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Constructing Invariant Tori for the Spatial Hill Lunar Problem

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Abstract In this paper, the spatial Hill lunar problem is investigated, and the existence of invariant tori of hyperbolic type in a neighborhood of its equilibrium is shown. Moreover, the author checks the non-degenerate condition analytically and obtains two-dimensional elliptic invariant tori on its central manifold as well.

Keywords Spatial Hill lunar problem, KAM theorem, Quasi-periodic orbits, Elliptic invariant tori
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1 Introduction and Main Results

The Hill lunar problem deals with the motions of two small masses under the mutual interactions perturbed by a massive body, and it covers some interesting astrodynamical systems, which include a satellite orbiting the planet and perturbed by the Sun. Stemming from the circular restricted three-body problem, Hill's approximation is achieved by translating the origin of the rotating reference frame to the planet and the unit of the length is scaled by $l^{\frac{1}{3}}$, where l is the mass parameter of the circular restricted three-body problem. For the planar circular Hill lunar problem, [5, 10] showed its rich dynamics by computing Poincaré surfaces of the section. In [9], Hénon discovered the main families of periodic orbits and computed the width of the stability regions. For the planar elliptic Hill lunar case, [18] established its dynamics about the periodic solutions and stability regions.

Similar Hill's approximation can be applied to the spatial Hill lunar problem (see [11] for reference). In [17], Villac derived the following normalized equation for the spatial Hill lunar problem by selecting proper length and time scales:

$$\ddot{x} - 2\dot{y} = -\frac{x}{r^3} + 3x, \tag{1.1}$$

$$\ddot{y} + 2\dot{x} = -\frac{y}{r^3}, \tag{1.2}$$

$$\ddot{z} = -\frac{z}{r^3} - z, \tag{1.3}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ denotes the magnitude of the relative position vector between the small masses. Moreover, he used this model in the space mission orbit design. In [7], accurate

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numerical procedures were developed to compute homoclinic and heteroclinic orbits in the spatial Hill lunar problem, while [6] computed the scattering map between the normally hyperbolic invariant manifolds associated to the equilibrium points in the spatial Hill lunar problem. [16] studied a single averaged model for the spatial Hill lunar problem and found some particular solutions.

Our purpose in this paper is to investigate the existence of invariant tori for the spatial Hill lunar problem. We rewrite the above spatial Hill lunar equation as a Hamiltonian system, and then its Hamiltonian takes the form

$$H = \frac{1}{2}[(p_x + y)^2 + (p_y - x)^2 + p_z^2] - \frac{1}{2}(3x^2 - z^2) - \frac{1}{r}, \tag{1.4}$$

and the corresponding 2-form is

$$\omega = \mathrm{d}x \wedge \mathrm{d}p_x + \mathrm{d}y \wedge \mathrm{d}p_y + \mathrm{d}z \wedge \mathrm{d}p_z.$$

It is natural to note that this Hamiltonian has two collinear equilibrium points along the x-axis, and we can obtain three pairs of eigenvalues of the linearized equation around the equilibrium solutions, which read

$$\lambda_1^\pm = \pm 2\mathrm{i}, \quad \lambda_2^\pm = \pm\sqrt{2\sqrt{7} - 1}\mathrm{i}, \quad \lambda_3^\pm = \pm\sqrt{2\sqrt{7} + 1}\,.$$

Thus we can put the quadratic term into the normal form

$$H_0 = 2I_1 + \sqrt{2\sqrt{7} - 1}I_2 + \sqrt{2\sqrt{7} + 1}p_3 q_3\,.$$

By the Birkhoff normal form lemma (see [3]), we can put this Hamiltonian into the partial Birkhoff normal form (that is, (2.5)), and then by making use of Moser's theorem (see [14, Theorem 5.2]), we can derive the hyperbolic invariant tori for the spatial Hill lunar problem. Moreover, the above Hamiltonian has 4-dimensional central manifold, and the standard KAM theorem guarantees the reduced Hamiltonian on its central manifold has 2-dimensional elliptic invariant tori, the motions on which are quasi-periodic. Thus our main results in this paper can be summarized as follows.

Theorem 1.1 *For the spatial Hill lunar problem* (1.4)*, there are hyperbolic invariant tori in the neighbourhood of the equilibrium, and the corresponding reduced Hamiltonian on the center manifold around the equilibrium has* 2*-dimensional elliptic invariant tori with quasi-periodic solutions along them.*

Let us make some comparisons with earlier papers. Villac [17] had never considered the reduced Hamiltonian on the central manifold, while Gómez, Marcote and Mondelo [7] proceeded with the analysis of the dynamics in the central manifold in a semi-analytical way, and provided accurate numerical procedures to compute homoclinic and heteroclinic orbits. However, in our work, we manage to check the non-degenerate condition of the standard KAM theorem analytically, and establish the existence of invariant tori in a neighbourhood of the collinear equilibrium points for the spatial Hill lunar problem.

The following paper is organized as follows. In Section 2, we derive the normal form for the spatial Hill lunar problem. In Section 3, we present the proof the our main theorem with the aid of the KAM theorem.

2 Normal Forms

In order to apply the KAM theorem to the spatial Hill lunar problem, we will derive the normal form of its Hamiltonian at the equilibrium points in this section.

2.1 The normal form for the quadratic term

Consider the spatial Hill lunar problem (1.4), and it is not hard to derive its equilibrium points

$$\begin{aligned} &L_1 : (3^{-\frac{1}{3}}, 0, 0, 3^{-\frac{1}{3}}, 0), \\ &L_2 : (-3^{-\frac{1}{3}}, 0, 0, -3^{-\frac{1}{3}}, 0). \end{aligned}$$

From now on we only take L_1 into consideration for convenience. After expanding H into Taylor series at L_1, we have

$$\begin{aligned} H = C &+ \frac{1}{2}[(p_x + y)^2 + (p_y - x)^2 + p_z^2] - \frac{1}{2}(3x^2 - z^2) \\ &+ \frac{1}{2}(-6x^2 + 3y^2 + 3z^2) \\ &- 3^{\frac{4}{3}}\Big(- x^3 + \frac{3}{2}xy^2 + \frac{3}{2}xz^2\Big) \\ &- 3^{\frac{5}{3}}\Big(x^4 - 3x^2y^2 - 3x^2z^2 + \frac{3}{4}y^2z^2 + \frac{3}{8}y^4 + \frac{3}{8}z^4\Big) + \cdots , \end{aligned} \tag{2.1}$$

where C is a dynamically irrelated constant, and we omit the terms with order higher than 4. Consider the quadratic term

$$\begin{aligned} Q &= \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + 2p_xy - 2xp_y - 8x^2 + 4y_2 + 4z^2) \\ &= \frac{1}{2}(x, y, z, p_x, p_y, p_z)S(x, y, z, p_x, p_y, p_z)^{\mathrm{T}}, \end{aligned}$$

where

$$S = \begin{bmatrix} -8 & 0 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the corresponding Hamiltonian matrix takes

$$A = JS = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 8 & 0 & 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{bmatrix}.$$

After simple computations, we derive the eigenvalues of A

$$\lambda_1^{\pm} = \pm 2\mathrm{i}, \quad \lambda_2^{\pm} = \pm\sqrt{2\sqrt{7} - 1}\mathrm{i}, \quad \lambda_3^{\pm} = \pm\sqrt{2\sqrt{7} + 1},$$

and $\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_3^-$ correspond to eigenvectors:

$$\boldsymbol{\xi_1} = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{i} \\ 0 \\ 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{\xi_2} = \begin{pmatrix} \dfrac{\sqrt{7}-3}{2}\sqrt{2\sqrt{7}-1} \\ \dfrac{(\sqrt{7}-5)\mathrm{i}}{2} \\ 0 \\ \dfrac{(11-4\sqrt{7})\mathrm{i}}{\sqrt{2\sqrt{7}-1}} \\ 0 \end{pmatrix},$$

$$\boldsymbol{\xi_3} = \begin{pmatrix} -\dfrac{(\sqrt{7}+3)\sqrt{2\sqrt{7}+1}}{2} \\ \dfrac{\sqrt{7}+5}{2} \\ 0 \\ \dfrac{-4\sqrt{7}-11}{\sqrt{2\sqrt{7}+1}} \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi_4} = \begin{pmatrix} -\dfrac{(\sqrt{7}+3)\sqrt{2\sqrt{7}+1}}{2} \\ -\dfrac{\sqrt{7}+5}{2} \\ 0 \\ \dfrac{11+4\sqrt{7}}{\sqrt{2\sqrt{7}+1}} \\ 0 \end{pmatrix}.$$

By making use of discussions about the canonical forms for Hamiltonian matrices in [12, 19], we seek a real symplectic matrix T to put A into the canonical form, such that

$$T^{-1}AT = B = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2\sqrt{7}-1} & 0 \\ 0 & 0 & \sqrt{2\sqrt{7}+1} & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2\sqrt{7}-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2\sqrt{7}+1} \end{pmatrix}.$$

Denote $T = (\boldsymbol{P_1^+}, \boldsymbol{P_2^+}, \boldsymbol{P_3^+}, \boldsymbol{P_1^-}, \boldsymbol{P_2^-}, \boldsymbol{P_3^-})$, and since $T^{\mathrm{T}}JT = J$, we have

$$\{\boldsymbol{P_i^+}, \boldsymbol{P_j^-}\} = \delta_i^j, \quad i, j = 1, 2, 3,$$

where

$$\delta_i^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

From the fact $T^{-1}AT = B$ and $AT = TB$, we have

$$\begin{cases} A\boldsymbol{P_1^+} = 2\boldsymbol{P_1^-}, \\ A\boldsymbol{P_1^-} = 2\boldsymbol{p_1^+}, \end{cases} \quad \begin{cases} A\boldsymbol{P_2^+} = -\sqrt{2\sqrt{7}-1}\boldsymbol{P_2^-}, \\ A\boldsymbol{P_2^-} = \sqrt{2\sqrt{7}-1}\boldsymbol{P_2^+}, \end{cases} \quad \begin{cases} A\boldsymbol{P_3^+} = \sqrt{2\sqrt{7}+1}\boldsymbol{P_3^+}, \\ A\boldsymbol{P_3^-} = -\sqrt{2\sqrt{7}+1}\boldsymbol{P_3^-}. \end{cases}$$

Denote $\boldsymbol{\xi_1} = \boldsymbol{u_1} + \mathrm{i}\boldsymbol{v_1}$. Then we have

$$A\boldsymbol{u_1} = -2\boldsymbol{v_1}, \quad A\boldsymbol{v_1} = 2\boldsymbol{u_1}, \quad \{\boldsymbol{u_1}, \boldsymbol{v_1}\} = 2,$$

and thus

$$\boldsymbol{P_1^+} = \frac{1}{\sqrt{2}}\boldsymbol{u_1}, \quad \boldsymbol{P_1^-} = \frac{1}{\sqrt{2}}\boldsymbol{v_1}.$$

We can construct $\boldsymbol{P_2^+}, \boldsymbol{P_2^-}, \boldsymbol{P_3^+}, \boldsymbol{P_3^-}$ similarly, so T takes the following form:

$$\begin{bmatrix} 0 & \frac{(\sqrt{7}-3)(2\sqrt{7}-1)^{\frac{1}{4}}}{2\sqrt{11\sqrt{7}-28}} & -\frac{(\sqrt{7}+3)\sqrt{2\sqrt{7}+1}}{2} & 0 & 0 & \frac{\sqrt{7}+3}{4(28+11\sqrt{7})} \\ 0 & 0 & \frac{\sqrt{7}+5}{2} & 0 & \frac{(\sqrt{7}-5)(2\sqrt{7}-1)^{-\frac{1}{4}}}{2\sqrt{11\sqrt{7}-28}} & \frac{\sqrt{7}+5}{4(28+11\sqrt{7})\sqrt{2\sqrt{7}+1}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -4\sqrt{7}-11 & 0 & \frac{(11-4\sqrt{7})(2\sqrt{7}-1)^{-\frac{1}{4}}}{\sqrt{11\sqrt{7}-28}} & -\frac{4\sqrt{7}+11}{2(28+11\sqrt{7})\sqrt{2\sqrt{7}+1}} \\ 0 & \frac{(2\sqrt{7}-1)^{\frac{1}{4}}}{\sqrt{11\sqrt{7}-28}} & \sqrt{2\sqrt{7}+1} & 0 & 0 & -\frac{1}{2(28+11\sqrt{7})} \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we can derive the normal form for the quadratic part of H,

$$\begin{aligned} Q &= \frac{1}{2}\boldsymbol{u}^{\mathrm{T}} S \boldsymbol{u} \\ &\overset{\boldsymbol{u}=T\boldsymbol{w}}{=} \frac{1}{2}\boldsymbol{w}^{\mathrm{T}}(T^{\mathrm{T}} S T)\boldsymbol{w} \\ &= \frac{1}{2}\boldsymbol{w}^{\mathrm{T}}(J^{\mathrm{T}} T^{-1} A T)\boldsymbol{w} \\ &= \frac{1}{2}\boldsymbol{w}^{\mathrm{T}}(J^{\mathrm{T}} B)\boldsymbol{w} \\ &= 2\cdot\frac{q_1^2+p_1^2}{2} + \sqrt{2\sqrt{7}-1}\cdot\frac{q_2^2+p_2^2}{2} + \sqrt{2\sqrt{7}+1}p_3 q_3, \end{aligned}$$

where

$$\boldsymbol{u} = (x, y, z, p_x, p_y, p_z)^{\mathrm{T}}, \quad \boldsymbol{w} = (p_1, p_2, p_3, q_1, q_2, q_3)^{\mathrm{T}}.$$

Next, we introduce a symplectic transformation

$$(p_1, p_2, p_3, q_1, q_2, q_3) \longmapsto (I_1, I_2, p_3, \varphi_1, \varphi_2, q_3),$$

where

$$p_k = \sqrt{I_k}\mathrm{e}^{\mathrm{i}\varphi_k}, \quad q_k = \mathrm{i}\sqrt{I_k}\mathrm{e}^{-\mathrm{i}\varphi_k}, \quad k = 1, 2.$$

Then Q takes the form

$$Q = 2I_1 + \sqrt{2\sqrt{7}-1}I_2 + \sqrt{2\sqrt{7}+1}p_3 q_3.$$

2.2 The partial Birkhoff normal form

To derive the partial Birkhoff normal form, we introduce the symplectic coordinates

$$p_k = \frac{1}{\sqrt{2}}(u_k - \mathrm{i}v_k), \quad q_k = \frac{1}{\sqrt{2}}(-\mathrm{i}u_k + v_k), \quad p_3 = p_3, \quad q_3 = q_3, \quad k = 1, 2.$$

Then

$$I_1 = -\mathrm{i}u_1 v_1, \quad I_2 = -\mathrm{i}u_2 v_2.$$

Under these variables, we have

$$\begin{cases} x = \dfrac{(\sqrt{7}-3)(2\sqrt{7}-1)^{\frac{1}{4}}}{2\sqrt{22\sqrt{7}-56}}(u_2 - \mathrm{i}v_2) - \dfrac{(\sqrt{7}+3)\sqrt{2\sqrt{7}+1}}{2}p_3 + \dfrac{\sqrt{7}+3}{4(28+11\sqrt{7})}q_3, \\ y = \dfrac{\sqrt{7}+5}{2}p_3 + \dfrac{(\sqrt{7}-5)(2\sqrt{7}-1)^{-\frac{1}{4}}}{2\sqrt{22\sqrt{7}-56}}(v_2 - \mathrm{i}u_2) + \dfrac{\sqrt{7}+5}{4(28+11\sqrt{7})\sqrt{2\sqrt{7}+1}}q_3, \\ z = -\dfrac{1}{\sqrt{2}}q_1 = -\dfrac{1}{2}(v_1 - \mathrm{i}u_1). \end{cases}$$

It is not difficult to note the fact that there are no integers k_1, k_2, k_3 satisfying

$$0 < |k_1| + |k_2| + |k_3| \leq 4,$$

such that

$$2k_1 + \sqrt{2\sqrt{7}-1}k_2 + \sqrt{2\sqrt{7}+1}k_3 = 0.$$

So by the Birkhoff normal form lemma (see [3]), we can put H into the partial Birkhoff normal form. As a result, we can find Hamiltonian $W = W_1 + W_2$, where W_i denotes the homogeneous polynomial of order $i+2$ about $u_1, v_1, u_2, v_2, p_3, q_3$, such that $H \circ X_W^1$ is in the partial Birkhoff normal form, where X_W^1 denotes the time-1 map of Hamiltonian vector field X_W.

To proceed, we introduce some notations. Denote

$$A = \frac{(\sqrt{7}-3)(2\sqrt{7}-1)^{\frac{1}{4}}}{2\sqrt{22\sqrt{7}-56}}, \quad B = -\frac{\sqrt{7}+3}{2}, \quad C = \frac{\sqrt{7}+3}{428+11\sqrt{7}}, \quad D = \frac{\sqrt{7}+5}{2},$$

$$E = \frac{(\sqrt{7}-5)(2\sqrt{7}-1)^{-\frac{1}{4}}}{2\sqrt{22\sqrt{7}-56}}, \quad F = \frac{\sqrt{7}+5}{4(28+11\sqrt{7})\sqrt{2\sqrt{7}+1}},$$

$$H = \sum_{i=0}^{\infty} \frac{1}{i!} H_i^0,$$

where H_i^0 denotes the homogeneous polynomial of order $i+2$. Then we have

$$\begin{aligned}
H_0^0 &= -2\mathrm{i}u_1v_1 - \sqrt{2\sqrt{7}-1}\mathrm{i}u_2v_2 + \sqrt{2\sqrt{7}+1}p_3q_3, \\
H_1^0 &= 3^{\frac{4}{3}}\Big[\Big(A^3 + \frac{3}{2}AE^2\Big)u_2^3 + \Big(-3\mathrm{i}A^3 + \frac{3}{2}\mathrm{i}AE^2\Big)u_2^2v_2 + \Big(-3A^3 + \frac{3}{2}AE^2\Big)u_2v_2^2 \\
&\quad + \Big(\mathrm{i}A^3 + \frac{3}{2}\mathrm{i}AE^2\Big)v_2^3 + \Big(3A^2C + 3\mathrm{i}AEF + \frac{3}{2}CE^2\Big)u_2^2q_3 + \frac{3}{8}Bp_3u_1^2 \\
&\quad + \Big(-3A^2C - \frac{3}{2}CE^2 + 3\mathrm{i}AEF\Big)v_2^2q_3 + \Big(3A^2B + \frac{3}{2}BE^2 + 3\mathrm{i}ADE\Big)u_2^2p_3 \\
&\quad + (-6\mathrm{i}A^2B + 3\mathrm{i}BE^2)u_2v_2p_3 + \Big(-3A^2B + 3\mathrm{i}ADE - \frac{3}{2}BE^2\Big)v_2^2p_3 \\
&\quad + \Big(3AC^2 - \frac{3}{2}AF^2 + 3\mathrm{i}CEF\Big)u_2q_3^2 + \Big(C^3 - \frac{3}{2}CF^2\Big)q_3^3 + \frac{3}{4}\mathrm{i}Cu_1v_1q_3 \\
&\quad + (6ABC - 3ADF + 3\mathrm{i}BEF + 3\mathrm{i}CDE)u_2p_3q_3 + (-6\mathrm{i}A^2C + 3\mathrm{i}CE^2)u_2v_2q_3 \\
&\quad + \frac{3}{4}\mathrm{i}Bu_1v_1p_3 + \frac{3}{8}Cu_1^2q_3 + (-6\mathrm{i}ABC + 3\mathrm{i}ADF - 3BEF - 3CDE)v_2p_3q_3 \\
&\quad + \frac{3}{4}A\mathrm{i}u_1u_2v_1 - \frac{3}{8}A\mathrm{i}u_1^2v_2 + \Big(3BC^2 - \frac{3}{2}BF^2 - 3CDF\Big)p_3q_3^2 + \frac{3}{8}Au_1^2u_2 \\
&\quad + \Big(3B^2C - 3BDF - \frac{3}{2}CD^2\Big)p_3^2q_3 - \frac{3}{8}Au_2v_1^2 + \Big(B^3 - \frac{3}{2}BD^2\Big)p_3^3 + \frac{3}{8}A\mathrm{i}v_1^2v_2 \\
&\quad + \Big(-3\mathrm{i}AC^2 + \frac{3}{2}\mathrm{i}AF^2 - 3CEF\Big)v_2q_3^2 + \Big(-3\mathrm{i}AB^2 + \frac{3}{2}\mathrm{i}AD^2 - 3BDE\Big)v_2p_3^2 \\
&\quad + \Big(3AB^2 - \frac{3}{2}AD^2 + 3\mathrm{i}BDE\Big)u_2p_3^2 - \frac{3}{8}Cv_1^2q_3 - \frac{3}{8}Bp_3v_1^2 + \frac{3}{4}Au_1v_1v_2\Big], \\
H_2^0 &= 2\cdot 3^{\frac{5}{3}}\Big[(24\mathrm{i}A^2BC_9\mathrm{i}DE^2F - 12\mathrm{i}A^2DF - 12\mathrm{i}BCE^2)u_2v_2p_3q_3 \\
&\quad - \Big(6B^2C^2 + \frac{9}{4}D^2F^2 - 3B^2F^2 - 12BCDF\Big)p_3^2q_3^2 + \frac{9}{64}u_1^2v_1^2
\end{aligned}$$

$$+\Big(6A^4+\frac{9}{4}E^4-6A^2E^2\Big)u_2^2v_2^2-\Big(3A^2-\frac{3}{4}E^2\Big)u_1v_1u_2v_2$$
$$-\Big(3\mathrm{i}BC-\frac{3}{4}\mathrm{i}DF\Big)u_1v_1p_3q_3+\cdots\Big].$$

From

$$\begin{aligned}H\circ X_W^1&=H_0^0\circ X_W^1+H_1^0\circ X_W^1+\frac{1}{2}H_2^0\circ X_W^1+\cdots\\&=H_0^0+\{H_0^0,W\}+\frac{1}{2}\{\{H_0^0,W\},W\}+H_1^0+\{H_1^0,W\}+\frac{1}{2}H_2^0+\cdots\\&=H_0^0+\{H_0^0,W_1\}+H_1^0+\frac{1}{2}[H_2^0+\{\{H_0^0,W_1\},W_1\}+2\{H_1^0,W_1\}+2\{H_0^0,W_2\}]+\cdots,\end{aligned}$$

we have

$$H_0^1=\{H_0^0,W_1\}+H_1^0=0, \tag{2.2}$$
$$H_0^2=H_2^0+\{\{H_0^0,W_1\},W_1\}+2\{H_1^0,W_1\}+2\{H_0^0,W_2\}. \tag{2.3}$$

Assume

$$W_1=\sum_{\substack{a_1+a_2+a_3\\+b_1+b_2+b_3=3}}\gamma^{a_1a_2a_3}_{b_1b_2b_3}u_1^{a_1}u_2^{a_2}p_3^{a_3}v_1^{b_1}v_2^{b_2}q_3^{b_3}.$$

Then

$$\begin{aligned}\{W_1,H_0^0\}&=\sum_{\substack{a_1+a_2+a_3\\+b_1+b_2+b_3=3}}\gamma^{a_1a_2a_3}_{b_1b_2b_3}\frac{u_1^{a_1}u_2^{a_2}p_3^{a_3}v_1^{b_1}v_2^{b_2}q_3^{b_3}}{2i(b_1-a_1)+\sqrt{2\sqrt{7}-1}\mathrm{i}(b_2-a_2)+\sqrt{2\sqrt{7}+1}(a_3-b_3)}\\&=H_1^0.\end{aligned}$$

Thus we can solve the above equation and get

$$\begin{aligned}W_1=3^{\frac{4}{3}}\Big[&\frac{(A^3+\frac{3}{2}AE^2)u_2^3}{-3\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{(-3\mathrm{i}A^3+\frac{3}{2}\mathrm{i}AE^2)u_2^2v_2}{-\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{(-3A^3+\frac{3}{2}AE^2)u_2v_2^2}{\sqrt{2\sqrt{7}-1}\mathrm{i}}\\&+\frac{(\mathrm{i}A^3+\frac{3}{2}\mathrm{i}AE^2)v_2^3}{3\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{(3A^2C+3\mathrm{i}AEF+\frac{3}{2}CE^2)u_2^2q_3}{-2\sqrt{2\sqrt{7}-1}\mathrm{i}-\sqrt{2\sqrt{7}+1}}+\frac{(-6\mathrm{i}A^2C+3\mathrm{i}CE^2)u_2v_2q_3}{-\sqrt{2\sqrt{7}+1}}\\&+\frac{(-3A^2C-\frac{3}{2}CE^2+3\mathrm{i}AEF)v_2^2q_3}{2\sqrt{2\sqrt{7}-1}\mathrm{i}-\sqrt{2\sqrt{7}+1}}+\frac{(3A^2B+\frac{3}{2}BE^2+3\mathrm{i}ADE)u_2^2p_3}{-2\sqrt{2\sqrt{7}-1}\mathrm{i}+\sqrt{2\sqrt{7}+1}}\\&+\frac{(-3A^2B+3\mathrm{i}ADE-\frac{3}{2}BE^2)v_2^2p_3}{2\sqrt{2\sqrt{7}-1}\mathrm{i}+\sqrt{2\sqrt{7}+1}}+\frac{(3A^2C-\frac{3}{2}AF^2+3\mathrm{i}CEF)u_2q_3^2}{-\sqrt{2\sqrt{7}-1}\mathrm{i}-2\sqrt{2\sqrt{7}+1}}\\&+\frac{(6ABC-3ADF+3\mathrm{i}BEF+3\mathrm{i}CDE)u_2p_3q_3}{-\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{(-6\mathrm{i}A^2B+3\mathrm{i}BE^2)u_2v_2p_3}{\sqrt{2\sqrt{7}+1}}\\&+\frac{(-6\mathrm{i}ABC+3\mathrm{i}ADF-3BEF-3CDE)v_2p_3q_3}{\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{(-3\mathrm{i}AC^2+\frac{3}{2}\mathrm{i}AF^2-3CEF)v_2q_3^2}{\sqrt{2\sqrt{7}-1}\mathrm{i}-2\sqrt{2\sqrt{7}+1}}\\&+\frac{(3AB^2-\frac{3}{2}AD^2+3\mathrm{i}BDE)u_2p_3^2}{-\sqrt{2\sqrt{7}-1}\mathrm{i}+2\sqrt{2\sqrt{7}+1}}+\frac{(-3\mathrm{i}AB^2+\frac{3}{2}\mathrm{i}AD^2-3BDE)v_2p_3^2}{\sqrt{2\sqrt{7}-1}\mathrm{i}+2\sqrt{2\sqrt{7}+1}}\\&+\frac{(C^3-\frac{3}{2}CF^2)q_3^3}{-3\sqrt{2\sqrt{7}+1}}+\frac{(3BC^2-\frac{3}{2}BF^2-3CDF)p_3q_3^2}{-\sqrt{2\sqrt{7}+1}}+\frac{(3B^2C-3BDF-\frac{3}{2}CD^2)p_3^2q_3}{\sqrt{2\sqrt{7}+1}}\end{aligned}$$

$$+\frac{(B^3-\frac{3}{2}BD^2)p_3^3}{3\sqrt{2\sqrt{7}+1}}-\frac{\frac{3}{8}Au_2v_1^2}{4\mathrm{i}-\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{\frac{3}{8}Au_1^2u_2}{-4\mathrm{i}-\sqrt{2\sqrt{7}-1}\mathrm{i}}+\frac{\frac{3}{4}A\mathrm{i}u_1u_2v_1}{-\sqrt{2\sqrt{7}-1}\mathrm{i}}$$
$$-\frac{\frac{3}{8}A\mathrm{i}u_1^2v_2}{\sqrt{2\sqrt{7}i-4i}}+\frac{\frac{3}{4}Au_1v_1v_2}{\sqrt{2\sqrt{7}-1}\mathrm{i}}-\frac{\frac{3}{8}Bv_1^2p_3}{\sqrt{2\sqrt{7}+4\mathrm{i}}}+\frac{\frac{3}{8}Bu_1^2p_3}{\sqrt{2\sqrt{7}+1}-4\mathrm{i}}+\frac{\frac{3}{4}\mathrm{i}Bu_1v_1p_3}{\sqrt{2\sqrt{7}+1}}$$
$$-\frac{\frac{3}{8}Cv_1^2q_3}{4\mathrm{i}-\sqrt{2\sqrt{7}+1}}+\frac{\frac{3}{8}Cu_1^2q_3}{-4\mathrm{i}-\sqrt{2\sqrt{7}+1}}+\frac{\frac{3}{4}C\mathrm{i}u_1v_1q_3}{-\sqrt{2\sqrt{7}+1}}+\frac{\frac{3}{8}A\mathrm{i}v_1^2v_2}{\sqrt{2\sqrt{7}-1}\mathrm{i}+4\mathrm{i}}\Big].$$

Meanwhile, (2.3) turns into

$$H_0^2=H_2^0+\{H_1^0,W_1\}+2\{H_0^0,W_2\},\tag{2.4}$$

and then by the Birkhoff normal form lemma (see [3]), we can derive W_2 similarly such that there are only homogeneous monomials about $u_1v_1,\ u_2v_2,\ p_3q_3$ in H_0^2. Thus we have

$$H_0^2=a_{11}u_1^2v_1^2+a_{12}u_1v_1u_2v_2+a_{22}u_2^2v_2^2+\alpha\mathrm{i}u_1v_1p_3q_3+\beta\mathrm{i}u_2v_2p_3q_3+\gamma p_3^2q_3^2,$$

where

$$\begin{aligned}
a_{11}&=3^{\frac{8}{3}}\Big(\frac{3}{32}+\frac{\frac{9}{16}A^2\sqrt{2\sqrt{7}-1}}{2\sqrt{7}-17}+\frac{\frac{9}{16}BC\sqrt{2\sqrt{7}+1}}{2\sqrt{7}+17}+\frac{\frac{9}{8}A^2}{\sqrt{2\sqrt{7}-1}}+\frac{\frac{9}{8}BC}{\sqrt{2\sqrt{7}+1}}\Big)\\
&=3^{\frac{8}{3}}\cdot\frac{1}{58},\\
a_{12}&=3^{\frac{8}{3}}\Big(\frac{9BCE^2-18A^2BC}{\sqrt{2\sqrt{7}+1}}-2A^2+\frac{1}{2}E^2+\frac{9A^2}{17-2\sqrt{7}}+\frac{9A^2E^2-18A^4}{\sqrt{2\sqrt{7}-1}}\Big)\\
&=3^{\frac{8}{3}}\cdot\frac{(70-3\sqrt{7})\sqrt{2\sqrt{7}-1}}{3654},\\
a_{22}&=3^{\frac{8}{3}}\Big[\frac{60A^6-36A^4E^2+27A^2E^4}{\sqrt{2\sqrt{7}-1}}+\frac{72A^4BC+18BCE^4-72A^2BCE^2}{\sqrt{2\sqrt{7}+1}}\\
&\quad+\frac{18}{3-10\sqrt{7}}\Big(-2\sqrt{2\sqrt{7}+1}A^4BC-4\sqrt{2\sqrt{7}-1}A^3BEF-2\sqrt{2\sqrt{7}+1}A^2BCE^2\\
&\quad-\frac{\sqrt{2\sqrt{7}+1}BCE^4}{2}-2\sqrt{2\sqrt{7}-1}ABE^3F+4\sqrt{2\sqrt{7}-1}A^3CDE\\
&\quad-2\sqrt{2\sqrt{7}+1}A^2DE^2F+2\sqrt{2\sqrt{7}-1}ACDE^3\Big)+4A^4+\frac{3E^4}{2}-4A^2E^2\Big]\\
&=3^{\frac{8}{3}}\frac{252218\sqrt{7}-246617}{6500928},
\end{aligned}$$

and α,β,γ are real numbers.

Hence, the partial Birkhoff normal form of the spatial Hill lunar problem is

$$\begin{aligned}
H_{\mathrm{new}}&=2I_1+\sqrt{2\sqrt{7}-1}I_2+\sqrt{2\sqrt{7}+1}p_3q_3\\
&\quad+\frac{1}{2}(-a_{11}I_1^2-a_{12}I_1I_2-a_{22}I_2^2)-\alpha I_1p_3q_3-\beta I_2p_3q_3+\gamma p_3^2q_3^2\\
&\quad+H^+(I_1,I_2,\varphi_1,\varphi_2,p_3,q_3),
\end{aligned}\tag{2.5}$$

where $\alpha,\beta,\gamma,a_{11},a_{12},a_{22}$ are real numbers.

3 Proof of Theorem 1.1

To prove the first part of our main theorem, we investigate the Hamiltonian $H_{\rm new} = H_0 + H^+$ in the neighbourhood of the equilibrium point L_1, where

$$\begin{aligned} H_0 &= F(I_1, I_2) + G(I_1, I_2, p_3, q_3), \\ F(I_1, I_2) &= 2I_1 + \sqrt{2\sqrt{7}-1}I_2 + \frac{1}{2}(-a_{11}I_1^2 - a_{12}I_1I_2 - a_{22}I_2^2), \\ G(I_1, I_2, p_3, q_3) &= \sqrt{2\sqrt{7}+1}p_3q_3 - \alpha I_1p_3q_3 - \beta I_2p_3q_3 + \gamma p_3^2q_3^2. \end{aligned}$$

Treat H^+ as a perturbation of H_0, for the non-perturbed Hamiltonian H_0, which has 2-dimensional invariant tori

$$\begin{aligned} \varphi_1 &= \Big(-2 + a_{11}I_1^0 + \frac{1}{2}a_{12}I_2^0 \Big)t, \quad I_1 = I_1^0,\ p_3 = 0, \\ \varphi_2 &= \Big(-\sqrt{2\sqrt{7}-1} + \frac{1}{2}a_{12}I_1^0 + a_{22}I_2^0 \Big)t, \quad I_2 = I_2^0,\ q_3 = 0. \end{aligned}$$

The Moser's theorem (see [14, Theorem 5.2]) ensures the preservation of those invariant tori with a slight deformation for the perturbed $H_{\rm new}$. Next we need to check the conditions of Moser's theorem:

(1) $H_{\rm new}$ is 2π periodic in $\varphi_i,\ i = 1, 2$, and

$$G(I_1, I_2, 0, 0) = G_{p_3}(I_1, I_2, 0, 0) = G_{q_3}(I_1, I_2, 0, 0) = 0.$$

(2) (The non-degenerate condition) If $p_3 = q_3 = 0$, then

$$\det\Big(\frac{\partial^2 F}{\partial I^2}\Big) = a_{11}a_{22} - \frac{1}{4}a_{12}^2 \approx 0.496 \neq 0.$$

(3) When $p_3 = q_3 = 0,\ I = (I_1, I_2)$ belongs to some open set in $\mathbb{R}^2$,

$$\begin{pmatrix} G_{q_3p_3} & G_{q_3q_3} \\ -G_{p_3p_3} & -G_{q_3p_3} \end{pmatrix}$$

is a diagonal matrix with real entries, so it does not have any pure imagine eigenvalue.

Hence all the conditions in Moser's theorem are fulfilled, which makes sure that the Moser's theorem can be applied. To proceed with the proof of the second part in the main theorem, we need the following KAM theorem.

Lemma 3.1 (see [2]) *Consider a Hamiltonian system with n degrees of freedom in a neighbourhood of an equilibrium point located at the origin, and $\omega_1, \cdots, \omega_n$ are its eigenfrequencies. Suppose that the Hamiltonian function has the following Birkhoff normal form:*

$$\begin{aligned} H &= H_0(\tau) + \text{h.o.t}, \\ H_0(\tau) &= \sum_{j=1}^{n} \omega_j\tau_j + \frac{1}{2}\sum_{i,j=1}^{n} \omega_{ij}\tau_i\tau_j, \quad \tau_i = \frac{1}{2}(p_i^2 + q_i^2). \end{aligned}$$

Then,

(i) *if the system fulfills the non-degenerate condition*

$$\det\left(\frac{\partial^2 H_0}{\partial \tau^2}\right) = \det(\omega_{ij}) \neq 0,$$

then, the Hamiltonian system has invariant tori close to the tori of the linearized system, and these tori form a set whose relative measure tends to 1 *as it tends to the origin.*

(ii) *if the iso-energetically non-degenerate condition*

$$\det\begin{pmatrix} \dfrac{\partial^2 H_0}{\partial \tau^2} & \dfrac{\partial H_0}{\partial \tau} \\ \dfrac{\partial H_0}{\partial \tau}^{\mathrm{T}} & 0 \end{pmatrix}_0 = \det\begin{pmatrix} \omega_{ij} & \omega_i \\ \omega_i & 0 \end{pmatrix} \neq 0$$

holds, the Hamiltonian system also has invariant tori, and such tori occupy a larger part of each energy level passing near the equilibrium position.

Remark 3.1 For the proof of this KAM theorem, see [1, 15].

The first step is to study the spatial Hill lunar problem on its central manifold. Under the symplectic variables $(I_1, I_2, \varphi_1, \varphi_2, p_3, q_3)$, the Hamiltonian for the spatial Hill lunar problem takes the form

$$H_{\text{new}} = H_2(I_1, I_2, p_3, q_3) + H_4(I_1, I_2, p_3, q_3) + H^+(I_1, I_2, \varphi_1, \varphi_2, p_3, q_3),$$

where

$$\begin{aligned} H_2(I_1, I_2, p_3, q_3) &= 2I_1 + \sqrt{2\sqrt{7}-1}I_2 + \sqrt{2\sqrt{7}+1}p_3 q_3, \\ H_4(I_1, I_2, p_3, q_3) &= \frac{1}{2}(-a_{11}I_1^2 - a_{12}I_1 I_2 - a_{22}I_2^2) - \alpha I_1 p_3 q_3 - \beta I_2 p_3 q_3 + \gamma p_3^2 q_3^2, \end{aligned}$$

and $H^+(I_1, I_2, \varphi_1, \varphi_2, p_3, q_3)$ denotes the terms with order higher than 4.

From the discussions about the central manifold in [4, 8, 13], we know that the spatial Hill lunar problem has the 4-dimensional central manifold at L_1, and moreover, a Hamiltonian reduced on its central manifold is still a Hamiltonian. Furthermore, if we take $p_3 = 0$, and $q_3 = 0$, we will obtain the reduced Hamiltonian on this central manifold, that is,

$$H_{\text{c}}(I_1, I_2, \varphi_1, \varphi_2) = H_{\text{c}0}(I_1, I_2) + H_{\text{c}1}(I_1, I_2, \varphi_1, \varphi_2),$$

where

$$H_{\text{c}0}(I_1, I_2) = 2I_1 + \sqrt{2\sqrt{7}-1}I_2 + \frac{1}{2}(-a_{11}I_1^2 - a_{12}I_1 I_2 - a_{22}I_2^2).$$

For the integral part $H_{\text{c}0}$, it has 2-dimensional elliptic invariant tori

$$\Pi \triangleq \{I_1 = \text{constant}\} \times \{I_2 = \text{constant}\}.$$

In the neighbourhood of the equilibrium point, we can treat $H_{\text{c}1}(I_1, I_2, \varphi_1, \varphi_2)$ as a small perturbation of $H_{\text{c}0}(I_1, I_2)$, and the above standard KAM theorem guarantees the preservation of the elliptic invariant tori Π.

The second step is to check the non-degenerate conditions:

(1) (the non-degenerate condition)

$$D_1 = \det\Big(\frac{\partial^2 H_{c0}}{\partial I^2}\Big) = \begin{vmatrix} -a_{11} & -\frac{1}{2}a_{12} \\ -\frac{1}{2}a_{12} & -a_{22} \end{vmatrix} = a_{11}a_{22} - \frac{1}{4}a_{12}^2 \approx 0.496 \neq 0,$$

(2) (the iso-energetically non-degenerate condition)

$$D_2 = \det \begin{pmatrix} \frac{\partial^2 H_{c0}}{\partial \tau^2} & \frac{\partial H_{c0}}{\partial \tau} \\ \frac{\partial H_{c0}}{\partial \tau}^{\mathrm{T}} & 0 \end{pmatrix}_0 .$$

Since D_2 continously depends on I_1, I_2, we only need to check the condition at the equilibrium

$$D_2 = \sqrt{2\sqrt{7}-1}a_{11} - (2\sqrt{7}-1)a_{12} + 4a_{22} \neq 0.$$

Thus Lemma 3.1 holds true, which guarantees the existence of 2-dimensional elliptic invariant tori for the reduced Hamiltonian of the spatial Hill lunar problem on its central manifold. Moreover, along these tori, the motions are quasi-periodic

$$\varphi_1(t) = \Big(-2 + a_{11}I_1^0 + \frac{1}{2}a_{12}I_2^0\Big)t, \tag{3.1}$$

$$\varphi_2(t) = \Big(-\sqrt{2\sqrt{7}-1} + \frac{1}{2}a_{12}I_1^0 + a_{22}I_2^0\Big)t. \tag{3.2}$$

Finally, we finish the proof of Theorem 1.1.

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