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Kinematic Formulas of Total Mean Curvatures for Hypersurfaces*

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Abstract By using the moving frame method, the authors obtain a kind of asymmetric kinematic formulas for the total mean curvatures of hypersurfaces in the n-dimensional Euclidean space.

Keywords Kinematic formula, Total mean curvature, Hypersurface
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1 Introduction

The kinematic formulas are the most beautiful and also useful formulas in integral geometry. At the beginning of his classical paper [4] Chern said: "One of the basic problems in integral geometry is to find explicit formulas for the integrals of geometric quantities over the kinematic density in terms of known integral invariants."

For instance, Chern proved in [3] the fundamental kinematic formula in the n-dimensional Euclidean space $\mathbb{E}^n$. Let D_0 and D_1 be two domains with smooth boundaries ∂D_0 and ∂D_1, respectively, in $\mathbb{E}^n$. If we denote by G the group of rigid motions of $\mathbb{E}^n$ with density $\mathrm{d}g$, and by O_{n-1} the volume of the unit sphere S^{n-1} in $\mathbb{E}^n$, then the fundamental kinematic formula is

$$\begin{aligned}&\int_{\{g\in G\,|\,D_0\cap gD_1\neq\emptyset\}}\chi(D_0\cap gD_1)\mathrm{d}g\\&=O_{n-2}\cdots O_1\Big[O_{n-1}\chi(D_0)\sigma(D_1)+O_{n-1}\chi(D_1)\sigma(D_0)\\&\quad+\frac{1}{n}\sum_{h=0}^{n-2}\binom{n}{h+1}\widetilde{H}_h(\partial D_0)\widetilde{H}_{n-2-h}(\partial D_1)\Big],\end{aligned}$$

where $\chi(\cdot)$ denotes the Euler characteristic, $\sigma(\cdot)$ the volume and $\widetilde{H}_i$ the i^{th} total mean curvature.

In [4], Chern gave the integral formulas of the quantities which appear in Weyl's formula for the volume of tubes. Let M_0 and M_1 be two closed smooth submanifolds of dimension p

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and q respectively in $\mathbb{E}^n$, and then

$$\int_{\{g\in G|M_0\cap gM_1\neq\emptyset\}}\mu_e(M_0\cap gM_1)\mathrm{d}g=\sum_{\substack{i=0\\ i\text{is even}}}^{e}c_i\mu_i(M_0)\mu_{e-i}(M_1),\tag{1.1}$$

where the quantities μ_e appear in Weyl's formula for the volume of tubes (see [14]), $0\le e\le p+q-n$ and e is even. The coefficients c_i are constants depending on $n,\ p,\ q$ and e.

These formulas can also be found in books [11–12].

In his paper [18], Zhou obtained the kinematic formulas for mean curvature power integrals in the n-dimensional Euclidean space, which are the generalization of the formulas of the 3-dimensional case in [1, 16] and are of the extrinsic type. Let S_i $(i=0,1)$ be two closed smooth hypersurfaces in $\mathbb{E}^n$. Denote by H the mean curvature of $S_0\cap gS_1$, and $0\le 2k\le n-1$. Then (see [18])

$$\int_{\{g\in G|S_0\cap gS_1\neq\emptyset\}}\Big(\int_{S_0\cap gS_1}H^{2k}\mathrm{d}\sigma\Big)\mathrm{d}g=\sum_{\substack{p,q,l\\ p+q+l=k\\ l\text{ is even}}}c_{pqkln}\widetilde{\kappa}_n^{l+2q}(S_0)\widetilde{\kappa}_n^{l+2p}(S_1),$$

where c_{pqkln} are constants depending on the indices, $\widetilde{\kappa}_n^{l+2q}(S_i)$ is a kind of total curvature of $S_i,\ i=0,1$. This is a remarkable work in which the moving frame method is effectively used and a successful application of the kinematic formulas is given.

The novel approach to study the containment problem and geometric inequalities by using kinematic formulas has been systematically developed. For the recent developments, interested readers can refer to [8–10, 15–23]. We also recommend the books [11–12] for the classical results of integral geometry and its applications.

In fact, integral geometry can be set up within the framework of the theory of homogeneous spaces.

Let G be a unimodular Lie group with kinematic density $\mathrm{d}g$ and H a closed subgroup of G. Let M and N be two compact submanifolds in the homogeneous space G/H, M fixed and gN the image of N under a motion $g\in G$. Let $I(M\cap gN)$ denote some kind of global geometric invariants of $M\cap gN$ which may be volume, curvature integral, etc. Then the goal of the kinematic formulas related to the invariant $I(M\cap gN)$ is to evaluate the following integral

$$\int_G I(M\cap gN)\mathrm{d}g$$

by the known integral invariants of M and N.

Howard proved in [7] a theorem of kinematic formulas in arbitrary homogeneous spaces. It indicates the computability of kinematic formulas with which the related invariants are of the following type

$$I(M\cap gN)=\int_{M\cap gN}\mathcal{P}(h)(x)\mathrm{d}\sigma(x),$$

where h is the second fundamental form of $M\cap gN$ in G/H, $\mathcal{P}$ is an invariant polynomial in the components of h with value $\mathcal{P}(h)(x)$ at $x\in M\cap gN$, and $d\sigma(x)$ is the volume element.

This theorem is general and implicit. However, it still requires concrete calculation to obtain explicit formulas, particularly when the kinematic formula is of the extrinsic type.

We prefer the formulas in the Euclidean case. First, because they are important from the point of view of their applications, and second, because they lead to more computable results.

In this paper, we give another kind of explicit kinematic formulas about two closed orientable hypersurfaces in $\mathbb{E}^n$, $n \geq 3$. The manifolds that appear are assumed to be smooth.

We will denote the group of rigid motions of $\mathbb{E}^n$ by G. The isotropic subgroup of G is denoted by G_0. Indeed G_0 is the special orthogonal group $SO(n)$. The groups G and G_0 are unimodular with canonical densities $\mathrm{d}g$ and $\mathrm{d}g_0$ respectively. Let $\mathrm{d}\sigma$ be the Lebesgue measure of $\mathbb{E}^n$, and then we have $\mathrm{d}g = \mathrm{d}\sigma \wedge \mathrm{d}g_0$. Moreover, the total volume of G_0 denoted by J_n is finite and given by

$$J_n = O_{n-1}O_{n-2}\cdots O_1,$$

where O_{i-1} is the volume of the $i-1$ dimensional unit sphere in $\mathbb{E}^i$ with the value

$$O_{i-1} = \frac{2\pi^{\frac{i}{2}}}{\Gamma\left(\frac{i}{2}\right)}. \tag{1.2}$$

Our motivation comes from the following formulas in [6, 12–13]. For example, let M be a closed hypersurface in $\mathbb{E}^n$, and then

$$\int_{M\cap L\neq\emptyset} \widetilde{H}_i^L(M\cap L)\mathrm{d}L = \frac{O_{n-2}O_{n-i}}{O_{n-1-i}}\widetilde{H}_i(M). \tag{1.3}$$

In this formula, the integer i satisfies $0 \leq i \leq n-2$, L is a random oriented hyperplane, and $\mathrm{d}L$ is the canonical invariant measure at L. $\widetilde{H}_i^L(M\cap L)$ denotes the i^{th} mean curvature integral of $M \cap L$ which is considered as a hypersurface in L. Similarly, $\widetilde{H}_i(M)$ is $\widetilde{H}_i^{\mathbb{E}^n}(M)$ for the sake of simplicity.

Taking an arbitrary orientable closed hypersurface in place of L, we obtain the following kinematic formula. It is not of Howard's type.

Theorem 1.1 *Let S_0 and S_1 be two closed oriented hypersurfaces in the n-dimensional Euclidean space $\mathbb{E}^n$. For any integer i satisfying $0 \leq i \leq n-2$, we have an asymmetric kinematic formula*

$$\int_{\{g\in G|S_0\cap gS_1\neq\emptyset\}} \widetilde{H}_i^{gS_1}(S_0\cap gS_1)\mathrm{d}g = \sum_{\substack{p=0\\ p \text{ is even}}}^{i} \mathrm{c}(p,i,n)\widetilde{H}_{i-p}(S_0)\widetilde{H}_p(S_1), \tag{1.4}$$

where the coefficients $\mathrm{c}(p,i,n)$ depend only on p, i and n with value

$$\mathrm{c}(p,i,n) = J_{n-1}\binom{i}{p}\frac{O_{n-2}O_{n-i+p}O_0}{O_pO_{n-1-i}}.$$

Note that the formula (1.3) can be viewed as a limiting case of (1.4).

Remark 1.1 By direct observation, if i is even, then (1.4) is just a special case of (1.1) for the hypersurfaces.

But the situation is different if i is odd. The simplest case is $i = 1$. Since $\widetilde{H}_0(\cdot)$ is actually the volume $\sigma(\cdot)$, we have the following result.

Corollary 1.1 *Let S_0 and S_1 be two closed oriented hypersurfaces in the n-dimensional Euclidean space $\mathbb{E}^n$. Then*

$$\int_{\{g\in G|S_0\cap gS_1\neq\emptyset\}}\widetilde{H}_1^{gS_1}(S_0\cap gS_1)\mathrm{d}g=J_n\sigma(S_1)\widetilde{H}_1(S_0). \tag{1.5}$$

More specifically, for $n=3$, $\widetilde{H}_1^{gS_1}(S_0\cap gS_1)$ is the total geodesic curvature of the curve $S_0\cap gS_1$ in gS_1, denoted as $\kappa^{gS_1}(S_0\cap gS_1)$. Then the above formula is

$$\int_{\{g\in G|S_0\cap gS_1\neq\emptyset\}}\kappa^{gS_1}(S_0\cap gS_1)\mathrm{d}g=8\pi^2\sigma(S_1)\widetilde{H}_1(S_0).$$

The formula (1.5) gives exactly an integral representation of the total mean curvature based on any given closed hypersurface.

2 Preliminaries

In the papers [3–4], Chern associated each given Riemannian manifold certain fiber bundle with canonical densities. These concepts and the moving frame method have been proved to be effective in integral geometry. We would like to follow this way in this paper.

We agree in this section on the following indix ranges:

$$1\le A,B,C\le n,\quad 1\le i,j,k\le m,\quad m+1\le\alpha,\beta,\gamma\le n,\quad 1\le a,b,c\le k.$$

2.1 Fundamental equations of submanifolds

Let M be an m-dimensional submanifold in an oriented n-dimensional Riemannian manifold N with Riemannian metric $\langle\ ,\ \rangle$, and $x:M\to N$ be the identity map.

First of all, we mention the local structure of M (see [5]). Locally M can be attached to a Darboux frame $e_1,\cdots,e_m,e_{m+1},\cdots,e_n$, which is a smooth orthonormal frame field and satisfies that $e_1,\cdots,e_m$ is tangent to M. In the rest of this article, when we mention frames tangent to an oriented manifold, it is always assumed that the orientation of the frames has been chosen to be compatible with the orientation of the given manifold. So here the orientation of $e_1,\cdots,e_m$ and $e_1,\cdots,e_m,e_{m+1},\cdots,e_n$ is the same as that of M and N respectively.

Let ω^A, ω_B^A respectively be the canonical forms and the Levi-Civita connection forms on the orthogonal group $SO(n)$ principle bundle $SO(N)$ under arbitrary extension of the given Darboux frame. These forms can be pulled back on M by the Darboux frame. The images are still denoted by ω^A, ω_B^A.

Let ∇ be the Levi-Civita connection on the tangent bundle TN, and then we have

$$\begin{aligned}&\mathrm{d}x=\omega^ie_i,\quad \omega^\alpha=0,\\&\nabla e_i=\omega_i^je_j+\omega_i^\alpha e_\alpha,\\&\nabla e_\alpha=\omega_\alpha^je_j+\omega_\alpha^\beta e_\beta\end{aligned}$$

and

$$\omega_i^\alpha\wedge\omega^i=0.$$

From the above equation, we obtain

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

The quantities h_{ji}^α are the components of the second fundamental form of M.

We review the definition of the mean curvature integrals of a hypersurfaces as follows. Let $m = n-1$, and then M is an oriented hypersurface in N, which has a chosen orientation. The i^{th} mean curvature is related to the following characteristic polynomial of (h_{ij}^n), or (h_{ij}) for simplicity in the hypersurface cases. Let

$$\det(\delta_{ij} + \lambda h_{ij}) = 1 + \psi_1\lambda + \psi_2\lambda^2 + \cdots + \psi_{n-1}\lambda^{n-1},$$

and then the i^{th} mean curvature is defined as

$$H_i = \binom{n-1}{i}^{-1} \psi_i,$$

and the i^{th} total mean curvature, if it exists, is denoted by

$$\widetilde{H}_i(M) = \int_M H_i \mathrm{d}\sigma.$$

In this paper, we write $\widetilde{H}_i^N(M)$ instead of $\widetilde{H}_i(M)$ to emphasize that the calculation is done in N, since M may be simultaneously a submanifold of other ambient spaces. Notation H_i^N is used for the same purpose.

2.2 Associated fiber bundles of *M*

For a fixed nonnegative integer k, we define a space $\mathcal{E}_k$ associated with M by the set

$$\{(x; e_1, \cdots, e_k) \mid x \in M, e_1, \cdots, e_k \in T_xM, \langle e_a, e_b\rangle = \delta_{ab}\}.$$

$\mathcal{E}_k$ is a fiber bundle with the base space M and the fiber $SO(m)/SO(m-k)$. $\mathcal{E}_k$ is called the tangent k-bundle of M here. In fact, $\mathcal{E}_k$ is an orientable differentiable manifold of dimension $\frac{1}{2}(k+1)(2m-k)$. At any point where $(x; e_1, \cdots, e_k) \in \mathcal{E}_k$, one extends $(x; e_1, \cdots, e_k)$ to be a Darboux frame $e_1, \cdots, e_n$ beside x, and then defines the following form:

$$\Theta = \bigwedge_i \omega^i \bigwedge_{a<j} \omega_a^j = \mathrm{d}\sigma \wedge \mathrm{d}V_k^m,$$

where $\mathrm{d}\sigma = \omega^1 \wedge \cdots \wedge \omega^m$ is the volume element of M, and the restriction of $\mathrm{d}V_k^m = \bigwedge_{a<j} \omega_a^j$ on the fibers is indeed the density of $SO(m)/SO(m-k)$. It is proved in [3], that Θ is independent of the choice of extension and is well defined on $\mathcal{E}_k$. It gives rise to a density of $\mathcal{E}_k$.

3 Proof of Theorem 1.1

Let S_0 and S_1 be two closed oriented hypersurfaces in the n-dimensional Euclidean space $\mathbb{E}^n$. The tangent $(n-2)$-bundles of S_0 and S_1 are denoted by $\mathcal{E}_{n-2,0}$ and $\mathcal{E}_{n-2,1}$, respectively.

For each $g \in G$ such that $\dim(S_0 \cap gS_1) = n-2$, $\mathcal{E}_{n-2,0} \cap g\mathcal{E}_{n-2,1}$ is the tangent $(n-2)$-bundle of $S_0 \cap gS_1$ with density Φ_g. Then $\Phi_g \wedge \mathrm{d}g$ is a density of the set $\mathcal{D} = \{(X, Y, g) \in \mathcal{E}_{n-2,0} \times \mathcal{E}_{n-2,1} \times G \mid X = gY\}$.

For any point $(x; e_1, \cdots, e_{n-2})$ of $\mathcal{E}_{n-2,0} \cap g\mathcal{E}_{n-2,1}$, we complement it into an orthonormal frame $(x; e_1, \cdots, e_{n-2}, e_{n-1}, e_n)$ such that e_n is orthogonal to S_0 and also into an orthonormal frame $(x; e_1, \cdots, e_{n-2}, e'_{n-1}, e'_n)$ such that e'_n is orthogonal to gS_1. Let $\theta \in (0, \pi)$ be the angle between e_n and e'_n, and then the following differential formula (see [3–4, 11–12]) is well known for the density of $\mathcal{D}$:

$$\Phi_g \wedge \mathrm{d}g = \sin^{n-1}\theta \mathrm{d}\theta \wedge \Theta_0 \wedge \Theta_1, \tag{3.1}$$

where Θ_0 and Θ_1 are the densities of $\mathcal{E}_{n-2,0}$ and $\mathcal{E}_{n-2,1}$, respectively. Indeed the right side of (3.1) is a density of $\widetilde{\mathcal{D}} = (0, \pi) \times \mathcal{E}_{n-2,0} \times \mathcal{E}_{n-2,1}$.

Observe that the subset of $\mathcal{D}$ is $\mathcal{E}_{n-2,0} \cap g\mathcal{E}_{n-2,1}$ for fixed $g \in G$, and the left side of (1.4) writes as

$$\frac{1}{J_{n-2}} \int_{\mathcal{D}} H_i^{gS_1} \Phi_g \wedge \mathrm{d}g.$$

By (3.1), we see that the above integration is

$$\frac{1}{J_{n-2}} \int_{\widetilde{\mathcal{D}}} H_i^{gS_1} \sin^{n-1}\theta \mathrm{d}\theta \wedge \Theta_0 \wedge \Theta_1. \tag{3.2}$$

We turn our attention to the computation of $H_i^{gS_1}$. The problem is to give $H_i^{gS_1}$ a representation by the curvatures of S_0 and gS_1 at $x \in S_0 \cap gS_1$, when the motion g is fixed.

From now on, the indexes i, j, k are agreed to range from 1 to $(n-2)$, and A, B from 1 to $(n-1)$.

The induced Levi-Civita connection of gS_1 is actually d of $\mathbb{E}^n$ projected on the tangent space of gS_1. Hence the curvature of $S_0 \cap gS_1$ with respect to gS_1 is given by

$$h_{ij}\omega^j = \langle \mathrm{d}e_i, e'_{n-1} \rangle.$$

We here denote by $\omega^1, \cdots, \omega^{n-2}, \omega'^{n-1}, \omega'^n$ the dual basis with respect to the frame $e_1, \cdots, e_{n-2}$, e'_{n-1}, e'_n, and by $\omega^1, \cdots, \omega^n$ with respect to $e_1, \cdots, e_n$.

From the relation

$$e_n = \sin\theta\ e'_{n-1} + \cos\theta\ e'_n,$$

e'_{n-1} can be represented as

$$e'_{n-1} = \sin^{-1}\theta e_n - \tan^{-1}\theta e'_n.$$

Then

$$h_{ij}\omega^j = \sin^{-1}\theta \langle \mathrm{d}e_i, e_n \rangle - \tan^{-1}\theta \langle \mathrm{d}e_i, e'_n \rangle. \tag{3.3}$$

If we attach an orthonormal frame field $v_1, \cdots, v_{n-1}$ on S_0 and $v'_1, \cdots, v'_{n-1}$ on gS_1 near x, the correspondent dual orthonormal frame fields are $\eta^1, \cdots, \eta^{n-1}$ and $\eta'^1, \cdots, \eta'^{n-1}$, and then there exist orthogonal matrices (c_A^B) and $({c'}_A^B)$ such that

$$e_i = c_i^B v_B = {c'}_i^B v'_B, \quad e_{n-1} = c_{n-1}^B v_B, \quad e'_{n-1} = {c'}_{n-1}^B v'_B.$$

Now we assume that v_A and v'_A are the principle tangent vectors of S_0 and gS_1, respectively, and then

$$\langle \mathrm{d}v_A, e_n \rangle = \kappa_A \eta^A, \quad \langle \mathrm{d}v'_A, e'_n \rangle = \kappa'_A \eta'^A,$$

where κ_A and κ'_A are the principle curvatures of S_0 and gS_1 at x, respectively.

With the aid of these notions, we reformulate the right side of (3.3) into

$$\begin{aligned}
&\sin^{-1}\theta\langle \mathrm{d}e_i, e_n\rangle - \tan^{-1}\theta\langle \mathrm{d}e_i, e'_n\rangle \\
&= \sin^{-1}\theta c_i^A\langle \mathrm{d}v_A, e_n\rangle - \tan^{-1}\theta {c'}_i^A\langle \mathrm{d}v'_A, e'_n\rangle \\
&= \sin^{-1}\theta\sum_A c_i^A\kappa_A\eta^A - \tan^{-1}\theta\sum_A {c'}_i^A\kappa'_A{\eta'}^A \\
&= \sin^{-1}\theta\sum_A c_i^A\kappa_A c_j^A\omega^j - \tan^{-1}\theta\sum_A {c'}_i^A\kappa'_A{c'}_j^A\omega^j.
\end{aligned}$$

The reason of the last equality is that we calculate in $S_0\cap gS_1$ and $\omega^{n-1} = {\omega'}^{n-1} = 0$.

By the linear independence of ω^i, we represent the curvature of $S_0\cap gS_1$ as

$$h_{ij} = \sin^{-1}\theta\sum_A c_i^A\kappa_A c_j^A - \tan^{-1}\theta\sum_A {c'}_i^A\kappa'_A{c'}_j^A.$$

Following Chern [3], we are going to expand the determinant of (h_{ij}) as the polynomial of κ_A and κ'_A. By the multilinearity of the determinant, one has

$$\begin{aligned}
&\det(h_{ij}) \\
&= \det\Big(\sin^{-1}\theta\sum_A c_i^A\kappa_A c_j^A - \tan^{-1}\theta\sum_A {c'}_i^A\kappa'_A{c'}_j^A\Big) \\
&= \sum_{p+q=n-2}\ \sum_{\substack{1\le A_1,\cdots,A_q\le n-1\\ 1\le B_1,\cdots,B_p\le n-1}} \Psi^{A_1A_2\cdots A_q,B_1B_2\cdots B_p}\sin^{-q}\theta(-\tan\theta)^{-p}\kappa_{A_1}\kappa_{A_2}\cdots\kappa_{A_q}\kappa'_{B_1}\kappa'_{B_2}\cdots\kappa'_{B_p},
\end{aligned}$$

where $\Psi^{A_1A_2\cdots A_q,B_1B_2\cdots B_p}$ is the sum of some $(n-2)\times(n-2)$-determinants, and one term in the summation is

$$\begin{vmatrix}
c_1^{A_1}c_1^{A_1} & c_1^{A_2}c_2^{A_2} & \cdots & c_1^{A_q}c_q^{A_q} & {c'}_1^{B_1}{c'}_{q+1}^{B_1} & {c'}_1^{B_2}{c'}_{q+2}^{B_2} & \cdots & {c'}_1^{B_p}{c'}_{n-2}^{B_p} \\
c_2^{A_1}c_1^{A_1} & c_2^{A_2}c_2^{A_2} & \cdots & c_2^{A_q}c_q^{A_q} & {c'}_2^{B_1}{c'}_{q+1}^{B_1} & {c'}_2^{B_2}{c'}_{q+2}^{B_2} & \cdots & {c'}_2^{B_p}{c'}_{n-2}^{B_p} \\
 & & \vdots & & & & \vdots & \\
c_{n-2}^{A_1}c_1^{A_1} & c_{n-2}^{A_2}c_2^{A_2} & \cdots & c_{n-2}^{A_q}c_q^{A_q} & {c'}_{n-2}^{B_1}{c'}_{q+1}^{B_1} & {c'}_{n-2}^{B_2}{c'}_{q+2}^{B_2} & \cdots & {c'}_{n-2}^{B_p}{c'}_{n-2}^{B_p}
\end{vmatrix}, \tag{3.4}$$

while others can be obtained by permuting some upscript A's with some upscript B's in (3.4), but preserving the orders of $A_1A_2\cdots A_q$ and $B_1B_2\cdots B_p$ to themselves. If some elements of $A_1A_2\cdots A_q$ or $B_1B_2\cdots B_p$ appear twice, then clearly

$$\Psi^{A_1A_2\cdots A_q,B_1B_2\cdots B_p} = 0. \tag{3.5}$$

By this observation, one has

$$\begin{aligned}
&\det(h_{ij}) \\
&= \sin^{2-n}\theta\sum_{p+q=n-2}(-1)^p\cos^p\theta\sum_{\substack{1\le A_1<\cdots<A_q\le n-1\\ 1\le B_1<\cdots<B_p\le n-1}}\widetilde{\Psi}^{A_1A_2\cdots A_q,B_1B_2\cdots B_p}\kappa_{A_1}\kappa_{A_2}\cdots\kappa_{A_q}\kappa'_{B_1}\kappa'_{B_2}\cdots\kappa'_{B_p},
\end{aligned}$$

where

$$\widetilde{\Psi}^{A_1A_2\cdots A_q,B_1B_2\cdots B_p} = \sum_{\sigma,\tau}\Psi^{A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(q)},B_{\tau(1)}B_{\tau(2)}\cdots B_{\sigma(p)}},$$

with σ and τ running through the permutation groups of order q and p, respectively.

The equations (3.5) then imply that

$$\begin{aligned}
&\widetilde{\Psi}^{A_1A_2\cdots A_q,B_1B_2\cdots B_p}\\
&=\begin{vmatrix}
\sum c_1^{A_s}c_1^{A_s}+\sum c'^{B_t}_1c'^{B_t}_1 & \sum c_1^{A_s}c_2^{A_s}+\sum c'^{B_t}_1c'^{B_t}_2 & \cdots & \sum c_1^{A_s}c_{n-2}^{A_s}+\sum c'^{B_t}_1c'^{B_t}_{n-2}\\
\sum c_2^{A_s}c_1^{A_s}+\sum c'^{B_t}_2c'^{B_t}_1 & \sum c_2^{A_s}c_2^{A_s}+\sum c'^{B_t}_2c'^{B_t}_2 & \cdots & \sum c_2^{A_s}c_{n-2}^{A_s}+\sum c'^{B_t}_2c'^{B_t}_{n-2}\\
\vdots & \vdots & \vdots & \vdots\\
\sum c_{n-2}^{A_s}c_1^{A_s}+\sum c'^{B_t}_{n-2}c'^{B_t}_1 & \sum c_{n-2}^{A_s}c_2^{A_s}+\sum c'^{B_t}_{n-2}c'^{B_t}_2 & \cdots & \sum c_{n-2}^{A_s}c_{n-2}^{A_s}+\sum c'^{B_t}_{n-2}c'^{B_t}_{n-2}
\end{vmatrix}\\
&=\begin{vmatrix}
c_1^{A_1} & c_1^{A_2} & \cdots & c_1^{A_q} & c'^{B_1}_1 & c'^{B_2}_1 & \cdots & c'^{B_p}_1\\
c_2^{A_1} & c_2^{A_2} & \cdots & c_2^{A_q} & c'^{B_1}_2 & c'^{B_2}_2 & \cdots & c'^{B_p}_2\\
 & & \vdots & & & & \vdots & \\
c_{n-2}^{A_1} & c_{n-2}^{A_2} & \cdots & c_{n-2}^{A_q} & c'^{B_1}_{n-2} & c'^{B_2}_{n-2} & \cdots & c'^{B_p}_{n-2}
\end{vmatrix}
\begin{vmatrix}
c_1^{A_1} & c_2^{A_1} & \cdots & c_{n-2}^{A_1}\\
c_1^{A_2} & c_2^{A_2} & \cdots & c_{n-2}^{A_2}\\
 & & \vdots & \\
c_1^{A_q} & c_2^{A_q} & \cdots & c_{n-2}^{A_q}\\
c'^{B_1}_1 & c'^{B_1}_2 & \cdots & c'^{B_1}_{n-2}\\
c'^{B_2}_1 & c'^{B_2}_2 & \cdots & c'^{B_2}_{n-2}\\
\vdots & \vdots & \vdots & \vdots\\
c'^{B_p}_1 & c'^{B_p}_2 & \cdots & c'^{B_p}_{n-2}
\end{vmatrix}\\
&=\begin{vmatrix}
c_1^{A_1} & c_1^{A_2} & \cdots & c_1^{A_q} & c'^{B_1}_1 & c'^{B_2}_1 & \cdots & c'^{B_p}_1\\
c_2^{A_1} & c_2^{A_2} & \cdots & c_2^{A_q} & c'^{B_1}_2 & c'^{B_2}_2 & \cdots & c'^{B_p}_2\\
 & & \vdots & & & & \vdots & \\
c_{n-2}^{A_1} & c_{n-2}^{A_2} & \cdots & c_{n-2}^{A_q} & c'^{B_1}_{n-2} & c'^{B_2}_{n-2} & \cdots & c'^{B_p}_{n-2}
\end{vmatrix}^2,
\end{aligned}$$

where the summation index s runs from 1 to q, and t from 1 to p. So we obtain the following formula about $\det(h_{ij})$:

$$\det(h_{ij})=\sin^{2-n}\theta\sum_{p=0}^{n-2}(-1)^p\cos^p\theta\mathscr{S}_p,$$

where

$$\begin{aligned}
\mathscr{S}_p&=\sum_{\substack{1\le A_1<\cdots<A_q\le n-1\\1\le B_1<\cdots<B_p\le n-1}}\widetilde{\Psi}^{A_1A_2\cdots A_q,B_1B_2\cdots B_p}\kappa_{A_1}\cdots\kappa_{A_q}\kappa'_{B_1}\cdots\kappa'_{B_p}\\
&=\sum_{\substack{1\le A_1<\cdots<A_q\le n-1\\1\le B_1<\cdots<B_p\le n-1}}\begin{vmatrix}
c_1^{A_1} & \cdots & c_1^{A_q} & c'^{B_1}_1 & \cdots & c'^{B_p}_1\\
 & \vdots & & & \vdots & \\
c_{n-2}^{A_1} & \cdots & c_{n-2}^{A_q} & c'^{B_1}_{n-2} & \cdots & c'^{B_p}_{n-2}
\end{vmatrix}^2\kappa_{A_1}\cdots\kappa_{A_q}\kappa'_{B_1}\cdots\kappa'_{B_p},
\end{aligned}$$

and $q=n-2-p$.

More generally, we will discuss the coefficients of the characteristic polynomial

$$\det(\delta_{ij}+\lambda h_{ij})=1+\psi_1\lambda+\psi_2\lambda^2+\cdots+\psi_{n-2}\lambda^{n-2}.$$

It is clear that $\det(h_{ij})=\psi_{n-2}$. By a result in linear algebra, the i^{th} coefficient is given by

$$\psi_i=\sum_{1\le j_1<\cdots<j_i\le n-2}\begin{vmatrix}
h_{j_1j_1} & \cdots & h_{j_1j_i}\\
\vdots & \vdots & \vdots\\
h_{j_ij_1} & \cdots & h_{j_ij_i}
\end{vmatrix}.$$

A similar discussion about $\det(h_{ij})$ gives that

$$\psi_i = \sin^{-i}\theta \sum_{p=0}^{i} (-1)^p \cos^p \theta \mathscr{S}_p^{(i)},$$

where

$$\begin{aligned}\mathscr{S}_p^{(i)} &= \sum_{\substack{1\le A_1<\cdots<A_{i-p}\le n-1\\ 1\le B_1<\cdots<B_p\le n-1\\ 1\le j_1<\cdots<j_i\le n-2}} \begin{vmatrix} c_{j_1}^{A_1} & \cdots & c_{j_1}^{A_{i-p}} & {c'}_{j_1}^{B_1} & \cdots & {c'}_{j_1}^{B_p} \\ & \vdots & & & \vdots & \\ c_{j_i}^{A_1} & \cdots & c_{j_i}^{A_{i-p}} & {c'}_{j_i}^{B_1} & \cdots & {c'}_{j_i}^{B_p} \end{vmatrix}^2 \kappa_{A_1}\cdots\kappa_{A_{i-p}}\kappa'_{B_1}\cdots\kappa'_{B_p} \\ &=: \sum_{\substack{1\le A_1<\cdots<A_{i-p}\le n-1\\ 1\le B_1<\cdots<B_p\le n-1}} \widetilde{\Psi}^{(i)\ A_1A_2\cdots A_{i-p},B_1B_2\cdots B_p} \kappa_{A_1}\cdots\kappa_{A_{i-p}}\kappa'_{B_1}\cdots\kappa'_{B_p}.\end{aligned}$$

One notes that the coefficients $\widetilde{\Psi}^{(i)\ A_1A_2\cdots A_{i-p},B_1B_2\cdots B_p}$ at x are only dependent on the relative position of S_0 and S_1, and they are invariant if the frame $e_1,\cdots,e_{n-2}$ is acted by an element of $O(n-2)$.

Then the integration (3.2) continues as follows:

$$\begin{aligned}&\frac{1}{J_{n-2}}\binom{n-2}{i}^{-1}\int_{\widetilde{\mathcal{D}}} \psi_i \sin^{n-1}\theta \mathrm{d}\theta\wedge\Theta_0\wedge\Theta_1 \\ &= \frac{1}{J_{n-2}}\binom{n-2}{i}^{-1}\sum_{p=0}^{i}\Big(\int_0^\pi (-1)^p \sin^{n-1-i}\theta\cos^p\theta \mathrm{d}\theta\Big)\Big(\int_{\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}} \mathscr{S}_p^{(i)}\Theta_0\wedge\Theta_1\Big) \\ &= \frac{1}{J_{n-2}}\binom{n-2}{i}^{-1}\sum_{\substack{p=0\\ p \text{ is even}}}^{i} \frac{\Gamma\Big(\dfrac{p+1}{2}\Big)\Gamma\Big(\dfrac{n-i}{2}\Big)}{\Gamma\Big(\dfrac{n-i+p+1}{2}\Big)}\Big(\int_{\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}} \mathscr{S}_p^{(i)}\Theta_0\wedge\Theta_1\Big) \\ &= \sum_{\substack{p=0\\ p \text{ is even}}}^{i} \mathrm{c}(p,i,n)\widetilde{H}_{i-p}(S_0)\widetilde{H}_p(S_1).\end{aligned}$$

The last equality deserves more explanations. Now we are going to discuss the integral of $\mathscr{S}_p^{(i)}$ over $\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}$. By fiber integration, we have

$$\begin{aligned}&\int_{\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}} \mathscr{S}_p^{(i)}\Theta_0\wedge\Theta_1 \\ &= \int_{\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}} \Big(\sum \widetilde{\Psi}^{(i)\ A_1A_2\cdots A_{i-p},B_1B_2\cdots B_p}\kappa_{A_1}\cdots\kappa_{A_{i-p}}\kappa'_{B_1}\cdots\kappa'_{B_p}\Big)\Theta_0\wedge\Theta_1 \\ &= \sum\int_{\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}} (\widetilde{\Psi}^{(i)\ A_1A_2\cdots A_{i-p},B_1B_2\cdots B_p}\kappa_{A_1}\cdots\kappa_{A_{i-p}}\kappa'_{B_1}\cdots\kappa'_{B_p})\Theta_0\wedge\Theta_1 \\ &= \sum\int_{S_0\times S_1}\Big(\int_P \widetilde{\Psi}^{(i)\ A_1A_2\cdots A_{i-p},B_1B_2\cdots B_p}\mathrm{d}V_{n-2,0}^{n-1}\times\mathrm{d}V_{n-2,1}^{n-1}\Big) \\ &\quad\cdot\kappa_{A_1}\cdots\kappa_{A_{i-p}}\kappa'_{B_1}\cdots\kappa'_{B_p}\mathrm{d}\sigma_0\wedge\mathrm{d}\sigma_1.\end{aligned}$$

One notes that the densities $\mathrm{d}V_{n-2,0}^{n-1}$ and $\mathrm{d}V_{n-2,1}^{n-1}$ are $SO(n-1)$ invariant respectively. One also remembers that c_i^A and ${c'}_i^B$ are coefficients of e_i and e'_i with respect to v_A and v'_B, respectively. For any combinations $1 \le j_1 < \cdots < j_i \le n-2$, there exists a rotation in $SO(n-1)$ by which $e_1, \cdots, e_i$ is rotated to the position of $e_{j_1}, \cdots, e_{j_i}$. So the fiber integral is

$$\begin{aligned}
&\int_P \widetilde{\Psi}^{(i)\ A_1A_2\cdots A_{i-p},B_1B_2\cdots B_p} \mathrm{d}V^{n-1}_{n-2,0} \times \mathrm{d}V^{n-1}_{n-2,1} \\
&= \sum_{0\le j_1<\cdots<j_i\le n-2} \int_P \begin{vmatrix} c^{A_1}_{j_1} & \cdots & c^{A_{i-p}}_{j_1} & {c'}^{B_1}_{j_1} & \cdots & {c'}^{B_p}_{j_1} \\ & \vdots & & & \vdots & \\ c^{A_1}_{j_i} & \cdots & c^{A_{i-p}}_{j_i} & {c'}^{B_1}_{j_i} & \cdots & {c'}^{B_p}_{j_i} \end{vmatrix}^2 \mathrm{d}V^{n-1}_{n-2,0} \times \mathrm{d}V^{n-1}_{n-2,1} \\
&= \binom{n-2}{i} \int_P \begin{vmatrix} c^{A_1}_{1} & \cdots & c^{A_{i-p}}_{1} & {c'}^{B_1}_{1} & \cdots & {c'}^{B_p}_{1} \\ & \vdots & & & \vdots & \\ c^{A_1}_{i} & \cdots & c^{A_{i-p}}_{i} & {c'}^{B_1}_{i} & \cdots & {c'}^{B_p}_{i} \end{vmatrix}^2 \mathrm{d}V^{n-1}_{n-2,0} \times \mathrm{d}V^{n-1}_{n-2,1}.
\end{aligned}$$

On the hypersurface S_0, for two sets of different indexes $A_1, \cdots, A_q$ and $\widetilde{A}_1, \cdots, \widetilde{A}_q$, there exists a rotation g in $SO(n-1)$, such that $\langle g(e_s), v^{\widetilde{A}_t}\rangle = \langle e_s, v^{A_t}\rangle$, $s, t = 1, \cdots, q$. The same statement is also valid for S_1. Let

$$\Omega(p,i,n) = \int_P \begin{vmatrix} c^{A_1}_{j_1} & \cdots & c^{A_{i-p}}_{j_1} & {c'}^{B_1}_{j_1} & \cdots & {c'}^{B_p}_{j_1} \\ & \vdots & & & \vdots & \\ c^{A_1}_{j_i} & \cdots & c^{A_{i-p}}_{j_i} & {c'}^{B_1}_{j_i} & \cdots & {c'}^{B_p}_{j_i} \end{vmatrix}^2 \mathrm{d}V^{n-1}_{n-2,0} \times \mathrm{d}V^{n-1}_{n-2,1},$$

and then $\Omega(p,i,n)$ are independent of the choices of $A_1, \cdots, A_{i-p}$ and $B_1, \cdots, B_p$.

If we denote the i^{th} elementary symmetric polynomial of the elements $a_1, a_2, \cdots, a_{n-1}$ as $\{a_{A_1} \cdots a_{A_i}\}$, then the integration of $\mathscr{S}^{(i)}_p$ on $\mathcal{E}_{n-2,0} \times \mathcal{E}_{n-2,1}$ is

$$\begin{aligned}
&\int_{\mathcal{E}_{n-2,0}\times\mathcal{E}_{n-2,1}} \mathscr{S}^{(i)}_p \Theta_0 \wedge \Theta_1 \\
&= \binom{n-2}{i} \Omega(p,i,n) \int_{S_0\times S_1} \{\kappa_{A_1} \cdots \kappa_{A_{i-p}}\}\{\kappa'_{B_1} \cdots \kappa'_{B_p}\} \mathrm{d}\sigma_0 \wedge \mathrm{d}\sigma_1 \\
&= \binom{n-2}{i}\binom{n-1}{i-p}\binom{n-1}{p} \Omega(p,i,n) \int_{S_0\times S_1} H_{i-p}(S_0) H_p(S_1) \mathrm{d}\sigma_0 \wedge \mathrm{d}\sigma_1.
\end{aligned}$$

Instead of direct calculation of $\Omega(p,i,n)$, we prefer to determine the universal constants $\mathrm{c}(p,i,n)$ by taking S_0 and S_1 to be hyperspheres, and then evaluate $\Omega(p,i,n)$.

The constants will be determined in the last section. We complete the proof of Theorem 1.1.

4 Determination of the Constants

Let $S_0 = S^{n-1}(1)$ and $S_1 = S^{n-1}(R)$ be hyperspheres in $\mathbb{E}^n$ of radii 1 and R $(0 < R < 1)$ respectively. Let S_0 be centered at the origin point O. For a fixed motion g, let y be the distance from O to the center of $gS_1 = gS^{n-1}(R)$.

When $S_0 \cap gS_1 \ne \emptyset$ which is equivalent to $1 - R \le y \le 1 + R$, $S_0 \cap gS_1$ is a hypersphere $S^{n-2}(r)$ in a hyperplane of radius

$$r = \frac{[4y^2 - (1 - R^2 + y^2)^2]^{\frac{1}{2}}}{2y}. \tag{4.1}$$

It can be proved that

$$H_i^{gS_1} = \frac{\left(1 - \dfrac{r^2}{R^2}\right)^{\frac{i}{2}}}{r^i}.$$

So we obtain

$$\widetilde{H}_i^{gS_1}(S_0 \cap gS_1) = O_{n-2} r^{n-2-i}\left(1 - \frac{r^2}{R^2}\right)^{\frac{i}{2}}.$$

In this specific case, the left side of (1.4) reads

$$O_{n-2}\int_{G_0} \mathrm{d}g_0 \int_{\mathbb{E}^n} r^{n-2-i}\left(1 - \frac{r^2}{R^2}\right)^{\frac{i}{2}} \mathrm{d}\sigma. \tag{4.2}$$

By the polar decomposition of Lebesgue measure $\mathrm{d}\sigma$ and using (4.1), the above integration is

$$\begin{aligned} & O_{n-1}O_{n-2}J_n \int_{1-R}^{1+R} r^{n-2-i}\left(1 - \frac{r^2}{R^2}\right)^{\frac{i}{2}} y^{n-1}\mathrm{d}y \\ = & \frac{O_{n-1}O_{n-2}J_n}{2^{n-2}R^i}\int_{1-R}^{1+R} y[4y^2 - (1-R^2+y^2)^2]^{\frac{n-2-i}{2}}[4y^2R^2 - 4y^2 + (1-R^2+y^2)^2]^{\frac{i}{2}}\mathrm{d}y. \end{aligned}$$

Putting $2Ru = y^2 - 1 - R^2$, we reformulate the integration as

$$\begin{aligned} & O_{n-1}O_{n-2}J_n R^{n-1-i}\int_{-1}^{1}(1-u^2)^{\frac{n-2-i}{2}}(R+u)^i \mathrm{d}u \\ = & O_{n-1}O_{n-2}J_n \sum_{\substack{p=0 \\ p \text{ is even}}}^{i} \binom{i}{p} R^{n-1-p}\int_{-1}^{1} u^p(1-u^2)^{\frac{n-2-i}{2}}\mathrm{d}u \\ = & O_{n-1}O_{n-2}J_n \sum_{\substack{p=0 \\ p \text{ is even}}}^{i} \binom{i}{p} R^{n-1-p}\frac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{n-i}{2}\right)}{\Gamma\left(\dfrac{n-i+p+1}{2}\right)}. \end{aligned}$$

By the formula (1.2), the kinematic formula for the hyperspheres $S_0 = S^{n-1}(1)$ and $S_1 = S^{n-1}(R)$ is proved to be

$$\int_{\{g|S_0\cap gS_1\neq\emptyset\}} \widetilde{H}_i^{gS_1}(S_0\cap gS_1)\mathrm{d}g = J_{n-1}\sum_{\substack{p=0 \\ p \text{ is even}}}^{i}\binom{i}{p}\frac{O_{n-2}O_{n-i+p}O_0}{O_pO_{n-1-i}}\widetilde{H}_{i-p}(S_0)\widetilde{H}_p(S_1).$$

So the universal coefficients in (1.4) are given by

$$\mathrm{c}(p,i,n) = J_{n-1}\binom{i}{p}\frac{O_{n-2}O_{n-i+p}O_0}{O_pO_{n-1-i}}.$$

As a consequence, one has

$$\Omega(p,i,n) = (J_{n-1})^2\binom{i}{p}\binom{n-1}{p}^{-1}\binom{n-1}{i-p}^{-1}.$$

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