circ u$. The composition of morphisms is defined by the composition of morphisms in $\mathcal{C}$.*

Dually, we define the slice category $(\mathcal{C} \downarrow X)$ over X.

Now let $\mathcal{C}$ be a closed model category. If we define a morphism in $(X \downarrow \mathcal{C})$ and $(\mathcal{C} \downarrow X)$ is a weak equivalence, cofibration, or fibration if it is one in $\mathcal{C}$, then both the coslice category $(X \downarrow \mathcal{C})$ and the slice category $(\mathcal{C} \downarrow X)$ become closed model categories (see [4, Theorem 7.6.5]). In this case, we call $(X \downarrow \mathcal{C})$ a model coslice category and $(\mathcal{C} \downarrow X)$ a model slice category.

Lemma 2.1 *Let $\mathcal{C}$ be a model category. Then*
(i) $(X \downarrow \mathcal{C})_c = \{u \in (X \downarrow \mathcal{C}) \mid u \in \mathrm{Cof}(\mathcal{C})\}$ *and* $(X \downarrow \mathcal{C})_f = \{X \xrightarrow{u} C \in (X \downarrow \mathcal{C}) \mid C \in \mathcal{C}_f\}$.
(ii) $(\mathcal{C} \downarrow X)_c = \{C \xrightarrow{u} X \in (\mathcal{C} \downarrow X) \mid C \in \mathcal{C}_c\}$ *and* $(\mathcal{C} \downarrow X)_f = \{u \in (\mathcal{C} \downarrow X) \mid u \in \mathcal{F}ib(\mathcal{C})\}$.

Proof (i) Note that, the initial object in $(X \downarrow \mathcal{C})$ is $X \xrightarrow{1_X} X$ and the terminal object is $X \to *$. From these, the statement (i) can be verified directly. Dually, we can prove (ii).

2.3 Quillen adjunctions between model (co)slice categories

Let $\mathcal{C}$ be a bicomplete category and $g : X \to Y$ be a morphism in $\mathcal{C}$. There are two adjunctions:

(1)
$$g_! : (X \downarrow \mathcal{C}) \rightleftarrows (Y \downarrow \mathcal{C}) : g^*,$$

where g^* takes the object $v : Y \to D$ to the composition $vg : X \to D$ and $h : v \to v'$ to h, $g_!$ sends $u : X \to C$ to its cobase change along g and $f : u \to u'$ to the induced morphism $Y \amalg_X C \to Y \amalg_X C'$ by the universal property of pushouts and they fit into the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "u"'] \arrow[rr, bend left, "u'"] \arrow[d, "g"'] & C \arrow[r, "f"'] \arrow[d, "s_u"] & C' \arrow[d] \\
Y \arrow[r] \arrow[rr, bend right] & Y \amalg_X C \arrow[r] & Y \amalg_X C'
\end{tikzcd}$$

with the three squares being pushouts. The unit of this adjunction is $s_u : u \to g^* g_!(u)$ for any $u \in (X \downarrow \mathcal{C})$, where s_u is defined in the above diagram.

(2)
$$g_* : (\mathcal{C} \downarrow X) \rightleftarrows (\mathcal{C} \downarrow Y) : g^!,$$

where the constructions of the functors g_* and $g^!$ are dual to (1). In fact, g_* takes the object $u : C \to X$ to the composition $gu : C \to Y$ and $f : u \to u'$ to f, and $g^!$ takes the object $v : D \to Y$ to its base change along g and $h : v' \to v$ to the induced morphism $D' \times_Y X \to D \times_Y X$ by the universal property of pullbacks and they fit into the following commutative diagram:

$$\begin{tikzcd}
D' \times_Y X \arrow[r] \arrow[rr, bend left] \arrow[d] & D \times_Y X \arrow[r] \arrow[d, "t_v"] & X \arrow[d, "g"] \\
D' \arrow[r, "h"] \arrow[rr, bend right, "v'"'] & D \arrow[r, "v"] & Y
\end{tikzcd}$$

with the three squares being pullbacks. The counit of this adjunction is $t_v : g_* g^!(v) \to v$ for any $v \in (\mathcal{C} \downarrow Y)$, where t_v is defined in the above diagram.

In particular, if we take $X = \emptyset$ to be the initial object of $\mathcal{C}$, then g^* is just the forgetful functor from $(Y \downarrow \mathcal{C})$ to $\mathcal{C} = (\emptyset \downarrow \mathcal{C})$, and if we take $Y = *$ to be the terminal object of $\mathcal{C}$, then g_* is just the forgetful functor from $(\mathcal{C} \downarrow X)$ to $\mathcal{C} = (\mathcal{C} \downarrow *)$.

If we already have an adjunction $(S, U; \varphi, \eta, \varepsilon) : \mathcal{C} \to \mathcal{D}$ between the bicomplete categories $\mathcal{C}$ and $\mathcal{D}$, then for any object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, (S, U) induces the following four adjunctions.

(3)
$$\overline{S}_X : (X \downarrow \mathcal{C}) \rightleftarrows (S(X) \downarrow \mathcal{D}) : \overline{U}_X,$$

where $\overline{S}_X$ sends $u: X \to C$ to $S(u)$ and $f: u \to u'$ to $S(f)$, while $\overline{U}_X$ sends $v: S(X) \to D$ to the composition $U(v)\eta_X : X \to U(D)$ and $h: v \to v'$ to $U(h)$. The unit of this adjunction is $\eta_C : u \to \overline{U}_X\overline{S}_X(u)$ for any $u: X \to C$ in $(X \downarrow \mathcal{C})$.

(4)
$$\underline{S}_X : (\mathcal{C} \downarrow X) \rightleftarrows (\mathcal{D} \downarrow S(X)) : \underline{U}_X,$$

where $\underline{S}_X$ sends $u: C \to X$ to $S(u): S(C) \to S(X)$ and $f: u \to u'$ to $S(f)$, $\underline{U}_X$ sends $v: D \to S(X)$ to the base change of $U(v)$ along the unit η_X and $h: v \to v'$ to the induced morphism from $U(D') \times_{US(X)} X \to U(D) \times_{US(X)} X$ by the universal property of pullbacks and they fit into the following commutative diagram:

$$\begin{tikzcd}
U(D') \times_{US(X)} X \arrow[r] \arrow[d] \arrow[rr, bend left] & U(D) \times_{US(X)} X \arrow[r] \arrow[d, "l_v"] & X \arrow[d, "\eta_X"] \\
U(D') \arrow[r, "U(h)"] \arrow[rr, bend right, "U(v')"'] & U(D) \arrow[r, "U(v)"] & US(X)
\end{tikzcd}$$

with the three squares being pullbacks. The counit of this adjunction is $\varepsilon_D S(l_v) : \underline{S}_X\underline{U}_X(v) \to v$ for any $v: D \to S(X) \in (\mathcal{D} \downarrow S(X))$, where l_v is defined in the above diagram.

(5)
$$\widetilde{S}_Y : (\mathcal{C} \downarrow U(Y)) \rightleftarrows (\mathcal{D} \downarrow Y) : \widetilde{U}_Y,$$

where the constructions of $\widetilde{S}_Y$ and $\widetilde{U}_Y$ are dual to (3). In fact, $\widetilde{S}_Y$ sends $u: C \to U(Y)$ to the composition $\varepsilon_Y S(u): S(C) \to Y$ and $f: u \to u'$ to $S(f)$, while $\widetilde{U}_Y$ sends v to $U(v)$ and $h: v \to v'$ to $U(h)$. The counit of this adjunction is $\varepsilon_D : \widetilde{S}_Y\widetilde{U}_Y(v) \to v$ for any $v: D \to Y$ in $(\mathcal{D} \downarrow Y)$.

(6)
$$\widetilde{S}'_Y : (U(Y) \downarrow \mathcal{C}) \rightleftarrows (Y \downarrow \mathcal{D}) : \widetilde{U}'_Y,$$

where the constructions of $\widetilde{S}'_Y$ and $\widetilde{U}'_Y$ are dual to (4). In fact, $\widetilde{U}'_Y$ sends $v: Y \to D$ to $U(v)$ and $h: v \to v'$ to $U(h)$, $\widetilde{S}'_Y$ sends $u: U(Y) \to C$ to the cobase change of $S(u)$ along the counit ε_Y and $f: u \to u'$ to the induced morphism from $Y \amalg_{SU(Y)} S(C) \to Y \amalg_{SU(Y)} S(C')$ by the universal property of pushouts and they fit into the following commutative diagram:

$$\begin{tikzcd}
SU(Y) \arrow[r, "S(u)"'] \arrow[d, "\varepsilon_Y"'] \arrow[rr, bend left, "S(u')"] & S(C) \arrow[r, "S(f)"'] \arrow[d, "m_u"'] & S(C') \arrow[d] \\
Y \arrow[r] \arrow[rr, bend right] & Y \amalg_{SU(Y)} S(C) \arrow[r] & Y \amalg_{SU(U)} S(C')
\end{tikzcd}$$

with the three squares being pushouts. The unit of this adjunction is $U(m_u)\eta_C : u \to \widetilde{U}'_Y\widetilde{S}'_Y(u)$ for any $u: U(Y) \to C$ in $(U(Y) \downarrow \mathcal{C})$, where m_u is defined in the above diagram.

We have the following proposition about these adjunctions, in which the first statement in some special case has been discussed in [5, Chapter 1, Section 3] and [7, Chapter 16, Section 2].

Proposition 2.2 (i) *Let $\mathcal{C}$ be a closed model category and $g: X \to Y$ be a morphism in $\mathcal{C}$. Then the adjunctions* (1) *and* (2) *are Quillen adjunctions.*

(ii) *Let $\mathcal{C}$ and $\mathcal{D}$ be two closed model categories. Assume that* $(S, U; \varphi, \eta, \varepsilon) : \mathcal{C} \to \mathcal{D}$ *is a Quillen adjunction and* $X \in \mathcal{C}$, $Y \in \mathcal{D}$. *Then the adjunctions* (3)–(6) *are Quillen adjunctions. Moreover we have*

$$(\underline{S}_X, \underline{U}_X) = (\widetilde{S}_{S(X)}, \widetilde{U}_{S(X)}) \circ ((\eta_X)_*, \eta_X^!)$$

and

$$(\widetilde{S}'_Y, \widetilde{U}'_Y) = ((\varepsilon_Y)_!, (\varepsilon_Y)^*) \circ (\overline{S}_{U(Y)}, \overline{U}_{U(Y)}),$$

where the compsoitions of the adjunctions are in the sense of Theorem IV.8.1 *of* [6].

Proof These statements can be proved directly. We leave the details to the interested reader. For example, since $g^*(h) = h$ for any morphism $h : v \to v'$ in $(Y \downarrow \mathcal{C})$ and h is a fibration or an acyclic fibration in $(Y \downarrow \mathcal{C})$ if and only if it is one in $\mathcal{C}$, g^* preserves fibrations and acyclic fibrations. Therefore the adjunction (1) is a Quillen adjunction.

Assume now that $\mathcal{C}$ and $\mathcal{D}$ are closed model categories and denote by $\mathcal{C}_* = (* \downarrow \mathcal{C})$ and $\mathcal{D}_* = (* \downarrow \mathcal{D})$ the slice categories of $\mathcal{C}$ and $\mathcal{D}$ induced by the terminal object $*$, respectively. If $(S, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction, M. Hovey defined a functor U_* from $\mathcal{D}_* \to \mathcal{C}_*$ by mapping object $* \xrightarrow{v} D$ to $U(*) = * \xrightarrow{U(v)} U(X)$ in [5]. This functor is a right Quillen functor as proved in [5]. Note that if we denote the morphism $S(*) \to *$ as g in $\mathcal{D}_*$, then U_* is the composition of the functors $\mathcal{D}_* \xrightarrow{g^*} (S(*) \downarrow \mathcal{D}) \xrightarrow{\overline{U}_*} \mathcal{C}_*$. So by Proposition 2.2, this functor has a left adjoint $S_* = \overline{S}_* \circ g_!$, and then we get Proposition 1.3.5 of [5] directly.

Corollary 2.1 (see [5, Proposition 1.3.5]) *A Quillen adjunction* $(S, U) : \mathcal{C} \to \mathcal{D}$ *induces a Quillen adjunction* $(S_*, U_*) : \mathcal{C}_* \to \mathcal{D}_*$.

3 Main Results

Proposition 3.1 (a) *Let $\mathcal{C}$ be a closed model category and $g : X \to Y$ be a morphism in $\mathcal{C}$. Then*

(i) *the adjunction* $(g_!, g^*) : (X \downarrow \mathcal{C}) \to (Y \downarrow \mathcal{C})$ *is a Quillen equivalence if and only if the cobase change of g along u for each cofibration $u : X \to C$ with $C \in \mathcal{C}$ is a weak equivalence;*

(ii) *dually, the adjunction* $(g_*, g^!) : (\mathcal{C} \downarrow X) \to (\mathcal{C} \downarrow Y)$ *is a Quillen equivalence if and only if the base change of g along v for each fibration $v : C \to Y$ with $Y \in \mathcal{C}$ is a weak equivalence.*

(b) *Let $\mathcal{C}$ and $\mathcal{D}$ be closed model categories and* $(S, U; \varphi, \eta, \varepsilon) : \mathcal{C} \to \mathcal{D}$ *be a Quillen equivalence. Then given any object $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,*

(i) *if $X \in \mathcal{C}_c$, the adjunction* $(\overline{S}_X, \overline{U}_X) : (X \downarrow \mathcal{C}) \to (S(X) \downarrow \mathcal{D})$ *is a Quillen equivalence;*

(ii) *if $Y \in \mathcal{D}_f$, the adjunction* $(\widetilde{S}_Y, \widetilde{U}_Y) : (\mathcal{C} \downarrow U(Y)) \to (\mathcal{D} \downarrow Y)$ *is a Quillen equivalence;*

(iii) *if $S(X) \in \mathcal{D}_f$, the adjunction* $(\underline{S}_X, \underline{U}_X) : (\mathcal{C} \downarrow X) \to (\mathcal{D} \downarrow S(X))$ *is a Quillen equivalence if and only if the base change of η_X along v for each fibration $v : D \to US(X)$ with $D \in \mathcal{D}$ is a weak equivalence;*

(iv) *if $U(Y) \in \mathcal{C}_c$, the adjunction* $(\widetilde{S}'_Y, \widetilde{U}'_Y) : (U(Y) \downarrow \mathcal{C}) \to (Y \downarrow \mathcal{D})$ *is a Quillen equivalence if and only if the cobase change of ε_Y along u for each cofibration $u : SU(Y) \to D$ with $D \in \mathcal{D}$ is a weak equivalence.*

Proof By Proposition 2.2, all the adjunctions in the statements are Quillen adjunctions. We only prove the assertions (i) of (a), (i) and (iii) of (b). The others can be proved dually.

The proof of (a)(i). On the one hand, since $g^*(h) = h$ for any morphism $h : v \to v'$ in $(Y \downarrow \mathcal{C})$, if $g^*(h)$ is a weak equivalence, so is h. On the other hand, for any $u : X \to C$ in $(X \downarrow \mathcal{C})_c = \{u \in (X \downarrow \mathcal{C}) \mid u \in \mathrm{Cof}(\mathcal{C})\}$ (see Lemma 2.1), the unit of the adjunction $(g_!, g^*)$ is

$s_u : C \to Y \amalg_X C$ (see the construction of this adjunction) which is the cobase change of g along u. So by (iv) of Proposition 2.2, $(g_!, g^*)$ is a Quillen equivalence if and only if the composite

$$C \xrightarrow{s_u} Y \amalg_X C \xrightarrow{r_{Y\amalg_X C}} R(Y \amalg_X C)$$

is a weak equivalence. Since $r_{Y\amalg_X C}$ is a weak equivalence, by the 2-out-of-3 axiom of weak equivalences, we know that $(g_!, g^*)$ is a Quillen equivalence if and only if s_u is a weak equivalence.

The proof of (b)(i). If $h : v \to v'$ is in $(S(X) \downarrow \mathcal{D})_f = \{S(X) \xrightarrow{v} D \in (S(X) \downarrow \mathcal{D}) \mid D \in \mathcal{D}_f\}$, then h is a morphism in $\mathcal{D}_f$, and $\overline{U}_X(h) = U(h)$. Recall that the unit of the adjunction $(\overline{S}_X, \overline{U}_X)$ is $\eta_C : C \to US(C)$ for any $u : X \to C$ (see the construction of this adjunction). If u is in $(X \downarrow \mathcal{C})_c = \{u \in (X \downarrow \mathcal{C}) \mid u \in \mathrm{Cof}(\mathcal{C})\}$, since $X \in \mathcal{C}_c$, we know that $C \in \mathcal{C}_c$. Thus by (iv) of Lemma 2.2, $(\overline{S}_X, \overline{U}_X)$ is a Quillen equivalence if and only if when $\overline{U}_X(h) = U(h)$ is a weakly equivalence, so is h and the composition $C \xrightarrow{\eta_C} US(C) \xrightarrow{U(r_{S(C)})} U(R(S(C)))$ is a weak equivalence. But again by (iv) of Lemma 2.2, this holds since(S, U) is a Quillen equivalence.

The proof of (b)(iii). By assumption and (ii) of (b), we know that $(\widetilde{S}_{S(X)}, \widetilde{U}_{S(X)})$ is a Quillen equivalence. Since $\underline{S}_X = \widetilde{S}_{S(X)}(\eta_X)_*$ by (ii) of Proposition 2.2, we know that $(\underline{S}_X, \underline{U}_X)$ is a Quillen equivalence if and only if $((\eta_X)_*, \eta_X^!)$ is a Quillen equivalence by Corollary 1.3.15 of [5]. Then the assertion follows from (ii) of (a).

Corollary 3.1 (see [5, Proposition 1.3.7]) *Suppose that $(S, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen equivalence, and suppose in addition that the terminal object $* \in \mathcal{C}_c$ and that S preserves the terminal object. Then $(S_*, U_*) : \mathcal{C}_* \to \mathcal{D}_*$ constructed as in Corollary* 2.1 *is a Quillen equivalence.*

Proof In this case, $(S_*, U_*) = (\overline{S}_*, \overline{U}_*)$ is a Quillen equivalence by Proposition 3.1.

Recall that a closed model category $\mathcal{C}$ is called left proper if every cobase change of a weak equivalence along a cofibration is a weak equivalence. Dually, $\mathcal{C}$ is called right proper if every base change of a weak equivalence along a fibration is a weak equivalence. By Proposition 3.1, we can redescribe the left or right properness of a model category by Quillen equivalences.

Corollary 3.2 [1] (1) *A closed model category $\mathcal{C}$ is left proper if and only if $(g_!, g^*)$ is a Quillen equivalence for every weak equivalence $g : X \to Y$.*

(2) *A closed model category $\mathcal{C}$ is right proper if and only if $(g_*, g^!)$ is a Quillen equivalence for every weak equivalence $g : X \to Y$.*

Remark 3.1 In general, if g is not a weak equivalence, and even $\mathcal{C}$ is proper, $(g_!, g^*)$ is not necessarily a Quillen equivalence. For example, take $\mathcal{C} = \mathrm{mod}k[x]/(x^2)$ where k is a field. Then $\mathcal{C}$ is a proper model category in which weak equivalences are stable equivalences and every object is bifibrant, and its homotopy category is the stable category $\underline{\mathrm{mod}}k[x]/(x^2)$. Take $g = 0 : 0 \to k$, and then $\eta = \binom{1}{0}$ is the unit of the adjunction $(g_!, g^*)$. In this case, every object is fibrant, the morphisms in (iv) of Proposition 2.2 are just η_C for any $C \in \mathcal{C}$. But $\eta_k : k \to k \oplus k$ is in no way a weak equivalence. So the Quillen adjunction $(g_!, g^*) : \mathcal{C} \to (k \downarrow \mathcal{C})$ is not a Quillen equivalence.

If $\mathcal{F}$ is a weakly idempotent complete Frobenius category, then $\mathcal{F}$ has a canonical model structure in which the cofibrations are the monomorphisms, the fibrations are the epimorphisms and the weak equivalences are the stable equivalences (see [2, Theorem 3.3]). Let A be any

[1] This should be the right version of Proposition 16.2.4 of [7], and there the authors claimed that a closed model category $\mathcal{C}$ is left proper or right proper if and only if $(g_!, g^*)$ or $(g_*, g^!)$ is Quillen equivalence for a given morphism g. For a counter example see Remark 3.1.

nonzero projective-injective object in $\mathcal{F}$. Take $g = 0 : 0 \to A$, and then $0_!(0 \to C) = C \overset{\binom{0}{1}}{\to} A \oplus C$ is a weak equivalence for any $C \in \mathcal{F}$. By Proposition 3.1, we have a Quillen equivalence $(0_!, 0^*) : (0 \downarrow \mathcal{F}) = \mathcal{F} \to (A \downarrow \mathcal{F})$. So the derived functors of $0_!$ and 0^* are equivalences of the homotopy categories between $\mathrm{Ho}(\mathcal{F})$ and $\mathrm{Ho}(A \downarrow \mathcal{F})$. Note that in this case, the homotopy category $\mathrm{Ho}(\mathcal{F})$ is just the stable category $\underline{\mathcal{F}}$ (see [3, Chapter I, Section 2.2]) which is a triangulated category by Theorem 2.6 of [3]. If we can show that the homotopy category $\mathrm{Ho}(A \downarrow \mathcal{F})$ is a triangulated category, then the derived adjunction $(\mathbb{L}0_!, \mathbb{R}0^*) : \underline{\mathcal{F}} \to \mathrm{Ho}(A \downarrow \mathcal{F})$ will be a triangle equivalence since Quillen equivalences are automatically triangle equivalences if the corresponding homotopy categories are triangulated categories (see [8, Chapter I, Theorem 3]).

Next we will show that the homotopy category $\mathrm{Ho}(A \downarrow \mathcal{F})$ is a triangulated category. And then we give the promised example as advertised in the introduction since the coslice category $(A \downarrow \mathcal{F})$ is not pointed by noting that its initial object is $A \overset{1_A}{\to} A$ and its terminal object is $A \to 0$.

Theorem 3.1 *The homotopy category* $\mathrm{Ho}(A \downarrow \mathcal{F})$ *is a triangulated category.*

Proof Since $\mathcal{F}_{cf} = \mathcal{F}$ and A is injective, we know that $(A \downarrow \mathcal{F})_f = (A \downarrow \mathcal{F})$ and cofibrant objects in $(A \downarrow \mathcal{F})$ are split monomorphisms in $\mathcal{F}$ with domain A. So for any object $u \in (A \downarrow \mathcal{F})_c$, we may write u as $A \overset{\binom{1}{0}}{\to} A \oplus C$ up to isomorphism. The morphisms from $u = A \overset{\binom{1}{0}}{\to} A \oplus C$ to $v = A \overset{\binom{1}{0}}{\to} A \oplus D$ are of the form $\left(\begin{smallmatrix} 1 & r \\ 0 & s \end{smallmatrix}\right)$. By Quillen's homotopy category theorem (see [8, Theorem I.1]), we know that the homotopy category $\mathrm{Ho}(A \downarrow \mathcal{F})$ can be realized as the quotient category $(A \downarrow \mathcal{F})_c / \sim$, where $\sim$ is the homotopy relation. For details, we refer the reader to Section 1 of Chapter I of [8] or Sections 4–5 of [1].

Let $(A \downarrow \mathcal{F})_c^0$ be the subcategory of $(A \downarrow \mathcal{F})_c$ consisting of the objects of the form $A \overset{\binom{1}{0}}{\to} A \oplus C$ and morphisms from $u = A \overset{\binom{1}{0}}{\to} A \oplus C$ to $v = A \overset{\binom{1}{0}}{\to} A \oplus D$ be of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$. The subcategory $(A \downarrow \mathcal{F})_c^0$ has zero object $A \overset{1_A}{\to} A$.

We claim that the inclusion $(A \downarrow \mathcal{F})_c^0 \hookrightarrow (A \downarrow \mathcal{F})_c$ induces an equivalence of quotient categories $(A \downarrow \mathcal{F})_c^0 / \sim \simeq (A \downarrow \mathcal{F})_c / \sim$. Firstly, note that $A \overset{\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)}{\to} A \oplus C \oplus I(C)$ is a very good cylinder object of $u = A \overset{\binom{1}{0}}{\to} A \oplus C$:

$$
\begin{tikzcd}
 & A \arrow[dl, bend right, "\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)"] \arrow[dr, bend left, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \\
A \oplus C \oplus C \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & 0 \end{smallmatrix}\right)"] & A \oplus C \oplus I(C) \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)"] & A \oplus C
\end{tikzcd}
$$

where $C \overset{i}{\to} I(C)$ is an injective preenvelope of C in $\mathcal{F}$. Given any morphism $\left(\begin{smallmatrix} 1 & r \\ 0 & s \end{smallmatrix}\right) : u = A \overset{\binom{1}{0}}{\to} A \oplus C \to v = A \overset{\binom{1}{0}}{\to} A \oplus D$, there is a morphism $r' : I(C) \to A$ such that $r'i = r$ since A is injective. Then $\left(\begin{smallmatrix} 1 & 0 & r' \\ 0 & s & 0 \end{smallmatrix}\right)$ is a cylinder homotopy from $\left(\begin{smallmatrix} 1 & r \\ 0 & s \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$ in $(A \downarrow \mathcal{F})_c$. That is $\left(\begin{smallmatrix} 1 & r \\ 0 & s \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$ in $(A \downarrow \mathcal{F})_c$. Meanwhile, it is easy to prove that given any two morphisms $\left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right) : u \to v$ in $(A \downarrow \mathcal{F})_c^0$, they are cylinder homotopic if and only if they are cylinder homotopic in $(A \downarrow \mathcal{F})_c$. So we have $(A \downarrow \mathcal{F})_c / \sim \simeq (A \downarrow \mathcal{F})_c^0 / \sim$. In particular, $\mathrm{Ho}(A \downarrow \mathcal{F}) \simeq \mathrm{Ho}((A \downarrow \mathcal{F})_c^0)$.

We can use Quillen's Theorem I.2 of [8] to the homotopy category $\mathrm{Ho}((A \downarrow \mathcal{F})_c^0)$. Recall that $A \overset{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)}{\to} A \oplus C \oplus I(C)$ is a very good cylinder object of $u = A \overset{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}{\to} A \oplus C$, where $I(C)$ is an injective preenvelope of C. By the construction of the suspension functor of the homotopy category $\mathrm{Ho}(A \downarrow \mathcal{F})$ (see [8, The proof of Theorem 2 in Chapter I]), we may define $\Sigma(u) = A \overset{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}{\to} A \oplus \Sigma^{\mathcal{F}}(C)$, where $\Sigma^{\mathcal{F}}$ is the suspension functor of the stable category $\underline{\mathcal{F}}$ which is an automorphism of $\underline{\mathcal{F}}$ (see [3, Chapter I, Proposition 2.2]). Then it can be verified directly that the suspension functor Σ defined above on the homotopy category $\mathrm{Ho}((A \downarrow \mathcal{F})_c^0)$ is an auto-equivalence and thus $\mathrm{Ho}(A \downarrow \mathcal{F})$ is a triangulated categories by Propositions 5–6 in Section I.3 of [8].

Corollary 3.3 *The derived functor* $\mathbb{L}0_! : \underline{\mathcal{F}} \to \mathrm{Ho}(A \downarrow \mathcal{F})$ *is a triangle equivalence with quasi-inverse* $\mathbb{R}0^*$*.*

Proof By construction, $(0_!, 0^*) : \mathcal{F} \to (A \downarrow \mathcal{F})$ is a Quillen equivalence. Then the derived adjunction $(\mathbb{L}0_!, \mathbb{R}0^*)$ is an equivalent adjunction and $\mathbb{L}0_!$ is a triangle equivalence by Theorem I.3 of [8].

Remark 3.2 Dually, we can construct a slice category $(\mathcal{F} \downarrow A)$ for a nonzero projective-injective object A, and there is a triangle equivalence $(\mathbb{L}0_*, \mathbb{R}0^!) : \underline{\mathcal{F}} \to \mathrm{Ho}(\mathcal{F} \downarrow A)$.

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