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The Pointwise Estimates of Solutions to the Cauchy Problem of a Chemotaxis Model*

Renkun SHI[1] Weike WANG[1]

Abstract This paper deals with an attraction-repulsion chemotaxis model (ARC) in multi-dimensions. By Duhamel's principle, the implicit expression of the solution to (ARC) is given. With the method of Green's function, the authors obtain the pointwise estimates of solutions to the Cauchy problem (ARC) for small initial data, which yield the $W^{s,p}$ ($1 \le p \le \infty$) decay properties of solutions.

Keywords Chemotaxis model, Pointwise estimates, Green's function, Decay rates
2000 MR Subject Classification 35B40, 35Q80

1 Introduction

Chemotaxis is a phenomenon describing the movements of bacteria or cells in response to some chemical substances. According to the orientation of the movements, we can describe the phenomenon by attractive chemotaxis and repulsive chemotaxis, respectively. The former occurs when the movement is toward a higher concentration of the chemical, and conversely, the latter occurs when the movement is in the opposite direction. One famous attractive chemotaxis model named Keller-Segel model was proposed by Keller and Segel [3] in the 1970s describing the aggregation process of amoebae by chemoattraction. In [9], Luca proposed a more general attraction-repulsion chemotaxis model to describe the aggregation of microglia and formation of local accumulations of chemicals observed in AD senile plaques. In this paper, we consider the following attraction-repulsion chemotaxis system (ARC):

$$\begin{cases} \partial_t u - \Delta u = \beta_1 \nabla \cdot (u^{m_1} \nabla w) - \beta_2 \nabla \cdot (u^{m_2} \nabla v), & x \in \mathbb{R}^n,\ t > 0, \\ \lambda_1 w - \Delta w = u, & x \in \mathbb{R}^n,\ t > 0, \\ \lambda_2 v - \Delta v = u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{ARC}$$

where the spatial dimension $n \ge 1$, $u(x,t)$ denotes the density of cells, $w(x,t)$ denotes the concentration of chemorepellents, and $v(x,t)$ denotes the concentration of chemoattractants; the parameters β_1 and β_2 represent the sensitivities of cells to the chemorepellents and the chemoattractants respectively; λ_i ($i = 1, 2$) are positive parameters and $m_i \ge 1$ ($i = 1, 2$) are

Manuscript received November 20, 2013. Revised October 10, 2014.
[1]Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China.
E-mail: blalbible@163.com wkwang@sjtu.edu.cn
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positive integers. The aim of this paper is to show the pointwise estimates of solutions to the system for small initial data and to obtain the decay rates of solutions in $W^{s,p}(\mathbb{R}^n)$ space.

When $\beta_1 = 0$, this model is just the well-known simplified Keller-Segel (KS) model, for which a number of works were carried out from various viewpoints. For these background knowledge and more information, we refer to [2, 10–11] and their references. We mention here some related works for the (KS) model with small initial data. In [14], Y. Sugiyama and H. Kunii proved that a global solution exists for the (KS) model with small initial data and obtained the $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ decay rates of the solution. For $n \ge 3$ and $m_2 = 1$, H. Kozono and Y. Sugiyama [4] considered local and global existence for the (KS) model with initial data $u_0 \in L_w^{\frac{n}{2}}(\mathbb{R}^n)$ and proved the existence of strong solution with $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for $n < q \le \infty$.

When $\beta_1, \beta_2 > 0$, because of the interaction of the two attraction and repulsion effects, the (ARC) model can represent much richer phenomena. When the repulsion prevails over the attraction, that is, when $\beta_1 > \beta_2$, one can expect the global existence of solutions for large initial data, while when $\beta_1 < \beta_2$, the finite time blow-up may occur. Such results can be seen in [15] where the authors considered the system (ARC) for $m_1 = m_2 = 1$ in a bounded domain. For other results of the attraction-repulsion chemotaxis model, we can refer to [6, 12] and the references therein. In this paper, we are interested in the Cauchy problem of the system (ARC) with small initial data. We prove that no matter in what parameter regime, the system (ARC) is always globally solvable for small initial data. More precisely, we study the pointwise estimates of solutions to the system, which is very helpful for us to better understand the behaviors of solutions to the system in both time and space. And we also derive the $W^{s,p}(\mathbb{R}^n)$ decay rates of solutions to (ARC).

For getting the precise pointwise estimates, to the authors' best knowledge, Green's function is one of the most effective tools to describe the pointwise estimates. In [8], Liu and Zeng investigated the pointwise estimates of solutions to general hyperbolic-parabolic systems in one space dimension. D. Hoff and K. Zumbrun [1] considered pointwise decay estimates for multidimensional Navier-Stokes equations with an artificial viscosity term. Then Liu and Wang [7] used the Green's function method in odd multi-dimensional Navier-Stokes equations, obtained long time behavior of solutions under small perturbation and proved the generalized Huygen's principle. After that more work involved in this field was done.

Most of the papers mentioned above are about hyperbolic-parabolic systems. For the hyperbolic-parabolic-elliptic coupled type, there are also some results recently (see [5, 16]). However, after the transformation of their systems, the elliptic effect is reflected in the nonlocal linear term, while for the system (ARC), the nonlocal term is nonlinear, which is a difficult point to deal with in this paper. Actually, we can solve the second and the third equations in (ARC) with the help of Bessel potentials $\frac{1}{\lambda_i - \Delta}$ $(i = 1, 2)$. Then we can easily get the boundedness estimates of w and v in the norms of u by the knowledge of pseudo-differential operators, while for the more precise estimates, such as pointwise estimates, we cannot obtain them so easily. Thus we must make more delicate calculations in our study. We shall show them later.

Before we show our results, we give some notations that are frequently used in the paper.

Notations Throughout the paper we denote generic constants by C which may vary line by line according to the context. $W^{s,p}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ represent the usual Sobolev spaces and the $L^p(\mathbb{R}^n)$ norm of a function is denoted by $\| \cdot \|_{L^p(\mathbb{R}^n)}$. For any multi-indexes $\alpha =$

$(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}^n$, we denote $\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$ by ∂_x^α. We also denote $\frac{1}{\lambda-\Delta}u = K_\lambda * u$ with

$$K_\lambda(x) = (4\pi)^{-\frac{n}{2}} \int_0^\infty \mathrm{e}^{-\lambda s - \frac{|x|^2}{4s}}\, s^{-\frac{n}{2}} \mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s} G(x,s) \mathrm{d}s, \tag{1.1}$$

where

$$G(x,t) = (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}} \tag{1.2}$$

is the fundamental solution to the heat equation. The expression (1.1) of the kernel of Bessel potential can be seen in [13, Chapter V, Section 3.1]. We also denote

$$B_m(|x|,t) \triangleq \Big(1 + \frac{|x|^2}{1+t}\Big)^{-m} \tag{1.3}$$

for any $m > \frac{n}{2}$ which can be seen as some substitution of the function $\mathrm{e}^{\frac{-|x|^2}{t}}$.

The main results of this paper are stated in the following.

Theorem 1.1 *Let (u, w, v) be the solutions to the Cauchy problem (ARC). Then there exists $\epsilon > 0$, such that if $u_0(x)$ satisfies*

$$|\partial_x^\alpha u_0(x)| \le \epsilon(1 + |x|^2)^{-r}, \tag{1.4}$$

where $r > \frac{n}{2}$, $|\alpha| \le h$, and h is a given positive integer, then we have the estimate

$$|\partial_x^\alpha u(x,t)| \le a(\epsilon)(1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t) \tag{1.5}$$

for all $t \ge 0$, $|\alpha| \le h$. $a(\epsilon)$ depends on n, h, r, ϵ and the parameters in the equations. $w(x,t)$ and $v(x,t)$ satisfy the same estimates as (1.5).

Corollary 1.1 *Let (u, w, v) be the solutions to the Cauchy problem (ARC), and $u_0 \in W^{s,p}(\mathbb{R}^n)$, where $s > 0$ is an integer, $p \in [1, \infty]$. Then there exists $\epsilon > 0$, such that if $u_0(x)$ satisfies*

$$|\partial_x^\alpha u_0(x)| \le \epsilon(1 + |x|^2)^{-r}, \tag{1.6}$$

where $r > \frac{n}{2}$, $|\alpha| \le s$, then we have the estimate

$$\|\partial_x^\alpha u\|_{L^p(\mathbb{R}^n)} \le a(\epsilon)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}} \tag{1.7}$$

for all $t \ge 0$, $|\alpha| \le s$. $a(\epsilon)$ depends on n, s, r, ϵ and the parameters in the equations. $w(x,t)$ and $v(x,t)$ satisfy the same estimates as (1.7).

From Theorem 1.1 and Corollary 1.1, we can conclude that both the attraction and repulsion effects can be dominated by the diffusion term under the smallness condition, and therefore the behavior of solutions is similar to the heat equation. We also state that our global existence result and decay rates of solutions are independent of the parameters m_i and β_i $(i = 1, 2)$ and the signs of β_i $(i = 1, 2)$.

For simplicity and convenience to carry on in this paper, we transform the system (ARC) into the following form:

$$\begin{cases} \partial_t u - \Delta u = \beta_1 \nabla \cdot \Big(u^{m_1} \dfrac{\nabla}{\lambda_1 - \Delta} u\Big) - \beta_2 \nabla \cdot \Big(u^{m_2} \dfrac{\nabla}{\lambda_2 - \Delta} u\Big), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

We investigate the pointwise estimates and $W^{s,p}(\mathbb{R}^n)$ decay rates of solution $u(x,t)$ to the problem (1.8), and we can prove that the solutions to the original system (ARC) satisfy the same estimates. In deriving these results, we also omit the proof of some lemmas to make this paper brief.

The rest of the paper is arranged as follows. In Section 2, we shall give some lemmas which are very useful for the proof of the main theorem. The pointwise estimates of solutions will be carried out by Duhamel's principle in Section 3.

2 Some Lemmas

In this section, we shall give some lemmas which will be used in proving the main theorem in the next section.

Lemma 2.1 *If $n_1, n_2 > \frac{n}{2}$, and $n_3 = \min\{n_1, n_2\}$, then*

$$\int_{\mathbb{R}^n} \Big(1 + \frac{|x-y|^2}{1+t}\Big)^{-n_1} (1+|y|^2)^{-n_2} \mathrm{d}y \le C\Big(1 + \frac{|x|^2}{1+t}\Big)^{-n_3}, \tag{2.1}$$

where C depends only on n_1, n_2 and n.

The proof of Lemma 2.1 can be seen in [17].

Lemma 2.2 *Assume that G is the heat kernel given by (1.2), and $f(x)$ satisfies*

$$f(x) \le (1+|x|^2)^{-m} \tag{2.2}$$

and

$$|\partial_x^\alpha f(x)| \le (1+|x|^2)^{-m} \tag{2.3}$$

for some multi-indexes $\alpha \in \mathbb{Z}^n$ and $m > \frac{n}{2}$. Then we have

$$|\partial_x^\alpha (G * f)(x,t)| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_m(|x|,t), \tag{2.4}$$

where C depends only on n, m and α, and $B_m(|x|,t)$ is defined by (1.3).

Proof When $t \ge 1$, we can easily deduce

$$|\partial_x^\alpha G(x,t)| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_{2m}(|x|,t). \tag{2.5}$$

Then by (2.2) and Lemma 2.1, it holds that

$$|\partial_x^\alpha (G * f)| = |(\partial_x^\alpha G) * f| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_m(|x|,t) \quad \text{for } t \ge 1. \tag{2.6}$$

When $t \le 1$, it follows from (2.3) that

$$|\partial_x^\alpha (G * f)| = |G * \partial_x^\alpha f| \le C \int_{\mathbb{R}^n} t^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} (1+|y|^2)^{-m} \mathrm{d}y. \tag{2.7}$$

Now denote

$$\mathcal{I} := \Big\{ y \in \mathbb{R}^n \;\Big|\; |y| \ge \frac{|x|}{2} \text{ or } |x-y| \le 1 \Big\}. \tag{2.8}$$

If $|y| \geq \frac{|x|}{2}$, it holds that

$$1+|y|^2 > \frac{1}{4}(1+|x|^2), \tag{2.9}$$

and if $|x-y| \leq 1$, it holds that

$$1+|x|^2 = 1+|x-y+y|^2 \leq 1+2|x-y|^2+2|y|^2 \leq 3(1+|y|^2). \tag{2.10}$$

Therefore, when $y \in \mathcal{I}$, we always have

$$(1+|y|^2)^{-m} \leq C(1+|x|^2)^{-m}, \tag{2.11}$$

hence

$$\int_{\mathcal{I}} t^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} (1+|y|^2)^{-m} \mathrm{d}y \leq C(1+|x|^2)^{-m}. \tag{2.12}$$

Else if $y \in \mathbb{R}^n \backslash \mathcal{I}$, then we have

$$|x-y| > \frac{|x|}{2}, \quad |x-y| > 1. \tag{2.13}$$

Therefore, we can deduce

$$\begin{aligned} t^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} &\leq t^{-\frac{n}{2}} \mathrm{e}^{-\frac{1}{8t}} \mathrm{e}^{-\frac{|x-y|^2}{8t}} \\ &\leq C\mathrm{e}^{-\frac{|x-y|^2}{8t}} \\ &\leq C\Big(1+\frac{|x-y|^2}{t}\Big)^{-m} \\ &\leq C(1+|x-y|^2)^{-m} \\ &\leq C(1+|x|^2)^{-m}, \end{aligned} \tag{2.14}$$

where we have also used the fact of $t \leq 1$. The corresponding integral satisfies

$$\int_{\mathbb{R}^n \backslash \mathcal{I}} t^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} (1+|y|^2)^{-m} \mathrm{d}y \leq C(1+|x|^2)^{-m}, \tag{2.15}$$

where $m > \frac{n}{2}$. From (2.7), (2.12) and (2.15), we can bound the whole integral:

$$|\partial_x^\alpha (G*f)| \leq C(1+|x|^2)^{-m} \leq C(1+t)^{-\frac{n+|\alpha|}{2}} B_m(|x|,t) \quad \text{for } t \leq 1. \tag{2.16}$$

By (2.6) and (2.16), we complete the proof.

To obtain the precise estimates of the nonlocal terms of equation (1.8), we next give a very useful lemma which plays an important role in our proof of the theorem.

Lemma 2.3 *If*

$$|f(x,t)| \leq (1+t)^{-b} B_m(|x|,t) \tag{2.17}$$

for some nonnegative constant b *and* $m > \frac{n}{2}$, *then*

$$|\nabla K_\lambda * f| \leq C(1+t)^{-b} B_m(|x|,t), \tag{2.18}$$

where C *depends only on* n, m *and* λ, K_λ *is given by* (1.1) *and* $B_m(|x|,t)$ *is defined by* (1.3).

Proof Without loss of generality, we take $b=0$.

$$\begin{aligned}|\nabla K_\lambda * f(x,t)| &= \Big|\int_{\mathbb{R}^n}\nabla K_\lambda(x-y)f(y,t)\mathrm{d}y\Big|\\
&= \Big|\int_{\mathbb{R}^n}\Big(\int_0^\infty \mathrm{e}^{-\lambda s}\nabla G(x-y,s)\mathrm{d}s\Big)f(y,t)\mathrm{d}y\Big|\\
&\le \int_0^\infty \mathrm{e}^{-\lambda s}(|\nabla G(\cdot,s)| * B_m(\cdot,t))\mathrm{d}s\\
&\le C\int_1^\infty \mathrm{e}^{-\lambda s}s^{-\frac12}\int_{\mathbb{R}^n}s^{-\frac n2}\mathrm{e}^{-\frac{|x-y|^2}{8s}}\Big(1+\frac{|y|^2}{1+t}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\
&\quad + C\int_0^1 \mathrm{e}^{-\lambda s}s^{-\frac12}\int_{|y-x|\le 1}s^{-\frac n2}\mathrm{e}^{-\frac{|x-y|^2}{8s}}\Big(1+\frac{|y|^2}{1+t}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\
&\quad + C\int_0^1 \mathrm{e}^{-\lambda s}s^{-\frac12}\int_{|y-x|\ge 1}s^{-\frac n2}\mathrm{e}^{-\frac{|x-y|^2}{8s}}\Big(1+\frac{|y|^2}{1+t}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\
&:= \mathrm{I}_1+\mathrm{I}_{2,1}+\mathrm{I}_{2,2}.\end{aligned}\tag{2.19}$$

We first make estimates for I_1. Since $s>1$, we have

$$\begin{aligned}s^{-\frac n2}\mathrm{e}^{-\frac{|x-y|^2}{8s}} &\le C\Big(1+\frac{|x-y|^2}{s}\Big)^{-2m}\\
&= Cs^{2m}(s+|x-y|^2)^{-2m}\\
&\le Cs^{2m}(1+|x-y|^2)^{-2m}.\end{aligned}\tag{2.20}$$

Then it follows from Lemma 2.1 that

$$\begin{aligned}\mathrm{I}_1 &\le C\int_1^\infty \mathrm{e}^{-\lambda s}s^{2m-\frac12}\int_{\mathbb{R}^n}(1+|x-y|^2)^{-2m}\Big(1+\frac{|y|^2}{1+t}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\
&\le C\int_1^\infty \mathrm{e}^{-\lambda s}s^{2m-\frac12}B_m(|x|,t)\mathrm{d}s\\
&\le CB_m(|x|,t).\end{aligned}\tag{2.21}$$

For $\mathrm{I}_{2,1}$, since $|y-x|\le 1$, it naturally holds that

$$\begin{aligned}1+\frac{|x|^2}{1+t} &= 1+\frac{|x-y+y|^2}{1+t}\\
&\le 1+\frac{2|x-y|^2}{1+t}+\frac{2|y|^2}{1+t}\\
&\le 3\Big(1+\frac{|y|^2}{1+t}\Big),\end{aligned}\tag{2.22}$$

so

$$\Big(1+\frac{|y|^2}{1+t}\Big)^{-m}\le C\Big(1+\frac{|x|^2}{1+t}\Big)^{-m},\quad \text{if } |y-x|\le 1,\tag{2.23}$$

which implies

$$\begin{aligned}\mathrm{I}_{2,1} &\le C\int_0^1 \mathrm{e}^{-\lambda s}s^{-\frac12}\int_{|y-x|\le 1}s^{-\frac n2}\mathrm{e}^{-\frac{|x-y|^2}{8s}}B_m(|x|,t)\mathrm{d}y\mathrm{d}s\\
&\le CB_m(|x|,t).\end{aligned}\tag{2.24}$$

And for $\mathrm{I}_{2,2}$, noticing that $s \in [0,1]$ and $|y-x| \geq 1$, we have

$$
\begin{aligned}
s^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{8s}} (1+|x-y|^2)^{2m} &\leq s^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{8s}} \Big(1+\frac{|x-y|^2}{s}\Big)^{2m} \\
&\leq \Big(1+\frac{|x-y|^2}{s}\Big)^{2m} \mathrm{e}^{-\frac{|x-y|^2}{8s}} \Big(\frac{|x-y|^2}{s}\Big)^{\frac{n}{2}} |x-y|^{-n} \\
&\leq \Big(1+\frac{|x-y|^2}{s}\Big)^{2m} \mathrm{e}^{-\frac{|x-y|^2}{8s}} \Big(\frac{|x-y|^2}{s}\Big)^{\frac{n}{2}} \\
&\leq C, \qquad (2.25)
\end{aligned}
$$

that is

$$
s^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{8s}} \leq C(1+|x-y|^2)^{-2m}, \quad \text{if } s \leq 1,\ |y-x| \geq 1. \qquad (2.26)
$$

Then by Lemma 2.1, we can get

$$
\begin{aligned}
\mathrm{I}_{2,2} &\leq C \int_0^1 \mathrm{e}^{-\lambda s} s^{-\frac{1}{2}} \int_{|y-x|\geq 1} (1+|x-y|^2)^{-2m} \Big(1+\frac{|y|^2}{1+t}\Big)^{-m} \mathrm{d}y\mathrm{d}s \\
&\leq C \int_0^1 \mathrm{e}^{-\lambda s} s^{-\frac{1}{2}} B_m(|x|,t)\mathrm{d}s \\
&\leq C B_m(|x|,t). \qquad (2.27)
\end{aligned}
$$

Combining all the estimates of above, we complete the proof.

The next lemma is also very important for our proof, which deals with the nonlinear parts in the equation.

Lemma 2.4 *If*

$$
|\partial_x^\alpha \widetilde{G}(x,t)| \leq C\, t^{-\frac{n+|\alpha|+1}{2}} \mathrm{e}^{-\frac{|x|^2}{8t}}, \qquad (2.28)
$$
$$
|\partial_x^\alpha S(x,t)| \leq C\,(1+t)^{-\frac{2n+|\alpha|}{2}} B_m(|x|,t), \qquad (2.29)
$$

where $m > \frac{n}{2}$, $n \geq 1$, then we have

$$
\begin{aligned}
\mathrm{I}_\alpha := \Big|\partial_x^\alpha \Big(\int_0^t \widetilde{G}(\cdot, t-s) * S(\cdot,s)\mathrm{d}s\Big)\Big| \\
\leq C\,(1+t)^{-\frac{n+|\alpha|}{2}} B_m(|x|,t). \qquad (2.30)
\end{aligned}
$$

Proof Because of the singularity of the function $\widetilde{G}(x,t)$ at $t=0$, we divide the integrating range into several parts and study them respectively. We have

$$
\begin{aligned}
\mathrm{I}_\alpha &\leq C \int_0^{\frac{t}{2}} |\partial_x^\alpha(\widetilde{G}(\cdot,t-s) * S(\cdot,s))|\mathrm{d}s + C\int_{\frac{t}{2}}^t |\widetilde{G}(\cdot,t-s) * \partial_x^\alpha S(\cdot,s)|\mathrm{d}s \\
&:= \mathrm{P} + \mathrm{Q}. \qquad (2.31)
\end{aligned}
$$

From the conditions (2.28) and (2.29), we have

$$\begin{aligned}\mathrm{P} &= C\int_0^{\frac{t}{2}} |\partial_x^\alpha \widetilde{G}(\cdot,t-s)*S(\cdot,s)|\mathrm{d}s\\ &\le C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}(t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-n}\mathrm{e}^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s, \qquad (2.32)\\ \mathrm{P} &= C\int_0^{\frac{t}{2}} |\widetilde{G}(\cdot,t-s)*\partial_x^\alpha S(\cdot,s)|\mathrm{d}s\\ &\le C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{e}^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s \qquad (2.33)\end{aligned}$$

and

$$\mathrm{Q} \le C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{e}^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s. \qquad (2.34)$$

We consider P first and study it in two different cases: $t \ge 1$ and $t \le 1$. When $t \ge 1$, noticing that $s \in [0, \frac{t}{2}]$, the following holds:

$$(t-s)^{-\frac{n+|\alpha|+1}{2}} \le C(1+t-s)^{-\frac{n+|\alpha|+1}{2}}. \qquad (2.35)$$

Therefore, it follows from (2.32) that

$$\mathrm{P} \le C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-n}\int_{\mathbb{R}^n}\mathrm{e}^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s. \qquad (2.36)$$

If $|x|^2 \le t$, it is easily shown that

$$1 \le 2^m\Big(1+\frac{|x|^2}{1+t}\Big)^{-m} = 2^m B_m(|x|,t), \qquad (2.37)$$

so

$$\begin{aligned}\mathrm{P} &\le C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-n}\int_{\mathbb{R}^n}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\ &= C(1+t)^{-\frac{n+|\alpha|+1}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2}}\int_{\mathbb{R}^n}(1+|z|^2)^{-m}\mathrm{d}z\mathrm{d}s\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\ge 1,\ |x|^2\le t. \qquad (2.38)\end{aligned}$$

Else if $|x|^2 \ge t$, we have estimates

$$\begin{aligned}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m} &\le C\Big(1+\frac{|x|^2}{1+s}\Big)^{-m}\\ &= C(1+s)^m(1+s+|x|^2)^{-m}\\ &\le C(1+s)^m\Big(1+\frac{t}{2}+\frac{|x|^2}{2}\Big)^{-m}\\ &\le C(1+s)^m(1+t+|x|^2)^{-m}\\ &\le C\Big(\frac{1+s}{1+t}\Big)^m\Big(1+\frac{|x|^2}{1+t}\Big)^{-m}, \quad \text{if } |y| \ge \frac{|x|}{2} \qquad (2.39)\end{aligned}$$

and

$$\begin{aligned} e^{-\frac{|x-y|^2}{8(t-s)}} &\le C\Big(1+\frac{|x-y|^2}{t-s}\Big)^{-m} \\ &\le C\Big(1+\frac{|x|^2}{t-s}\Big)^{-m} \\ &= C(t-s)^m(t-s+|x|^2)^{-m} \\ &\le C(t-s)^m\Big(\frac{1}{3}+\frac{t}{3}+\frac{|x|^2}{3}\Big)^{-m} \\ &\le C\Big(\frac{t-s}{1+t}\Big)^m\Big(1+\frac{|x|^2}{1+t}\Big)^{-m}, \quad \text{if } |y|\le\frac{|x|}{2}, \end{aligned} \tag{2.40}$$

where we have also used the fact of $|x|^2\ge t\ge 1$. (2.39)–(2.40) imply

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y \\ &\le C\int_{\{|y|\ge\frac{|x|}{2}\}} e^{-\frac{|x-y|^2}{8(t-s)}}\Big(\frac{1+s}{1+t}\Big)^m\Big(1+\frac{|x|^2}{1+t}\Big)^{-m}\mathrm{d}y \\ &\quad + C\int_{\{|y|\le\frac{|x|}{2}\}}\Big(\frac{t-s}{1+t}\Big)^m\Big(1+\frac{|x|^2}{1+t}\Big)^{-m}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y \\ &\le C\Big[(t-s)^{\frac{n}{2}}\Big(\frac{1+s}{1+t}\Big)^m+(1+s)^{\frac{n}{2}}\Big(\frac{t-s}{1+t}\Big)^m\Big]B_m(|x|,t) \end{aligned} \tag{2.41}$$

for $|x|^2\ge t\ge 1$. Substituting (2.41) into (2.36) and noticing that $s\in[0,\frac{t}{2}]$, we get

$$\begin{aligned} \mathrm{P} &\le CB_m(|x|,t)\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-n}\Big[(t-s)^{\frac{n}{2}}\Big(\frac{1+s}{1+t}\Big)^m+(1+s)^{\frac{n}{2}}\Big(\frac{t-s}{1+t}\Big)^m\Big]\mathrm{d}s \\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\ge 1,\ |x|^2\ge t. \end{aligned} \tag{2.42}$$

Combining (2.38) and (2.42), we have

$$\mathrm{P}\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\ge 1. \tag{2.43}$$

Now let us consider the other case $t\le 1$ for P. In this case, the following estimates hold:

$$\begin{aligned} \Big(1+\frac{|y|^2}{1+s}\Big)^{-m} &\le C\Big(1+\frac{|x|^2}{1+s}\Big)^{-m} \\ &\le C\Big(1+\frac{|x|^2}{1+t}\Big)^{-m}, \quad \text{if } |y-x|\le 1, \end{aligned} \tag{2.44}$$

$$e^{-\frac{|x-y|^2}{8(t-s)}}(t-s)^{-\frac{n+1}{2}}\le C\Big(1+|x-y|^2\Big)^{-2m}, \quad \text{if } t\le 1,\ |y-x|\ge 1, \tag{2.45}$$

where (2.44) and (2.45) can be proved in exactly the same way as (2.23) and (2.26), alternatively. We mention that (2.44) always holds for any $t\ge s\ge 0$ if $|y-x|\le 1$. From (2.33) and (2.44)–(2.45), we can obtain

$$\begin{aligned} \mathrm{P} &\le C\int_0^{\frac{t}{2}}\int_{|y-x|\le 1}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}e^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|x|^2}{1+t}\Big)^{-m}\mathrm{d}y\mathrm{d}s \\ &\quad + C\int_0^{\frac{t}{2}}\int_{|y-x|\ge 1}(1+s)^{-\frac{2n+|\alpha|}{2}}\Big(1+|x-y|^2\Big)^{-2m}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s \\ &\qquad \text{for } t\le 1. \end{aligned} \tag{2.46}$$

By using Lemma 2.1, we can get

$$\begin{aligned}\mathrm{P} &\le CB_m(|x|,t)\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{d}s + CB_m(|x|,t)\int_0^{\frac{t}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{d}s\\ &\le CB_m(|x|,t)\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\le 1.\end{aligned} \tag{2.47}$$

Combining (2.43) and (2.47), we have

$$\mathrm{P} \le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t). \tag{2.48}$$

Next we consider the other term Q. Obviously, from (2.34), we have

$$\begin{aligned}\mathrm{Q} &\le C\int_{\frac{t}{2}}^t\int_{|y-x|\le 1}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{e}^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\ &\quad + C\int_{\frac{t}{2}}^t\int_{|y-x|\ge 1}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{e}^{-\frac{|x-y|^2}{8(t-s)}}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\ &:= \mathrm{Q}_1+\mathrm{Q}_2.\end{aligned} \tag{2.49}$$

For Q_1, it follows from (2.44) that

$$\begin{aligned}\mathrm{Q}_1 &\le CB_m(|x|,t)\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\int_{|y-x|\le 1}(t-s)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}\mathrm{d}y\mathrm{d}s\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t).\end{aligned} \tag{2.50}$$

For Q_2, when $t\le 1$, it follows from (2.45) that

$$\begin{aligned}\mathrm{Q}_2 &\le C\int_{\frac{t}{2}}^t\int_{|y-x|\ge 1}(1+s)^{-\frac{2n+|\alpha|}{2}}(1+|x-y|^2)^{-2m}\Big(1+\frac{|y|^2}{1+s}\Big)^{-m}\mathrm{d}y\mathrm{d}s\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\le 1,\end{aligned} \tag{2.51}$$

where we have also used Lemma 2.1. When $t\ge 1$ and $|x|^2\le t$, by virtue of (2.37), we have

$$\begin{aligned}\mathrm{Q}_2 &\le C\int_{\frac{t}{2}}^t\int_{|y-x|\ge 1}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}\mathrm{d}y\mathrm{d}s\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\ge 1,\ |x|^2\le t.\end{aligned} \tag{2.52}$$

And when $|x|^2\ge t\ge 1$, we follow (2.41) and Lemma 2.1 to derive

$$\begin{aligned}\mathrm{Q}_2 &\le CB_m(|x|,t)\int_{\frac{t}{2}}^t(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{2n+|\alpha|}{2}}\Big[(t-s)^{\frac{n}{2}}\Big(\frac{1+s}{1+t}\Big)^m+(1+s)^{\frac{n}{2}}\Big(\frac{t-s}{1+t}\Big)^m\Big]\mathrm{d}s\\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t) \quad \text{for } t\ge 1,\ |x|^2\ge t.\end{aligned} \tag{2.53}$$

Then it follows from (2.51)–(2.53) that

$$\mathrm{Q}_2 \le C(1+t)^{-\frac{n+|\alpha|}{2}}B_m(|x|,t). \tag{2.54}$$

Combining (2.49)–(2.50) and (2.54), we have

$$\mathrm{Q} \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_m(|x|,t). \tag{2.55}$$

Finally, from (2.48) and (2.55), we obtain

$$\mathrm{I}_\alpha \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_m(|x|,t). \tag{2.56}$$

The proof is complete.

3 Proof of the Main Theorem

Now, we can prove Theorem 1.1 on the basis of the above lemmas.

Proof of Theorem 1.1 By (1.8) and Duhamel's principle, the solution u can be expressed by the formula

$$u(x,t) = G * u_0 + \int_0^t G(\cdot, t-s) * \nabla \cdot [\beta_1 u^{m_1}(\nabla K_{\lambda_1} * u) - \beta_2 u^{m_2}(\nabla K_{\lambda_2} * u)](\cdot, s)\mathrm{d}s. \tag{3.1}$$

Therefore, we consider the following successive approximation forms:

$$\begin{aligned} u_{j+1}(x,t) = u_1(x,t) + \int_0^t & G(\cdot, t-s) * \nabla \cdot [\beta_1 u_j^{m_1}(\nabla K_{\lambda_1} * u_j) \\ & - \beta_2 u_j^{m_2}(\nabla K_{\lambda_2} * u_j)](\cdot, s)\mathrm{d}s, \quad j = 1, 2, \cdots, \end{aligned} \tag{3.2}$$

where

$$u_1(x,t) = G * u_0.$$

Let

$$M_j(t) := \sup_{\substack{0 \le \tau \le t \\ |\alpha| \le h \\ x \in \mathbb{R}^n}} |\partial_x^\alpha u_j(x,\tau)| \varphi_\alpha^{-1}(x,\tau), \tag{3.3}$$

where

$$\varphi_\alpha(x,t) = (1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t). \tag{3.4}$$

From the assumption and Lemma 2.2, it holds that

$$|\partial_x^\alpha u_1| = |\partial_x^\alpha G * u_0| \le C\epsilon(1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t). \tag{3.5}$$

By Lemma 2.3, we have

$$\begin{aligned} |\partial_x^\alpha (u_j^{m_i}(\nabla K_{\lambda_i} * u_j))(x,t)| &= \Big| \sum_{\eta_1 + \cdots + \eta_{m_i} + \gamma = \alpha} \partial_x^{\eta_1} u_j \cdots \partial_x^{\eta_{m_i}} u_j \cdot (\nabla K_{\lambda_i} * \partial_x^\gamma u_j) \Big| \\ &\le C M_j^{m_i+1} \varphi_{\eta_1}(x,t) \cdots \varphi_{\eta_{m_i}}(x,t) \varphi_\gamma(x,t) \\ &= C M_j^{m_i+1} (1+t)^{-\frac{n}{2}(m_i+1) - \frac{|\alpha|}{2}} B_{(m_i+1)r}(|x|,t), \quad i = 1, 2, \end{aligned} \tag{3.6}$$

and obviously,

$$|\partial_x^\alpha (\nabla G)(x,t)| \le Ct^{-\frac{n+|\alpha|+1}{2}} \mathrm{e}^{-\frac{|x|^2}{8t}}. \tag{3.7}$$

Then by Lemma 2.4, we have

$$\begin{aligned} &\Big|\partial_x^\alpha \Big(\int_0^t G(\cdot,t-s) * \nabla\cdot (u_j^{m_i}(\nabla K_{\lambda_i} * u_j))(\cdot,s)\mathrm{d}s\Big)\Big| \\ &\le CM_j^{m_i+1}(1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t), \quad i=1,2 \end{aligned} \tag{3.8}$$

(3.2), (3.5) and (3.8) yield

$$|\partial_x^\alpha u_{j+1}(x,t)| \le C(\epsilon + M_j^{m_1+1}(t) + M_j^{m_2+1}(t))(1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t). \tag{3.9}$$

Recall the prior assumption (3.3), which holds that

$$M_{j+1}(t) \le C(\epsilon + M_j^{m_1+1}(t) + M_j^{m_2+1}(t)). \tag{3.10}$$

Now choose ϵ sufficiently small such that the equation $\mathbf{x} = C(\epsilon + \mathbf{x}^{m_1+1} + \mathbf{x}^{m_2+1})$ has a positive solution $a(\epsilon)$ satisfying $C\epsilon \le a(\epsilon) < 1$. Then we have the following uniform estimate:

$$M_j(t) \le a(\epsilon), \quad j=1,2,\cdots. \tag{3.11}$$

Therefore,

$$|\partial_x^\alpha u_j(x,t)| \le a(\epsilon)(1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t). \tag{3.12}$$

Next we show that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ for $1 \le p \le +\infty$. Let

$$\omega_{j+1}(t) := u_{j+1}(t) - u_j(t) \quad \text{for } j=1,2,\cdots$$

and

$$\omega_1(t) = u_1(t).$$

By (3.2), we have

$$\begin{aligned} &\omega_{j+1}(t) \\ &= \int_0^t \operatorname{div} G(\cdot,t-s) * \Big(\omega_j \sum_{k=1}^{m_1} u_j^{m_1-k} u_{j-1}^{k-1}\Big(\nabla K_{\lambda_1} * u_j\Big) + u_{j-1}^{m_1}\Big(\nabla K_{\lambda_1} * \omega_j\Big)\Big)(\cdot,s)\mathrm{d}s \\ &\quad - \int_0^t \operatorname{div} G(\cdot,t-s) * \Big(\omega_j \sum_{k=1}^{m_2} u_j^{m_2-k} u_{j-1}^{k-1}\Big(\nabla K_{\lambda_2} * u_j\Big) + u_{j-1}^{m_2}\Big(\nabla K_{\lambda_2} * \omega_j\Big)\Big)(\cdot,s)\mathrm{d}s. \end{aligned} \tag{3.13}$$

Let

$$a_j := \sup_{0\le t<\infty} \|\omega_j(t)\|_{L^p(\mathbb{R}^n)}.$$

Then from (3.12)–(3.13), we can deduce that

$$\begin{aligned} \| \omega_{j+1}(t) \|_{L^p(\mathbb{R}^n)} &\le C\, a(\epsilon) a_j \int_0^t (t-s)^{-\frac12}(1+s)^{-\frac{n}{2}}\mathrm{d}s \\ &\le C\, a(\epsilon) a_j \int_0^t (t-s)^{-\frac12} s^{-\frac12}\mathrm{d}s \\ &= CB\Big(\frac12,\frac12\Big)a(\epsilon)a_j \\ &= C\pi a(\epsilon)a_j, \end{aligned}$$

that is

$$a_{j+1} \leq C\pi a(\epsilon) a_j$$

with

$$a_1 = \sup_{0 \leq t < \infty} \|\omega_1(t)\|_{L^p(\mathbb{R}^n)} \leq C\epsilon.$$

Take ϵ sufficiently small, such that

$$C\pi a(\epsilon) \leq \frac{1}{2},$$

which yields that $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ $(1 \leq p \leq +\infty)$. Therefore, there exists a limit $u(t,\cdot) \in L^p(\mathbb{R}^n)$ that satisfies

$$\|u_j(t) - u(t)\|_{L^p(\mathbb{R}^n)} \to 0 \quad \text{as } j \to +\infty.$$

Now letting $j \to +\infty$ in (3.2), we see that $u(t,x)$ satisfies (3.1). By induction and a similar process, we can prove that $\{\partial_x^\alpha u_j\}_{j=1}^{\infty}$ is also a Cauchy sequence in $L^p(\mathbb{R}^n)$ for every $1 \leq p \leq +\infty$ and $|\alpha| \leq h$ and the limit is obviously the weak partial derivative of $u(x,t)$. Finally, from (3.12) we obtain

$$|\partial_x^\alpha u(x,t)| \leq a(\epsilon)(1+t)^{-\frac{n+|\alpha|}{2}} B_r(|x|,t). \tag{3.14}$$

By Lemma 2.3, the solutions $w(x,t)$ and $v(x,t)$ satisfy the same estimate as (3.14). The proof is complete.

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