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Characterization of Groups $L_2(q)$ by NSE Where $q \in \{17, 27, 29\}$*

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Abstract The authors show that linear simple groups $L_2(q)$ with $q \in \{17, 27, 29\}$ can be uniquely determined by $\mathrm{nse}(L_2(q))$, which is the set of numbers of elements with the same order.

Keywords Finite groups, Orders, Simple groups, Linear groups
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1 Introduction

Throughout this paper, all groups are finite and G is always a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, and by $\pi_e(G)$ the set of element orders of G. If r is a prime divisor of the order of G, then P_r denotes a Sylow r-subgroup of G and $n_r(G)$ denotes the number of Sylow r-subgroups of G. Let n be an integer. We denote by $\varphi(n)$ the Euler function of n. G is called a simple K_n-group if G is simple such that $|\pi(G)| = n$.

The prime graph $\mathrm{GK}(G)$ of a group G is defined as a graph with the vertex set $\pi(G)$. Two distinct primes $p, q \in \pi(G)$ are adjacent if G contains an element of order pq. Moreover, the connected components of $\mathrm{GK}(G)$ are denoted by $\pi_i, 1 \le i \le t(G)$, where $t(G)$ is the number of connected components of G. In particular, we define by π_1 the component containing the prime 2 for a group of even order.

The motivation of this article is to investigate Thompson's Problem as follows (see [1, Problem 12.37]).

Write $M_t(G) := \{g \in G \mid g^t = 1\}$. G_1 and G_2 are of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$, $t = 1, 2, \cdots$.

Thompson's Problem *Suppose that G_1 and G_2 are of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable*?

Unfortunately, so far, no one could prove it completely, or even give a counterexample.

Let $k \in \pi_e(G)$ and $m_k(G)$ be the number of elements of order k in G. Let $\mathrm{nse}(G) := \{m_k(G) \mid k \in \pi_e(G)\}$, the set of numbers of elements with the same order. If groups G_1 and G_2 are of the same order type, we clearly see that $|G_1| = |G_2|$ and $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$. So it is natural to investigate the Thompson's Problem by $|G|$ and $\mathrm{nse}(G)$. Notice that not all groups

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can be characterized by $\mathrm{nse}(G)$ and $|G|$. For instance, in 1987, Thompson gave an example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be two maximal subgroups of M_{23}, where M_{23} is a Mathieu group of degree 23. Then $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$. Unfortunately, $G_1 \ncong G_2$.

The authors of [2] proved that all simple K_4-groups G can be uniquely determined by $\mathrm{nse}(G)$ and $|G|$. Later, Asboei et al. [3] characterized sporadic simple groups by $\mathrm{nse}(G)$ and $|G|$. The authors of this paper proved (see [4]) that linear groups $L_2(q)$ are characterizable by their orders $|L_2(q)|$ and the set $\mathrm{nse}(L_2(q))$, if $q = 2^a - 1$ or $2^a + 1$ is a prime. On the other hand, some groups can be determined uniquely by the set nse. For instance, it is proved (see [5]) that $L_2(3)$, $L_2(4) \cong L_2(5)$ and $L_2(9)$ are uniquely determined by $\mathrm{nse}(G)$. Khatami, Khosravi and Akhlaghi [6] proved that simple groups $L_2(q)$ are characterizable uniquely by the set $\mathrm{nse}(L_2(q))$ if $q \in \{7, 8, 11, 13\}$. Moreover, Zhang and Shi [7], Asboei and Amiri [8] proved that $L_2(q)$ can be characterized uniquely by the set $\mathrm{nse}(L_2(q))$, where $q \in \{16, 25\}$. In this present paper, by introducing the prime graph of a group as a different method, we go on characterizing linear groups $L_2(q)$ when $q \in \{17, 27, 29\}$. Our result is the following theorem 1.1.

Theorem 1.1 *Let G be a group and $q \in \{17, 27, 29\}$. Then $G \cong L_2(q)$ if and only if $\mathrm{nse}(G) = \mathrm{nse}(L_2(q))$.*

We denote $n_r(G)$ by n_r and $m_k(G)$ by m_k if there is no confusion. Further unexplained notation is standard, and readers may refer to [9].

2 Preliminaries

In this section, we give some lemmas which will be used in the sequel.

Lemma 2.1 *Let G be a group. If $1 \neq n \in \mathrm{nse}(G)$ and $2 \nmid n$, then the following statements hold:*

(1) $2 \mid |G|$;

(2) $m_2 = n$;

(3) *for any* $2 < t \in \pi_e(G)$, $m_t \neq n$.

Proof Let $1 \neq t \in \pi_e(G)$ and k be the number of cyclic subgroups of G with order t. Then $m_t = k\varphi(t)$. If $t > 2$, then $\varphi(t)$ is even, so is m_t. Hence $m_t \neq n$ since n is odd. As a result, $m_2 = n$ and $2 \in \pi(G)$, as required.

Lemma 2.2 (see [10]) *Let G be a group and m be a positive integer dividing $|G|$. If $M_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |M_m(G)|$.*

Lemma 2.3 (see [4, Lemma 2.3]) *Let G be a group and P be a cyclic Sylow p-subgroup of G. Assume further that $|P| = p^a$ and r is an integer such that $p^a r \in \pi_e(G)$. Then $m_{p^a r} = m_r(C_G(P)) m_{p^a}$. In particular, $\varphi(r) m_{p^a} \mid m_{p^a r}$.*

Lemma 2.4 (see [11]) *Let G be a group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Recall that G is a 2-Frobenius group if G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with K/H and H being Frobenius kernels, respectively.

Lemma 2.5 (see [12, Theorem 2]) *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$,*

$|G/K| \mid |Aut\ (K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$ and G is solvable.

Lemma 2.6 (see [13, Theorem A]) *Let G be a group such that $t(G) \geq 2$. Then G has one of the following structures:*

(a) *G is a Frobenius or 2-Frobenius group.*

(b) *G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that $\pi(N) \cup \pi(G/G_1) \subseteq \pi_1$ and G_1/N is a nonabelian simple group.*

Lemma 2.7 *Let G be a simple group. If $\pi(G) = \{2, 3, 17\}$, then $G \cong L_2(17)$; if $\pi(G) = \{2, 3, 7, 13\}$, then $G \cong L_2(13)$ or $L_2(27)$; if $\pi(G) = \{2, 3, 5, 7, 29\}$, then $G \cong L_2(29)$.*

Proof This follows immediately by [14, Theorem 2], [15, Corollary 1] and [16, Theorem].

Lemma 2.8 (see [2, Lemma 2.5]) *Let G be a group with a normal series: $K \trianglelefteq L \trianglelefteq G$. Suppose that $P \in Syl_p(G)$, where $p \in \pi(G)$. If $P \leq L$ and $p \nmid |K|$, then the following statements hold:*

(1) *$|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;*

(2) *$|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t. Furthermore, $|N_K(P)|t = |K|$.*

3 Proof of Theorem 1.1

Proof of Theorem 1.1 The necessity is obvious, so we only prove the sufficiency. Let $n \in \pi_e(G)$ and k be the number of cyclic subgroups of order n in G. Then $m_n = k \cdot \varphi(n)$. In particular,

$$\varphi(n) \mid m_n. \tag{3.1}$$

We divide the proof into three cases.

Case 1 $\text{nse}(G) = \{1, 153, 272, 306, 612, 816, 288\} = \text{nse}(L_2(17))$.

By Lemma 2.1, we see that $2 \in \pi(G)$ and $m_2 = 153$. Further, Lemma 2.2 indicates that $\pi(G) \subseteq \{2, 3, 7, 13, 17, 19, 43, 307, 613\}$. If $13 \in \pi(G)$, then Lemma 2.2 implies that $m_{13} = 272$ and $|P_{13}| = 13$, yielding $n_{13} = \frac{m_{13}}{\varphi(13)} = \frac{272}{12}$, which is not an integer, a contradiction. Similarly, we have $7 \notin \pi(G)$. Suppose $307 \in \pi(G)$. Then $m_{307} = 306$ and $|P_{307}| = 307$ by Lemma 2.2. We claim that P_{307} acts fixed-point-freely on $\Omega_2 :=$ {all elements of order 2 in G}. If not, then $307 \cdot 2 \in \pi_e(G)$. By Lemma 2.3, we have $m_{307} \mid m_{2 \cdot 307}$, which leads to $m_{2 \cdot 307} = 306$ or 612. However, both cases indicate that $2 \cdot 307 \nmid (1 + m_2 + m_{307} + m_{2 \cdot 307})$, a contradiction to Lemma 2.2. Analogously, we obtain that $19, 43, 613 \notin \pi(G)$, and thus $\pi(G) \subseteq \{2, 3, 17\}$. Now we prove that the equality holds.

Assume that $\exp(P_2) = 2^s$. Then by (3.1) we have $\varphi(2^s) \mid m_{2^s}$, yielding $s \leq 6$. Analogously, if $3 \in \pi(G)$, then $\exp(P_3) \leq 3^3$ with $m_3 = 272$, $m_9 = 816$, $m_{27} = 612$ or 288; if $17 \in \pi(G)$, then $m_{17} = 288$, $|P_{17}| \leq 17^2$ and $\exp(P_{17}) = 17$. Further, if $\exp(P_3) = 3^2$ or 3^3, then P_3 is cyclic by Lemma 2.4.

First we say that G is not a 2-group. Otherwise, $\exp(P_2) = 2^6$ because $|\text{nse}(G)| = 7$. This implies that $|G| = \sum_{i \in \text{nse}(G)} i = 2548$, which is not a power of 2, a contradiction. On the other hand, suppose that G is a $\{2, 3\}$-group. Note that $\exp(P_3) \leq 3^3$. If $\exp(P_3) = 3^3$, then P_3 is cyclic by Lemma 2.4. Moreover, $2^2 \cdot 3^3 \notin \pi_e(G)$ by Lemma 2.3. If $2^2 \in \pi_e(G)$, then P_3 acts fixed-point-freely on $\Omega_4 :=$ {all elements of order 4 in G}, indicating that $|P_3| \mid |\Omega_4|$. Notice that $|\Omega_4| = m_4 = 306 = 2 \cdot 3^2 \cdot 17$ by Lemma 2.2, which is a contradiction. Hence $\exp(P_2) = 2$. Furthermore, $|P_2| \mid (1 + m_2)$ yields that $|P_2| = 2$ and thus $|G| = 2 \cdot 3^3$, also a contradiction. Hence $\exp(P_3) = 3^2$. Recall that P_3 is cyclic. It follows that $n_3 = \frac{m_9}{\varphi(9)} = 2^3 \cdot 17 \mid |G|$, contradicts our assumption. Similarly, we rule out the case $\exp(P_3) = 3$. Consequently, $\pi(G) \neq \{2, 3\}$.

Suppose that $\pi(G) = \{2, 17\}$. Notice that $|P_{17}| \leq 17^2$ and $\exp(P_{17}) = 17$. We show that $|P_{17}| = 17^2$. If not, $n_{17}(G) = m_{17}/\varphi(17) = 2 \cdot 3^2 \mid |G|$, a contradiction to our assumption. Further, $17 \cdot 2^3 \notin \pi_e(G)$ by (3.1). As a result, $|P_{17}| \mid m_{2^i}$ with $i \geq 3$ because P_{17} acts fixed-point-freely on $\Omega_{2^i} := \{$all elements of order 2^i in $G\}$, again a contradiction to Lemma 2.2. This shows that $\exp(P_2) \leq 2^2$. In this case, Lemma 2.2 implies that $|P_2| \mid (1 + m_2 + m_4)$, leading to $|P_2| \mid 2^2$. However, $|G| \leq 2^2 \cdot 17^2 = 1156 < \sum_{i \in \mathrm{nse}(G)} i = 2458$, a contradiction. Therefore, $\pi(G) = \{2, 3, 17\}$, as required.

We prove that $|P_{17}| = 17$. Assume that this is false. Then $|P_{17}| = 17^2$. If $\exp(P_3) = 3^3$, then P_3 is cyclic, implying $n_3 = \frac{m_{3^3}}{\varphi(3^3)} = 2^4$. Hence $17 \mid |N_G(P_3)|$. Let $A \leq N_G(P_3)$ be a group of order 17. Then $P_3 \rtimes A = P_3 \times A$ by Sylow's theorem, implying $17 \cdot 3^3 \in \pi_e(G)$. It follows by Lemma 2.3 that $16 \cdot m_{3^3} \mid m_{3^3 \cdot 17}$. Note that $m_{27} = 612$ or 288, which is a contradiction. Suppose that $\exp(P_3) = 3$. By Lemma 2.2, $|P_3| = 3$. Moreover, $|G : N_G(P_3)| = 2^3 \cdot 17$. By the same argument as above there is a contradiction, leading to $\exp(P_3) = 3^2$ and $|P_3| = 3^2$. Moreover, $|G : N_G(P_3)| = \frac{m_9}{\varphi(9)} = 2^3 \cdot 17$ and thus $17 \mid |N_G(P_3)|$. Let $A \in \mathrm{Syl}_3(N_G(P_3))$. Then $P_3 \rtimes A \leq G$. By Sylow's theorem, we obtain that $P_3 \rtimes A = P_3 \times A$ and thus $3 \cdot 17 \in \pi_e(G)$. However, $51 \nmid (1 + m_3 + m_{17} + m_{51})$, a contradiction to Lemma 2.2. Hence $|P_{17}| = 17$, as required.

If there is some prime $r \neq 17$ such that $17r \in \pi_e(G)$, then $(r-1)m_{17} \mid m_{17r}$ by Lemma 2.3. Further, $r = 2$ and $m_{34} = m_{17}$. However, $34 \nmid (1 + m_2 + m_{17} + m_{34})$, contradicting Lemma 2.2. As a result, $t(G) \geq 2$.

Assume first that $G = K \rtimes H$ is a Frobenius group with the kernel K and a complement H. As $t(G) = 2$, we see that $\pi_1 = \{2, 3\}$ and $\pi_2 = \{17\}$ as there is no element of order $17r$ for each prime r distinct from 17. Then either $|H| = 17$ or $|K| = 17$. If the latter holds, then $|H| \mid 16$ and $|G| \mid 16 \cdot 17$, a contradiction. Hence $|H| = 17$. Moreover, $K_3 \rtimes H$ is also a Frobenius group with a kernel K_3 and a complement H, yielding to $|H| \mid (|K_3| - 1)$. However, $|K_3| \mid 3^3$, which is a contradiction. Let G be a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = 17$ and $|G/K| \mid |\mathrm{Aut}(K/H)|$. Hence $H_3 \in \mathrm{Syl}_3(G)$ and thus $H_3 \rtimes C_{17}$ is also a Frobenius group with a Frobenius kernel H_3 and a complement C_{17}. By the same reasoning as above, this is also a contradiction. Hence by Lemma 2.6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple K_3-group since $|\pi(G)| = 3$. By Lemma 2.7, we get $K/H \cong L_2(17)$. Moreover, Lemma 2.8 implies that $n_{17}(K/H)t = n_{17}$ and $|N_H(P_{17})|t = |H|$. Since $n_{17}(K/H) = n_{17}$, we have $t = 1$ and thus $H = N_H(P_{17})$. So we obtain that $HP_{17} = H \times P_{17}$. Since $17r \notin \pi_e(G)$, we get $H = 1$ and $K \cong L_2(17)$. Note that $|\mathrm{Out}(L_2(17))| = 2$. Then we have $G = K \cdot 2$ or $G = K$. If $G = K \cdot 2$, then by [9], we obtain that $m_2 = 289 \neq 153$, a contradiction. Hence $G = K \cong L_2(17)$.

Case 2 $\mathrm{nse}(G) = \{1, 351, 728, 2106, 4536\} = \mathrm{nse}(L_2(27))$.

By Lemma 2.1, we see that $2 \in \pi(G)$ and $m_2 = 351$. Notice that $1 + 728 = 3^6$, $1 + 2106 = 7^2 \cdot 43$, $1 + 4536 = 13 \cdot 349$. Then $\pi(G) \subseteq \{2, 3, 7, 13, 43, 349\}$ by Lemma 2.2. Assume that $43 \in \pi(G)$. Then Lemma 2.2 implies that $m_{43} = 2106$. Further, $\exp(P_{43}) = 43$ and $|P_{43}| \mid (1 + m_{43})$, which leads to that P_{43} is a cyclic group of order 43. Hence $n_{43} = \frac{m_{43}}{\varphi(43)} = \frac{2106}{42}$, which is not an integer, a contradiction. Similarly, $349 \notin \pi(G)$ and thus $\pi(G) \subseteq \{2, 3, 7, 13\}$. We show that the equality holds.

Assume $\exp(P_2) = 2^s$. Then by (3.1) we obtain that $\varphi(2^s) \mid m_{2^s} \in \{728, 2106, 4536\}$, leading to $s \le 4$. If the equality holds, then $m_{2^2} \in \{728, 2106, 4536\}$, $m_{2^3} \in \{728, 2106, 4536\}$, $m_{2^4} \in \{728, 2106, 4536\}$, which is contrary to the fact that $2^4 \mid (1 + m_2 + m_{2^2} + m_{2^3} + m_{2^4})$ by Lemma 2.2. Thus $\exp(P_2) \le 2^3$. Furthermore, $|P_2| \le 2^6$. On the other hand, if $3, 7, 13 \in \pi(G)$, then $m_3 = 728, m_7 = 2106$ and $m_{13} = 4536$ according to Lemma 2.2. Further, $\exp(P_3) \le 3^4$, $|P_3| \le 3^6$, $\exp(P_7) = 7$, $|P_7| \le 7^2$ and $|P_{13}| = 13$ by a similar argument as above. Assume that $13 \in \pi(G)$. Then $n_{13} = \frac{m_{13}}{\varphi(13)} = 2 \cdot 3^3 \cdot 7$, yielding that $\pi(G) = \{2, 3, 7, 13\}$, as required. Suppose $7 \in \pi(G)$. If $|P_7| = 7$, then $n_7 = \frac{m_7}{\varphi(7)} = 3^3 \cdot 13$, which also implies that $\pi(G) = \{2, 3, 7, 13\}$. Hence we may assume that $13 \notin \pi(G)$ and $|P_7| = 7^2$ if $7 \in \pi(G)$. Note that G is neither a 2-group nor a $\{2, 7\}$-group because $|G| \le 2^6 \cdot 7^2 < \sum_{i \in \mathrm{nse}(G)} i = 7722$. As a result, $3 \in \pi(G)$. If $7 \in \pi(G)$, then G is a $\{2, 3, 7\}$-group with $|\pi_e(G)| \le 4 \cdot 5 \cdot 2 = 40$. Therefore,

$$|G| = 2^a \cdot 3^b \cdot 7^2 = 7722 + 728k_1 + 2106k_2 + 4536k_3, \tag{3.2}$$

where a, b, k_1, k_2 and k_3 are non-negative integers such that $\sum_{i=1}^{3} k_i \le 40 - 5 = 35$, $1 \le a \le 6$ and $1 \le b \le 6$ as $|P_2| \le 2^6$ and $|P_3| \le 3^6$. We see easily that the equation (3.2) is equivalent to

$$2^3 \cdot 7 \cdot 13k_1 + 2 \cdot 3^4 \cdot 13k_2 + 2^3 \cdot 3^4 \cdot 7k_3 = 2^a \cdot 3^b \cdot 7^2 - 2 \cdot 3^3 \cdot 11 \cdot 13. \tag{3.3}$$

Assume first that $b \ge 3$. Then we see clearly that $27 \mid k_1$, leading to either $k_1 = 0$ or $k_1 = 27$ since $\sum_{i=1}^{3} k_i \le 35$. If the former holds, we divide $2 \cdot 3^3$ at both sides of (3.3), and then

$$3 \cdot 13k_2 + 2^2 \cdot 3 \cdot 7k_3 = 2^{a-1} \cdot 3^{b-3} \cdot 7^2 - 11 \cdot 13 \tag{3.4}$$

indicating $b = 3$, since, otherwise, $3 \mid 11 \cdot 13$, a contradiction. Moreover, $13 \mid (2^{a-1} \cdot 7 - 2^2 \cdot 3k_3)$, implying $13 \mid (2^{a-3} \cdot 7 - 3k_3)$. It follows that either $a = 5, k_3 = 5$ or $a = 6, k_3 = 10$ as $a \le 6$. However, in these two cases, no integer k_2 satisfies (3.4). Hence $k_1 = 27$. Then (3.3) is equivalent to

$$3 \cdot 13k_2 + 2^2 \cdot 3 \cdot 7k_3 = 2^{a-1} \cdot 3^{b-3} \cdot 7^2 - 3 \cdot 13^2. \tag{3.5}$$

Thus $7 \mid (k_2 + 13)$, implying $k_2 = 1$ or $k_2 = 8$. If the latter holds, then $k_3 = 0$. However, $k_1 = 27,\ k_2 = 8,\ k_3 = 0$ is not a solution of (3.2). Hence $k_2 = 1$. However, in this case, (3.5) is equivalent to

$$2^2 \cdot 3 \cdot 7 \cdot k_3 = 2^{a-1} \cdot 3^{b-3} \cdot 7^2 - 2 \cdot 3 \cdot 7 \cdot 13.$$

This shows that $2k_3 = 2^{a-2} \cdot 3^{b-4} - 13$, and thus $a = 2$. Moreover, $2k_3 = 3^{b-4} - 13$, contradicting the fact $b \le 6$. Therefore, $b = 1$ or 2. The similar argument as above will also deduce a contradiction.

The remaining case is $\pi(G) \subseteq \{2, 3\}$. As G is not a 2-group, we see that $|\pi_e(G)| \le 4 \cdot 5 = 20$. Moreover,

$$|G| = 2^a 3^b = 7722 + 728k_1 + 2106k_2 + 4536k_3, \tag{3.6}$$

where a, b, k_1, k_2 and k_3 are non-negative integers such that $\sum_{i=1}^{3} k_i \le 20 - 5 = 15$, $1 \le a \le 6$ and $5 \le b \le 6$. That is,

$$2^3 \cdot 7 \cdot 13k_1 + 2 \cdot 3^4 \cdot 13k_2 + 2^3 \cdot 3^4 \cdot 7k_3 = 2^a \cdot 3^b - 2 \cdot 3^3 \cdot 11 \cdot 13. \tag{3.7}$$

Easily, $5 \le b \le 6$ implies that $k_1 = 0$ and thus (3.7) is equivalent to

$$3 \cdot 13k_2 + 2^2 \cdot 3 \cdot 7k_3 = 2^{a-1} \cdot 3^{b-3} - 11 \cdot 13, \tag{3.8}$$

leading to $3 \mid 11 \cdot 13$, again a contradiction. As a result, $\pi(G) = \{2, 3, 7, 13\}$, as required.

Recall that $|P_{13}| = 13$. We claim that $13s \notin \pi_e(G)$ for each $s \in \pi(G)$ distinct from 13. Otherwise, Lemma 2.3 indicates that $s = 2$ and $m_{26} = m_{13}$. But $26 \nmid (1 + m_2 + m_{13} + m_{26})$, a contradiction to Lemma 2.2. Hence $t(G) \ge 2$. Assume first that G is a Frobenius group. Then $t(G) = 2$ with $\pi_1 = \{2, 3, 7\}$ and $\pi_2 = \{13\}$. Write $G = K \rtimes H$. Suppose first that $13 \mid |K|$. Since K is nilpotent, we obtain that $m_{13} = |K_{13}| - 1 = |P_{13}| - 1 = 12$, where K_{13} is the Sylow 13-subgroup of K, a contradiction. Hence $13 \mid |H|$ and thus $2, 3, 7 \in \pi(K)$. Let K_7 be a Sylow 7-subgroup of K. Then $K_7 \rtimes H$ is also a Frobenius group with a kernel K_7 and a complement H. This implies that $13 \mid (|K_7| - 1)$, contrary to the fact that $|K_7| = |P_7| \le 7^2$. Suppose further that G is a 2-Frobenius group. Then by Lemma 2.5 we see that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ with $|K/H| = 13$ and $|G/K| \mid 12$, leading to $7 \mid |H|$. Since H is nilpotent, we get $m_7 = 6$ or 48, again a contradiction.

Consequently, by Lemma 2.6 we see that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is either a simple K_3 or K_4-group and $13 \mid |K/H|$. Assume first that K/H is a simple K_3-group. Then $K/H \cong L_3(3)$ by [14, Lemma 2]. Note that $|G/K| \mid |\mathrm{Out}(K/H)| = 2$. Then we obtain that $7 \mid |H|$. Since H is nilpotent, we get $m_7 = 6$ or 48, a contradiction. Hence K/H is a simple K_4-group. By Lemma 2.7, we have $K/H \cong L_2(13)$ or $L_2(27)$. If $K/H \cong L_2(13)$, then $n_7(K/H)t = n_7$ by Lemma 2.8. Since $n_7 = 3^3 \cdot 13$ and $n_7(K/H) = 2 \cdot 3 \cdot 13$ according to [9, p. 8], this is a contradiction. Hence $K/H \cong L_2(27)$. Again by applying Lemma 2.8, we obtain that $n_{13}(K/H)t = n_{13}$ and $|N_H(P_{13})|t = |H|$. Hence $t = 1$ since $n_{13}(K/H) = n_{13}$. Moreover, $H = N_H(P_{13})$, leading to $HP_{13} = H \times P_{13} \le G$. Note that $13r \notin \pi_e(G)$. This implies that $H = 1$ and therefore, $K \cong L_2(27)$, and $|G/K| \mid |\mathrm{Out}(K)| = 6$. Assume $|G/K| = 2$, and then $G = L_2(27) \cdot 2$. By [9], $m_2 = 729$, a contradiction. Similarly, the cases $|G/K| = 3$ and $|G/K| = 6$ also imply a contradiction. This shows $G = K \cong L_2(27)$, as wanted.

Case 3 $\mathrm{nse}(G) = \{1, 435, 2610, 812, 1624, 3248, 840\} = \mathrm{nse}(L_2(29))$.

Similar to the proof of Case 1, we obtain that $2 \in \pi(G) \subseteq \{2, 3, 5, 7, 29\}$ and $m_2 = 435$. Moreover, if $3, 5, 7, 29 \in \pi(G)$, then $m_3 \in \{812, 3248\}$, $m_5 = 1624$, $m_7 = 2610$ and $m_{29} = 840$. Suppose that $\exp(P_2) = 2^a$. Since $\varphi(2^a) \mid m_{2^a}$ and $m_{2^a} \in \mathrm{nse}(G)$, along with Lemma 2.2, we get $a \le 5$. Moreover, $|P_2| \le 2^7$. By the same reasoning, if $3, 5, 7, 29 \in \pi(G)$, we obtain that $|P_3| \le 3^3$, $\exp(P_3) \le 3^2$, $|P_5| \le 5^3$, $\exp(P_5) = 5$, $|P_7| = 7$, $|P_{29}| \le 29^2$ and $\exp(P_{29}) = 29$.

We prove that $\pi(G) = \{2, 3, 5, 7, 29\}$. Assume first $7 \in \pi(G)$. Then $n_7 = \frac{m_7}{6} = 3 \cdot 5 \cdot 29$, which implies that $\pi(G) = \{2, 3, 5, 7, 29\}$, and we are done. As a result, we may assume that $\pi(G) \subseteq \{2, 3, 5, 29\}$.

We see that G is neither a 2-group nor a $\{2, 3\}$-group because $|G| \le 2^7 \cdot 3^2 < \sum_{i \in \mathrm{nse}(G)} i = 9120$. Suppose that $\pi(G) = \{2, 5\}$. Then $\pi_e(G) \subseteq \{1, 2, 2^2, \cdots, 2^5\} \cup \{5, 5 \cdot 2, 5 \cdot 2^2\}$, which leads to

$$|G| = 2^a 5^b = 9570 + 2610k_1 + 812k_2 + 1624k_3 + 3248k_4 + 840k_5, \tag{3.9}$$

where a, b, k_1, k_2, k_3, k_4 and k_5 are non-negative integers such that $1 \le a \le 7$, $1 \le b \le 2$ and $\sum_{i=1}^{5} k_i \le 6 \cdot 2 - 7 = 5$. As $29 \nmid 2^a 5^b$, we obtain that $k_5 \ne 0$ and thus the equation has no solutions. Suppose that $\pi(G) = \{2, 29\}$. If $|P_{29}| = 29^2$, then

$$|G| = 2^a 29^2 = 9570 + 2610k_1 + 812k_2 + 1624k_3 + 3248k_4 + 840k_5, \tag{3.10}$$

where a, k_1, k_2, k_3, k_4 and k_5 are non-negative integers such that $1 \leq a \leq 7$ and $\sum_{i=1}^{5} k_i \leq 5$. Easily, $k_5 = 0$. Then (3.10) becomes

$$2^{a-1} \cdot 29 = 3 \cdot 5 \cdot 11 + 3^2 \cdot 5k_1 + 2 \cdot 7k_2 + 2^2 \cdot 7k_3 + 2^3 \cdot 7k_4. \tag{3.11}$$

Then k_1 must be odd. If $k_1 = 1$, then

$$2^{a-1} \cdot 29 = 2 \cdot 3 \cdot 5 \cdot 7 + 2 \cdot 7k_2 + 2^2 \cdot 7k_3 + 2^3 \cdot 7k_4 \tag{3.12}$$

has no solutions. If $k_1 = 3$, then (3.11) becomes

$$2^{a-1} \cdot 29 = 2^2 \cdot 3 \cdot 5^2 + 2 \cdot 7k_2 + 2^2 \cdot 7k_3 + 2^3 \cdot 7k_4, \tag{3.13}$$

leading to $a = 2$, which also is impossible. Hence, $|P_{29}| = 29$ yielding $n_{29} = \frac{m_{29}}{28} = 2 \cdot 3 \cdot 5$, a contradiction.

Suppose that $\pi(G) = \{2, 3, 5\}$. If $\exp(P_3) = 3^2$ and P_3 is cyclic, then $|P_3| = 3^2$. If $|P_5| = 5$, then $n_5 = 2 \cdot 7 \cdot 29$, a contradiction. Assume that $|P_5| = 5^2$. By Lemma 2.3, we have $45 \notin \pi_e(G)$. Hence P_5 acts fixed-point-freely on $\Omega_9 :=$ {all elements of order 9 in G}. So we have $5^2 \mid |\Omega_9|$, which is contrary to $|\Omega_9| = m_9$. The same argument implies that $|P_5| \neq 5^3$. Moreover, P_3 is non-cyclic. Note that $m_3 = 3248$. Then $15 \notin \pi_e(G)$, since, otherwise, $15 \nmid (1+m_3+m_5+m_{15})$, contrary to Lemma 2.2. As a result, P_3 acts fixed-point-freely on $\Omega_5 :=$ {all elements of order 5 in G}. Thus $|P_3| \mid |\Omega_5|$, which is a contradiction since $|\Omega_5| = m_5 = 1624$. Hence $\exp(P_3)$=3. If P_3 is cyclic, then $n_3 = \frac{m_3}{2} = 2 \cdot 7 \cdot 29$ or $2^3 \cdot 7 \cdot 29$ because $m_3 = 812$ or 3248, also a contradiction. As a consequence, $|P_3| = 3^2$ and $m_3 = 3248$. If $15 \in \pi_e(G)$, then $m_{15} = 1624, 3248$ or 840. By Lemma 2.2, we see that $15 \nmid (1+m_3+m_5+m_{15})$, which is a contradiction. Then P_5 acts fixed-point-freely on $\Omega_3 :=$ {all elements of order 3 in G}. So we have that $5^2 \mid |\Omega_3|$, contradicting $|\Omega_3| = m_3$. This indicates that $|P_5| = 5^3$. By the same reasoning, there is also a contradiction, leading to that $\pi(G) \neq \{2, 3, 5\}$. Analogously, $\pi(G) \neq \{2, 3, 29\}, \{2, 5, 29\}, \{2, 3, 5, 29\}$ and therefore, $\pi(G) = \{2, 3, 5, 7, 29\}$, as required.

Recall that $|P_7| = 7$. Then $n_7 = \frac{m_7}{6} = 3 \cdot 5 \cdot 29$. We prove that $|P_{29}| = 29$. If not, $|P_{29}| = 29^2$, implying $29 \mid |N_G(P_7)|$. Let $N \in \mathrm{Syl}_{29}(N_G(P_7))$. Then $N \trianglelefteq P_7N$ by Sylow's theorem. Hence $N \times P_7 \leq G$ and thus $7 \cdot 29 \in \pi_e(G)$. By Lemma 2.3, it follows that $28m_7 \mid m_{7\cdot 29}$, contradicting $m_{7\cdot 29} \in \mathrm{nse}(G)$. We prove that $29r \notin \pi_e(G)$ for each $r \in \pi(G)$ distinct from 29. Otherwise, $\varphi(r)m_{29} \mid m_{29r}$ by Lemma 2.3. This forces $r = 2$. However, $2 \cdot 29 \nmid (1 + m_2 + m_{29} + m_{2\cdot 29})$, contrary to Lemma 2.2. Consequently, $n_{29} = \frac{m_{29}}{28} = 2 \cdot 3 \cdot 5$. Assume that $N \in \mathrm{Syl}_5(N_G(P_{29}))$ and $|N| > 1$. Then $N \trianglelefteq P_{29}N$ by Sylow's theorem. So we have $P_{29}N = N \times P_{29}$ and thus $5 \cdot 29 \in \pi_e(G)$, also a contradiction. Hence $|P_5| = 5$.

Therefore, $t(G) \geq 2$. Assume first that G is a Frobenius group. Thus $t(G) = 2$ with $\pi_1 = \{2, 3, 5, 7\}$ and $\pi_2 = \{29\}$. Write $G = K \rtimes H$. If $29 \mid |K|$, then $m_{29} = |P_{29}| - 1 = 28$, contrary to our assumption. Thus $|H| = 29$ and $7 \mid |K|$. Let K_7 be a Sylow 7-subgroup of K. As K is nilpotent, we see that $m_7 = |K_7| - 1 = 6$. This contradiction implies that G is a 2-Frobenius group. Moreover, Lemma 2.5 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = 29$ and $|G/K| \mid 28$, leading to $5 \mid |H|$. Since H is nilpotent and $|P_5| = 5$, we obtain that $m_5 = |H_5| = 4$, which contradicts our assumption. Further, Lemma 2.5 indicates that G is non-solvable and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group, $29 \mid |K/H|$ and $\pi(H) \cup \pi(G/K) \subseteq \pi_1$. Because there is no simple K_3-group whose order is divisible by 29, we see that K/H is a simple K_4 or K_5-group. By Lemma 2.7, we see that $K/H \cong L_2(29)$. On the other hand, Lemma 2.8 implies that $n_{29} = n_{29}(K/H)t$ and

$|N_H(P_{29})|t = |H|$. Thus $t = 1$ and $H = N_H(P_{29})$, yielding to $H \times P_{29} \leq G$. Note that there is no element of order $29r$ for $r \in \pi(G)$. Then $H = 1$ and thus $K \cong L_2(29)$. Moreover, $G = K \cdot 2$ or $G = K$. If the former holds, it follows by [9] that $m_2 = 841$, a contradiction. Hence $G = K \cong L_2(29)$ and the theorem is established.

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