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Solution of Center-Focus Problem for a Class of Cubic Systems*

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Abstract For a class of cubic systems, the authors give a representation of the nth order Liapunov constant through a chain of pseudo-divisions. As an application, the center problem and the isochronous center problem of a particular system are considered. They show that the system has a center at the origin if and only if the first seven Liapunov constants vanish, and cannot have an isochronous center at the origin.

Keywords Center variety, Isochronous center, Center conditions, Integrating factor
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1 Introduction

Consider a two-dimensional system of differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + P_n(x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + Q_n(x, y), \tag{1.1}$$

where P_n, Q_n are real polynomials of degree n without constant and linear terms. The singularity at the origin is a weak focus (surrounded by spirals) or a center (surrounded by closed trajectories). The Poincaré center-focus problem (see [1]) is to determine conditions on the coefficients of P_n and Q_n, under which an open neighborhood of the origin is covered by closed trajectories of system (1.1).

Although the center-focus problem of system (1.1) has attracted intensive attentions, the characterization of centers for cubic systems is far from complete. Malkin [2] found necessary and sufficient conditions for a cubic vector field with no quadratic terms to have a center. For a cubic system of the form:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\lambda\, y + X_2(x, y) + X_3(x, y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \lambda\, x + Y_2(x, y) + Y_3(x, y), \end{cases}$$

where $X_s(x, y)$ and $Y_s(x, y)$, $s = 2, 3$, are homogeneous polynomials of degree s, satisfying $x\, Y_3(x, y) - y\, X_3(x, y) \equiv 0$, Chavarriga and Giné [3] gave a complete center characterization.

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For a cubic system of the form:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y(1 + D\,x + P\,x^2), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + A\,x^2 + 3B\,xy + C\,y^2 + K\,x^3 + 3L\,x^2y + M\,xy^2 + N\,y^3, \end{cases}$$

Sadovskii and Shcheglova [4] presented a complete solution to the center-focus problem. For a special case of the above system with $P = 0$, Hill, Lloyd and Pearson [5–7] obtained the necessary and sufficient conditions for the origin to be a center and an isochronous center, respectively.

A center of (1.1) is called to be isochronous if all cycles near it have the same period. It is well known that isochronous centers are non-degenerate. The problem to determine whether the center is isochronous or not is called the isochronicity problem. Although this problem has attracted the attentions of many authors, the characterization of isochronous centers even for cubic systems is far from complete. Recently, the isochronous center problem of time-reversible cubic systems was completely solved by Chen and Romanovski [8]. For other developments of some polynomial differential systems, we refer to [9–15] and the references therein.

In the case $n = 3$, system (1.1) can be written as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + a_{2,0}x^2 + a_{2,1}xy + a_{2,2}y^2 + a_{3,0}x^3 + a_{3,1}x^2y + a_{3,2}xy^2 + a_{3,3}y^3 \\ \qquad = y + P_2(x,y) + P_3(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 + b_{2,1}xy + b_{2,2}y^2 + b_{3,0}x^3 + b_{3,1}x^2y + b_{3,2}xy^2 + b_{3,3}y^3 \\ \qquad = -x + Q_2(x,y) + Q_3(x,y), \end{cases} \tag{1.2}$$

where $P_k(x,y), Q_k(x,y)$ are homogeneous polynomials of degree k.

By the method of [16], there exists a unique formal power series of the form:

$$H(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty}\Big(\sum_{j=0}^{k} B_{k,j}x^{k-j}y^j\Big) = x^2 + y^2 + H_3(x,y) + H_4(x,y) + \cdots, \tag{1.3}$$

where $B_{k,k} = 0$ with k even and $H_k(x,y)$ is a homogeneous polynomial of degree k, so that

$$\frac{\mathrm{d}H}{\mathrm{d}t}\Big|_{(1.2)} = \sum_{n=1}^{\infty} W_n y^{2\,(n+1)}, \tag{1.4}$$

where W_n is called the nth Liapunov constant of system (1.2).

The classical Poincaré-Liapunov method gives the usual version of definition. There exists a unique formal power series of the form:

$$G(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty}\Big(\sum_{j=0}^{k} C_{k,j}x^{k-j}y^j\Big) = x^2 + y^2 + G_3(x,y) + G_4(x,y) + \cdots,$$

where $C_{k,k} = 0$ with k even and $G_k(x,y)$ is a homogeneous polynomial of degree k, so that

$$\frac{\mathrm{d}G}{\mathrm{d}t}\Big|_{(1.2)} = \sum_{n=1}^{\infty} V_n (x^2 + y^2)^{(n+1)},$$

where V_n is also called the nth Liapunov constant of system (1.2).

In a general setting, the computational problems which appear in the computation of the Liapunov constants were discussed (see [16–18]).

Proposition 1.1 *For system* (1.2) *and any natural number* n, $V_1 = V_2 = \cdots = V_n = 0$ *if and only if* $W_1 = W_2 = \cdots = W_n = 0$.

Proof Suppose that $V_1 = V_2 = \cdots = V_n = 0$, and then according to the work of Zhang et al. [17], we can uniquely determine $G_3(x,y), G_4(x,y), \cdots, G_{2n+2}(x,y)$ such that

$$x\frac{\partial G_k}{\partial y} - y\frac{\partial G_k}{\partial x} = \Psi_{k-1}(P_2, P_3, Q_2, Q_3, G_3, G_4, \cdots, G_{k-1}), \quad 3 \le k \le 2n+2,$$

where

$$\Psi_{k-1} = 2(xP_{k-1} + yQ_{k-1}) + \sum_{s=3}^{k-1}\Big(P_{k-s+1}\frac{\partial G_s}{\partial x} + Q_{k-s+1}\frac{\partial G_s}{\partial y}\Big)$$

and $P_{k-1} = Q_{k-1} = 0$ when $k \ge 5$.

For any positive integer m satisfying $1 \le m \le n$, we set $H_k(x,y) = G_k(x,y)$, $3 \le k \le 2m+2$. From [16], it is known that the quantity W_m is determined by the equation

$$\begin{aligned} & x\frac{\partial H_{2m+2}}{\partial y} - y\frac{\partial H_{2m+2}}{\partial x} + W_m y^{2m+2} \\ = {} & \Psi_{2m+1}(P_2, P_3, Q_2, Q_3, H_3, H_4, \cdots, H_{2m+1}), \quad 1 \le m \le n, \end{aligned}$$

and thus $W_m = 0,\ 1 \le m \le n$.

The converse direction can be proved in the same way. Suppose that $W_1 = W_2 = \cdots = W_n = 0$, and then according to [16], we can uniquely determine $H_3(x,y), H_4(x,y), \cdots, H_{2n+2}(x,y)$ such that

$$x\frac{\partial H_k}{\partial y} - y\frac{\partial H_k}{\partial x} = \Psi_{k-1}(P_2, P_3, Q_2, Q_3, H_3, H_4, \cdots, H_{k-1}), \quad 3 \le k \le 2n+2.$$

For any positive integer m satisfying $1 \le m \le n$, we set $G_k(x,y) = H_k(x,y)$, $3 \le k \le 2m+2$. From [17], it is known that the quantity V_m is determined by the equation

$$\begin{aligned} & x\frac{\partial G_{2m+2}}{\partial y} - y\frac{\partial G_{2m+2}}{\partial x} + V_m(x^2+y^2)^{(m+1)} \\ = {} & \Psi_{2m+1}(P_2, P_3, Q_2, Q_3, G_3, G_4, \cdots, G_{2m+1}), \quad 1 \le m \le n, \end{aligned}$$

and thus $V_m = 0,\ 1 \le m \le n$.

Corollary 1.1 *For system* (1.2), *the center varieties obtained from the polynomial ideals* $\langle W_k : k \ge 1\rangle$ *and* $\langle V_k : k \ge 1\rangle$ *are the same.*

Proposition 1.2 *The* n*th Liapunov constant of system* (1.2) *is given by the formula*

$$W_n = (2n+1)B_{2n+1,2n+1}b_{2,2} + B_{2n,2n-1}a_{3,3} + B_{2n+1,2n}a_{2,2} + B_{2n+2,2n+1}, \tag{1.5}$$

where $B_{2,0} = B_{2,2} = 1$ *and* $B_{2,1} = 0$.

Proof Let

$$U(x,y)=\frac{\mathrm{d}H}{\mathrm{d}t}\Big|_{(1.2)}=\frac{\partial H}{\partial x}(y+P_2+P_3)+\frac{\partial H}{\partial y}(-x+Q_2+Q_3).$$

Then the nth Liapunov constant of system (1.2) is

$$W_n=\frac{1}{(2n+2)!}\frac{\partial^{2n+2}U}{\partial y^{2n+2}}(0,0).$$

So by the general Leibniz rule, we get the formula (1.5).

We remark that each $B_{i,j}$ on the right hand side of (1.5) is a polynomial in the coefficients of system (1.2), and can be uniquely determined by using the identity (1.4).

In the case $a_{2,2}=a_{3,3}=b_{2,2}=b_{3,3}=0$, system (1.2) becomes

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=y+a_{2,0}x^2+a_{2,1}xy+a_{3,0}x^3+a_{3,1}x^2y+a_{3,2}xy^2,\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x+b_{2,0}x^2+b_{2,1}xy+b_{3,0}x^3+b_{3,1}x^2y+b_{3,2}xy^2.\end{cases}\tag{1.6}$$

Applying Proposition 1.2 to system (1.6), we have the following corollary.

Corollary 1.2 *The nth Liapunov constant of system* (1.6) *is given by the formula*

$$W_n=B_{2n+2,2n+1}.\tag{1.7}$$

In Section 2, the computation of the nth Liapunov constants in the coefficients of cubic system (1.6) is considered. In Section 3, the solutions to the center-focus problem and the isochronous center problem for a particular case of system (1.6) are given.

2 The nth Liapunov Constant for System (1.6) and the Main Results

Let the time derivative of H along the orbits of system (1.6) be

$$\frac{\mathrm{d}H}{\mathrm{d}t}\Big|_{(1.6)}=\sum_{j=3}^{\infty}\Big(\sum_{k=0}^{j}f_{j,k}x^{j-k}y^k\Big),\tag{2.1}$$

where $f_{j,k}$ are polynomials in the coefficients of system (1.6) and the function $H(x,y)$ in (1.3). As a consequence of Corollary 1.2, for each s we have

$$f_{2s+2,2s+2}=B_{2s+2,2s+1}.\tag{2.2}$$

Suppose that (1.6) has a weak focus of order n at the origin, and then the following conditions hold:

$$\begin{cases}f_{j,k}=0, & 0\le k\le j,\ 3\le j\le 2n+1,\\ f_{2n+2,k}=0, & 0\le k\le 2n+1,\\ B_{2n+2,2n+1}\ne 0.\end{cases}\tag{2.3}$$

Inspired by the algorithm of Wang [16, 19] and the corresponding Maple procedure licon [20] in Epsilon[miscel]package, we can represent the nth Liapunov constant W_n, that is, $B_{2n+2,2n+1}$, in the coefficients of system (1.6) by succesive pseudo-divisions.

Let $\mathcal{A} = \{A_1, A_2, \cdots, A_r\}$ be a polynomial set in the variables $x_1, x_2, \cdots, x_n$, for which the main variable of A_j is x_{p_j}. The pseudo-remainder of a polynomial P (in the variables $x_1, x_2, \cdots, x_n$) with respect to $\mathcal{A}$ is

$$\operatorname{prem}(P, \mathcal{A}, \mathcal{X}) = \begin{cases} 0, & \text{if } r = 1 \text{ and } p_1 = 0, \\ \operatorname{prem}(\cdots \operatorname{prem}(P, A_r, x_{p_r}), \cdots, A_1, x_{p_1}), & \text{otherwise,} \end{cases}$$

where $\mathcal{X} = \{x_{p_r}, x_{p_{r-1}}, \cdots, x_{p_1}\}$ and $\operatorname{prem}(P, A_r, x_{p_r})$ denotes the pseudo-remainder of P modulo A_r with respect to the variable x_{p_r}.

Let

$$\begin{aligned} &\mathcal{A}_{2\,n+2} = \{f_{2n+2,2\,n}, f_{2n+2,2\,n-2}, \cdots, f_{2n+2,0}\}, \\ &\mathcal{C}_{2\,k+1} = \{f_{2\,k+1,2\,k}, f_{2k+1,2\,k-2}, \cdots, f_{2k+1,0}, f_{2k+1,1}, f_{2k+1,3}, \cdots, f_{2k+1,2k+1}\}, \quad 1 \le k \le n, \\ &\mathcal{C}_{2\,k} = \{f_{2k,1}, f_{2k,3}, \cdots, f_{2k,2k-1}, f_{2k,2k-4}, f_{2k,2k-6}, \cdots, f_{2k,0}\}, \quad 2 \le k \le n, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{X}_{2\,n+2} = \{B_{2n+2,2n+1}, B_{2n+2,2n-1}, \cdots, B_{2n+1,1}\}, \\ &\mathcal{Y}_{2\,k+1} = \{B_{2k+1,2k+1}, B_{2k+1,2k-1}, \cdots, B_{2k+1,1}, B_{2k+1,0}, B_{2k+1,2}, \cdots, B_{2k+1,2k}\}, \quad 1 \le k \le n, \\ &\mathcal{Y}_{2\,k} = \{B_{2k,0}, B_{2k,2}, \cdots, B_{2k,2k-2}, B_{2k,2k-3}, B_{2k,2k-5}, \cdots, B_{2k,1}\}, \quad 2 \le k \le n. \end{aligned}$$

Set the first $(n-1)$ Liapunov constants to be all zero, i.e., $W_k = B_{2k+2,2k+1} = 0$ for all $1 \le k \le n-1$ and let $V_n = B_{2n+2,2n+1} + v$, where v is a dummy variable, so then the nth Liapunov constant can be obtained through a chain of pseudo-divisions:

$$\begin{aligned} S_{2n+2}^{(2n+2)} &:= \operatorname{prem}(V_n, \mathcal{A}_{2n+2}, \mathcal{X}_{2n+2}), \\ S_{2n+2}^{(2n+1)} &:= \operatorname{prem}(S_{2n+2}^{(2n+2)}, \mathcal{C}_{2n+1}, \mathcal{Y}_{2n+1}), \\ S_{2n+2}^{(2n)} &:= \operatorname{prem}(S_{2n+2}^{(2n+1)}, \mathcal{C}_{2n}, \mathcal{Y}_{2n}), \\ S_{2n+2}^{(2n-1)} &:= \operatorname{prem}(S_{2n+2}^{(2n)}, \mathcal{C}_{2n-1}, \mathcal{Y}_{2n-1}), \\ &\ \vdots \\ S_{2n+2}^{(4)} &:= \operatorname{prem}(S_{2n+2}^{(5)}, \mathcal{C}_4, \mathcal{Y}_4), \\ S_{2n+2}^{(3)} &:= \operatorname{prem}(S_{2n+2}^{(4)}, \mathcal{C}_3, \mathcal{Y}_3), \\ W_n &:= -v + \frac{S_{2n+2}^{(3)}}{\operatorname{coeff}(S_{2n+2}^{(3)}, v)}. \end{aligned}$$

The last equation gives the nth Liapunov constant in the coefficients of system (1.6), where $\operatorname{coeff}(S_{2n+2}^{(3)}, v)$ stands for the coefficient of v in the polynomial $S_{2n+2}^{(3)}$.

Although the basic idea of the previous formulae and the ones implicit in the Maple procedure licon (see [20]) are the same, there are two differences: First, our formulae are designed just for system (1.6), while the Maple procedure licon is designed for the general polynomial differential systems; secondly, we find that it takes $2n^2 + 4n$ times of pseudo-divisions to compute the nth order Liapunov constant by using our formulae, while in licon it takes $2n^2 + 6n$ times.

Since the solution of the center-focus problem for system (1.6) is very difficult, we restrict our attention to a special system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + c_{2,0}x^2 + c_{3,0}x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + d_{2,0}x^2 + d_{2,1}xy + d_{3,0}x^3 + d_{3,1}x^2y + d_{3,2}xy^2 \end{cases} \tag{2.4}$$

with $c_{3,0} \neq 0$.

If $c_{3,0} > 0$, then using the scaling: $x \to \frac{x}{k},\ y \to \frac{y}{k}$, system (2.4) can be transformed into

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + a_{2,0}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 + b_{2,1}xy + b_{3,0}x^3 + b_{3,1}x^2y + b_{3,2}xy^2, \end{cases} \tag{2.5}$$

in which

$$c_{2,0} = a_{2,0}\, k, \quad c_{3,0} = k^2, \quad d_{2,0} = b_{2,0}\, k, \tag{2.6}$$

$$d_{2,1} = b_{2,1}\, k, \quad d_{3,0} = b_{3,0}\, k^2, \quad d_{3,1} = b_{3,1}\, k^2, \quad d_{3,2} = b_{3,2}\, k^2. \tag{2.7}$$

If $c_{3,0} < 0$, then using the scaling: $x \to \frac{x}{k},\ y \to \frac{y}{k}$, system (2.4) can be transformed into

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + a_{2,0}x^2 - x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 + b_{2,1}xy + b_{3,0}x^3 + b_{3,1}x^2y + b_{3,2}xy^2, \end{cases} \tag{2.8}$$

in which

$$c_{2,0} = a_{2,0}\, k, \quad c_{3,0} = -k^2, \quad d_{2,0} = b_{2,0}\, k, \tag{2.9}$$

$$d_{2,1} = b_{2,1}\, k, \quad d_{3,0} = b_{3,0}\, k^2, \quad d_{3,1} = b_{3,1}\, k^2, \quad d_{3,2} = b_{3,2}\, k^2. \tag{2.10}$$

Considering the coefficient conditions (2.6)–(2.7) and (2.9)–(2.10), a discussion of (2.5) and (2.8) in the next section gives the following results.

Theorem 2.1 *For system* (2.4) *with $c_{3,0} \neq 0$, the origin is a center if and only if one of the following conditions holds:*

(1) $c_{2,0} = a_{2,0}k,\ c_{3,0} = k^2,\ d_{2,0} = b_{2,0}k,\ d_{2,1} = -2a_{2,0}k,\ d_{3,0} = b_{3,0}k^2,\ d_{3,1} = -3k^2,$ $d_{3,2} = 0$;

(2) $c_{2,0} = a_{2,0}k,\ c_{3,0} = k^2,\ d_{2,0} = 0,\ d_{2,1} = 3a_{2,0}k,\ d_{3,0} = b_{3,0}k^2,\ d_{3,1} = -3k^2,\ d_{3,2} = 0$;

(3) $c_{2,0} = -\frac{3k}{2b_{2,0}},\ c_{3,0} = k^2,\ d_{2,0} = b_{2,0}k,\ d_{2,1} = -\frac{b_{3,1}k}{b_{2,0}},\ d_{3,0} = 0,\ d_{3,1} = b_{3,1}k^2,\ d_{3,2} = 0$;

(4) $c_{2,0} = -\frac{(5+b_{2,0}b_{2,1})k}{2b_{2,0}},\ c_{3,0} = k^2,\ d_{2,0} = b_{2,0}k,\ d_{2,1} = b_{2,1}k,\ d_{3,0} = -\frac{(6+5b_{2,0}b_{2,1}+b_{2,1}^2 {b_{2,0}}^2)k^2}{2{b_{2,0}}^2},$ $d_{3,1} = 2k^2,\ d_{3,2} = 0$;

(5) $c_{2,0} = -\frac{(6+b_{2,0}b_{2,1})k}{2b_{2,0}},\ c_{3,0} = k^2,\ d_{2,0} = b_{2,0}k,\ d_{2,1} = b_{2,1}k,\ d_{3,0} = 0,\ d_{3,1} = 3k^2,$ $d_{3,2} = \frac{3(3+b_{2,0}b_{2,1})k^2}{{b_{2,0}}^2}$;

(6) $c_{2,0} = a_{2,0}k,\ c_{3,0} = -k^2,\ d_{2,0} = b_{2,0}k,\ d_{2,1} = -2a_{2,0}k,\ d_{3,0} = b_{3,0}k^2,\ d_{3,1} = 3k^2,$ $d_{3,2} = 0$;

(7) $c_{2,0} = a_{2,0}k,\ c_{3,0} = -k^2,\ d_{2,0} = 0,\ d_{2,1} = 3a_{2,0}k,\ d_{3,0} = b_{3,0}k^2,\ d_{3,1} = 3k^2,\ d_{3,2} = 0$;

(8) $c_{2,0}=\frac{3k}{2b_{2,0}},\ c_{3,0}=-k^2,\ d_{2,0}=b_{2,0}k,\ d_{2,1}=-\frac{b_{3,1}k}{b_{2,0}},\ d_{3,0}=0,\ d_{3,1}=b_{3,1}k^2,\ d_{3,2}=0;$

(9) $c_{2,0}=-\frac{(-6+b_{2,0}b_{2,1})k}{2b_{2,0}},\ c_{3,0}=-k^2,\ d_{2,0}=b_{2,0}k,\ d_{2,1}=b_{2,1}k,\ d_{3,0}=0,\ d_{3,1}=-3k^2,$ $d_{3,2}=-\frac{3(-3+b_{2,0}b_{2,1})k^2}{{b_{2,0}}^2};$

(10) $c_{2,0}=-\frac{(-5+b_{2,0}b_{2,1})k}{2b_{2,0}},\ c_{3,0}=-k^2,\ d_{2,0}=b_{2,0}k,\ d_{2,1}=b_{2,1}k,$ $d_{3,0}=-\frac{(6-5b_{2,0}b_{2,1}+b_{2,1}^2{b_{2,0}}^2)k^2}{2{b_{2,0}}^2},\ d_{3,1}=-2k^2,\ d_{3,2}=0,$
where k is a non-zero real parameter.

Theorem 2.2 *System* (2.4) *with* $c_{3,0}\neq 0$ *cannot have an isochronous center at the origin.*

3 Proof of Main Results

According to the recursive formulae in Section 2, we obtain the first seven Liapunov constants of system (2.5):

$$
\begin{aligned}
W_1 &= \frac{2}{3}\,b_{3,1}+2+\frac{2}{3}\,b_{2,0}b_{2,1}+\frac{4}{3}\,a_{2,0}b_{2,0},\\
W_2 &= -\frac{6}{5}\,b_{3,2}-2\,b_{3,0}+\frac{4}{15}\,b_{2,1}b_{3,1}a_{2,0}+\frac{6}{5}\,a_{2,0}b_{2,1}-\frac{2}{5}\,b_{2,1}^2-\frac{4}{5}\,b_{3,1}a_{2,0}^2\\
&\quad -\frac{4}{3}\,b_{3,2}a_{2,0}b_{2,0}+\frac{2}{15}\,(-b_{3,2}-5\,b_{3,0})b_{3,1},\\
W_3 &= -\frac{12}{5}+\frac{4}{15}\,a_{2,0}b_{3,1}^2b_{2,0}+\frac{118}{525}\,b_{3,2}b_{3,1}a_{2,0}^2+\frac{2}{15}\,b_{3,2}b_{3,1}b_{2,0}^2+\frac{2}{15}\,b_{3,2}b_{3,1}b_{3,0}\\
&\quad +\frac{46}{175}\,b_{3,2}a_{2,0}b_{2,1}-\frac{46}{525}\,b_{3,2}b_{2,1}b_{3,1}a_{2,0}+\frac{4}{15}\,b_{3,1}a_{2,0}b_{2,0}+\frac{6}{5}\,b_{3,0}b_{3,2}-\frac{2}{525}\,b_{3,2}^2b_{3,1}\\
&\quad +\frac{8}{175}\,b_{3,2}b_{2,1}^2-\frac{6}{7}\,b_{3,2}a_{2,0}^2-\frac{2}{15}\,b_{3,2}b_{2,0}^2-\frac{8}{5}\,a_{2,0}b_{2,0}+\frac{2}{5}\,b_{3,1}^2\\
&\quad +\frac{34}{175}\,b_{3,2}^2+\frac{2}{5}\,b_{3,1},\\
W_4 &= \frac{3124}{2625}\,b_{2,1}b_{3,1}a_{2,0}+\frac{25454}{23625}\,b_{3,1}^2b_{2,1}a_{2,0}-\frac{3784}{23625}\,b_{2,1}b_{3,1}^3a_{2,0}-\frac{136}{4725}\,b_{3,2}^2b_{3,1}b_{3,0}\\
&\quad -\frac{21913}{18375}\,b_{3,2}^2a_{2,0}b_{2,1}-\frac{44}{45}\,b_{3,2}b_{3,1}a_{2,0}^4-\frac{22}{135}\,b_{3,2}b_{3,1}b_{3,0}^2+\frac{1012}{525}\,b_{3,2}a_{2,0}^3b_{2,1}\\
&\quad +\frac{55}{21}\,b_{3,2}a_{2,0}^2b_{2,1}^2-\frac{253}{525}\,b_{3,2}b_{2,1}^3a_{2,0}+\frac{35282}{18375}\,b_{3,2}^2b_{3,1}a_{2,0}^2-\frac{88}{1575}\,b_{3,0}b_{3,2}b_{2,1}^2\\
&\quad +\frac{6644}{1575}\,b_{3,0}b_{3,2}a_{2,0}^2-\frac{9658}{875}\,a_{2,0}b_{2,1}+\frac{19316}{2625}\,b_{3,1}a_{2,0}^2-\frac{528}{875}\,b_{3,1}^2a_{2,0}^2-\frac{2332}{2625}\,b_{3,1}^3a_{2,0}^2\\
&\quad +\frac{1892}{7875}\,b_{3,1}^2b_{3,1}^3+\frac{1892}{23625}\,b_{3,2}b_{3,1}^3+\frac{166}{55125}\,b_{3,2}^3b_{3,1}-\frac{592}{55125}\,b_{3,2}^2b_{2,1}^2\\
&\quad +\frac{52466}{11025}\,b_{3,2}^2a_{2,0}^2-\frac{44}{189}\,b_{3,2}^2b_{2,0}^2-\frac{264}{35}\,b_{3,2}a_{2,0}^4+\frac{6364}{7875}\,b_{3,2}b_{3,1}^2-\frac{22}{15}\,b_{3,0}^2b_{3,2}\\
&\quad +\frac{1276}{945}\,b_{3,0}b_{3,1}^2-\frac{1208}{1575}\,b_{3,0}b_{3,2}^2+\frac{374}{525}\,b_{3,1}b_{3,0}+\frac{404}{1575}\,b_{3,2}b_{3,1}+\frac{3014}{7875}\,b_{3,1}b_{2,1}^2\\
&\quad +\frac{1892}{4725}\,b_{3,1}^3b_{3,0}-\frac{253}{675}\,b_{3,0}b_{3,2}b_{2,1}b_{3,1}a_{2,0}-\frac{13507}{55125}\,b_{3,2}^2b_{2,1}b_{3,1}a_{2,0}\\
&\quad -\frac{352}{225}\,b_{3,0}b_{3,2}b_{3,1}a_{2,0}^2+\frac{4301}{1575}\,b_{3,0}b_{3,2}a_{2,0}b_{2,1}-\frac{2866}{55125}\,b_{3,2}^3-\frac{96}{125}\,b_{3,2}\\
&\quad +\frac{418}{525}\,b_{3,0}-\frac{374}{875}\,b_{2,1}^2,
\end{aligned}
$$

and $W_5,\ W_6,\ W_7$ have 66, 105, 64 terms, respectively. We would not present them here due to their lengthy expressions, but one can easily calculate them by using our formulae with the Maple computer algebra system.

Let $\mathcal{J}=\langle W_1,W_2,\cdots\rangle$, and then the ideal $\mathcal{J}$ is called the Bautin ideal of system (2.5). And let $\mathcal{J}_k=\langle W_1,W_2,\cdots,W_k\rangle$ be the polynomial ideal generated by $W_1,W_2,\cdots,W_k$. The affine variety $V(\mathcal{J})$ is called the center variety for the singular point at the origin of system (2.5). Computing a Gröbner basis G of the ideal $\mathcal{J}_7$ with respect to the graded reverse lexicographical order with

$$b_{2,0}\succ b_{3,0}\succ b_{2,1}\succ a_{2,0}\succ b_{3,1}\succ b_{3,2},$$

we obtain a list of polynomials:

$$\begin{aligned}
J_1&=b_{3,2}b_{3,1}-3\,b_{3,2},\\
J_2&=b_{3,0}b_{3,2},\\
J_3&=b_{3,1}+3+b_{2,0}b_{2,1}+2\,a_{2,0}b_{2,0},\\
J_4&=2\,b_{2,0}b_{3,2}^2-3\,b_{2,1}b_{3,2}+6\,b_{3,2}a_{2,0},\\
J_5&=b_{3,2}b_{2,1}^2-4\,b_{3,2}a_{2,0}^2+4\,b_{3,2}^2,\\
J_6&=b_{3,0}b_{3,1}^2+b_{3,1}b_{3,0}-6\,b_{3,0},\\
J_7&=2\,b_{2,1}b_{3,1}a_{2,0}-6\,b_{3,1}a_{2,0}^2-10\,b_{3,2}a_{2,0}b_{2,0}-3\,b_{2,1}^2+9\,a_{2,0}b_{2,1}\\
&\quad-5\,b_{3,1}b_{3,0}-15\,b_{3,0}-12\,b_{3,2},\\
J_8&=2\,a_{2,0}b_{3,1}^3-3\,b_{2,1}b_{3,1}^2+2\,a_{2,0}b_{3,1}^2-3\,b_{2,1}b_{3,1}-12\,a_{2,0}b_{3,1}+12\,b_{2,0}b_{3,2}+18\,b_{2,1},\\
J_9&=2\,a_{2,0}b_{3,1}^2b_{2,0}-18+2\,b_{3,1}a_{2,0}b_{2,0}+2\,b_{3,2}b_{2,0}^2-12\,a_{2,0}b_{2,0}+3\,b_{3,1}^2+3\,b_{3,1},\\
J_{10}&=10\,b_{2,0}a_{2,0}^2b_{3,1}+10\,b_{2,0}^2a_{2,0}b_{3,2}+30\,b_{2,0}a_{2,0}^2+5\,b_{2,0}b_{3,0}b_{3,1}+2\,a_{2,0}b_{3,1}^2+15\,b_{2,0}b_{3,0}\\
&\quad-3\,b_{2,1}b_{3,1}+21\,a_{2,0}b_{3,1}+12\,b_{2,0}b_{3,2}-9\,b_{2,1}+45\,a_{2,0},\\
J_{11}&=2\,b_{3,0}b_{3,1}b_{2,1}^2-18\,b_{3,0}b_{3,1}a_{2,0}^2-7\,b_{3,0}b_{2,1}^2+13\,b_{3,0}a_{2,0}b_{2,1}+24\,b_{3,0}a_{2,0}^2-5\,b_{3,1}b_{3,0}^2\\
&\quad-15\,b_{3,0}^2,\\
J_{12}&=3\,b_{3,0}b_{2,1}^3-7\,b_{3,0}b_{2,1}^2a_{2,0}-14\,b_{3,0}b_{2,1}a_{2,0}^2+24\,b_{3,0}a_{2,0}^3+5\,b_{3,0}^2b_{2,1}b_{3,1}\\
&\quad+10\,b_{3,0}^2a_{2,0}b_{3,1}+15\,b_{3,0}^2b_{2,1}+30\,b_{3,0}^2a_{2,0}.
\end{aligned}$$

Before proving the main results, we start with four lemmas.

Lemma 3.1 *The center variety of system* (2.5) *is the variety of the ideal* $\mathcal{J}_7$ *generated by the first seven Liapunov constants, and is composed of the following five components:*

(1) $V_1=V(I_1)$, *where* $I_1=\langle b_{3,2},b_{2,1}+2a_{2,0},b_{3,1}+3\rangle$;
(2) $V_2=V(I_2)$, *where* $I_2=\langle b_{2,0},b_{3,2},3a_{2,0}-b_{2,1},b_{3,1}+3\rangle$;
(3) $V_3=V(I_3)$, *where* $I_3=\langle 2a_{2,0}b_{2,0}+3,b_{3,0},b_{3,2},2a_{2,0}b_{3,1}-3b_{2,1}\rangle$;
(4) $V_4=V(I_4)$, *where*

$$I_4=\langle 5+b_{2,0}b_{2,1}+2a_{2,0}b_{2,0},12a_{2,0}^2-13a_{2,0}b_{2,1}+3b_{2,1}^2+25b_{3,0},b_{3,2},b_{3,1}-2\rangle;$$

(5) $V_5=V(I_5)$, *where* $I_5=\langle 6+b_{2,0}b_{2,1}+2a_{2,0}b_{2,0},b_{3,0},-b_{2,1}^2+4a_{2,0}^2-4b_{3,2},b_{3,1}-3\rangle$.

Proof Using the radical ideal membership test, we can verify that

$$W_2\notin\sqrt{\langle W_1\rangle},\quad W_3\notin\sqrt{\langle W_1,W_2\rangle},\quad\cdots,\quad W_7\notin\sqrt{\langle W_1,W_2,\cdots,W_6\rangle},$$

$$W_8 \in \sqrt{\langle W_1, W_2, \cdots, W_7\rangle}.$$

Thus we expect that $V(\mathcal{J}) = V(\mathcal{J}_7)$. To verify it, first we find that

$$V(\mathcal{J}_7) = V(G) = \bigcup_{k=1}^{5} V(I_k),$$

and then prove that every system from V_j, $1 \le j \le 5$ has a center at the origin.

Any system from the component V_1 has the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + a_{2,0}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 - 2\,a_{2,0}xy + b_{3,0}x^3 - 3\,x^2 y. \end{cases} \tag{3.1}$$

Since (3.1) is a Hamiltonian system, it has a center at the origin.

Any system from the component V_2 has the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + a_{2,0}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + 3\,a_{2,0}xy + b_{3,0}x^3 - 3\,x^2 y. \end{cases} \tag{3.2}$$

By the transformation

$$X = x, \quad Y = y + a_{2,0}x^2 + x^3,$$

system (3.2) becomes a time reversible system, and hence it has a center at the origin.

Any system from the component V_3 has the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y - \dfrac{3}{2b_{2,0}}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 - \dfrac{b_{3,1}}{b_{2,0}}xy + b_{3,1}x^2 y, \end{cases} \tag{3.3}$$

which admits an integrating factor

$$\mu(x,y) = \frac{b_{2,0}}{(-2\,b_{2,0}^2 - 6\,b_{2,0}y + (-3\,b_{3,1} + 9)x^2 + (2\,b_{3,1}b_{2,0} - 6\,b_{2,0})x^3)(b_{2,0} + b_{3,1}y)},$$

so it has a center at the origin.

Any system from the component V_4 has the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y - \dfrac{(b_{2,0}b_{2,1} + 5)}{2b_{2,0}}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 + b_{2,1}xy - \dfrac{(b_{2,0}^2{b_{2,1}}^2 + 5\,b_{2,0}b_{2,1} + 6)}{2{b_{2,0}}^2}x^3 + 2\,x^2 y, \end{cases} \tag{3.4}$$

which admits an integrating factor

$$\mu(x,y) = \Big(\frac{1}{2}\,b_{2,0} + y - \frac{(b_{2,0}b_{2,1} + 2)}{2b_{2,0}}x^2\Big)^{-\frac{5}{2}},$$

so it has a center at the origin.

Any system from the component V_5 has the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y - \dfrac{(b_{2,0}b_{2,1}+6)}{2b_{2,0}}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 + b_{2,1}xy + 3\,x^2y + \dfrac{3(b_{2,0}b_{2,1}+3)}{b_{2,0}^2}xy^2, \end{cases} \tag{3.5}$$

which admits an integrating factor

$$\mu(x,y) = \frac{9}{(b_{2,0}+3y)^2},$$

and hence it has a center at the origin.

With the same principle, we have the following result about system (2.8).

Lemma 3.2 *The center variety of system* (2.8) *is the variety of the ideal* $\mathcal{K}_7$ *generated by the first seven Liapunov constants, and is composed of the following five components:*

(1) $V_1 = V(I_1)$, *where* $I_1 = \langle b_{3,2}, b_{2,1}+2a_{2,0}, b_{3,1}-3\rangle$;

(2) $V_2 = V(I_2)$, *where* $I_2 = \langle b_{2,0}, b_{3,2}, 3a_{2,0}-b_{2,1}, b_{3,1}-3\rangle$;

(3) $V_3 = V(I_3)$, *where* $I_3 = \langle 2a_{2,0}b_{2,0}-3, b_{3,0}, b_{3,2}, 2b_{3,1}a_{2,0}+3b_{2,1}\rangle$;

(4) $V_4 = V(I_4)$, *where* $I_4 = \langle -6+b_{2,0}b_{2,1}+2a_{2,0}b_{2,0}, b_{3,0}, -b_{2,1}^2+4a_{2,0}^2-4b_{3,2}, b_{3,1}+3\rangle$;

(5) $V_5 = V(I_5)$, *where*

$$I_5 = \langle -5+b_{2,0}b_{2,1}+2a_{2,0}b_{2,0}, 12a_{2,0}^2-13a_{2,0}b_{2,1}+3b_{2,1}^2+25b_{3,0}, b_{3,2}, b_{3,1}+2\rangle.$$

Lemma 3.3 *System* (2.5) *cannot have an isochronous center at the origin.*

Proof Using the general algorithm in paper [9], we obtain the first two isochronous constants p_2, p_4 for system (2.5):

$$\begin{aligned} p_2 &= \frac{1}{12}\,\pi\,(-5\,a_{2,0}b_{2,1} + 4\,a_{2,0}^2 + 10\,b_{2,0}^2 + b_{2,1}^2 + 9\,b_{3,0} + 3\,b_{3,2}), \\ p_4 &= -\frac{1}{1152}\,\pi\,(-243 - 15\,b_{2,1}^2a_{2,0}^2 + 14\,b_{2,1}^3a_{2,0} - 259\,b_{2,1}^2b_{2,0}^2 - 304\,b_{2,1}a_{2,0}^3 \\ &\quad + 164\,a_{2,0}^2b_{2,0}^2 + 480\,a_{2,0}^2b_{3,0} - 144\,a_{2,0}^2b_{3,2} - 1980\,b_{2,0}^2b_{3,0} - 84\,b_{2,0}^2b_{3,2} - 42\,b_{2,1}^2b_{3,0} \\ &\quad - 6\,b_{2,1}^2b_{3,2} + 738\,b_{2,1}b_{2,0} - 90\,b_{3,0}b_{3,2} - 1836\,a_{2,0}b_{2,0} + 342\,b_{3,1} + b_{2,1}^4 + 304\,a_{2,0}^4 \\ &\quad - 140\,b_{2,0}^4 - 99\,b_{3,1}^2 - 459\,b_{3,0}^2 - 27\,b_{3,2}^2 - 498\,b_{2,1}b_{2,0}b_{3,1} + 426\,b_{2,1}a_{2,0}b_{3,0} \\ &\quad + 150\,b_{2,1}a_{2,0}b_{3,2} + 824\,b_{2,1}a_{2,0}b_{2,0}^2 + 1260\,a_{2,0}b_{2,0}b_{3,1}). \end{aligned}$$

The simultaneous vanishing of polynomials in $G \cup \{p_2, p_4\}$ gives rise to only one case

$$\begin{aligned} &a_{2,0} = \frac{-81+56\,b_{2,0}^4}{378\,b_{2,0}}, \quad b_{2,1} = -\frac{(-81+56\,b_{2,0}^4)}{189\,b_{2,0}}, \\ &b_{3,0} = -\frac{1}{71442}\,\frac{(6561+70308\,b_{2,0}^4+3136\,b_{2,0}^8)}{b_{2,0}^2}, \quad b_{3,1} = -3, \quad b_{3,2} = 0, \end{aligned}$$

where G is the Gröbner basis of $\mathcal{J}_7$. For this case, (2.5) has the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + \dfrac{1}{378}\,\dfrac{(-81+56\,b_{2,0}^4)}{b_{2,0}}x^2 + x^3, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + b_{2,0}x^2 - \dfrac{1}{189}\,\dfrac{(-81+56\,b_{2,0}^4)}{b_{2,0}}xy \\ \qquad -\dfrac{1}{71442}\,\dfrac{(6561+70308\,b_{2,0}^4+3136\,b_{2,0}^8)}{b_{2,0}^2}x^3 - 3\,x^2y. \end{cases} \tag{3.6}$$

Again using the general algorithm in paper [9], we get the third-order isochronous constant of system (3.6):

$$p_6 = \frac{5(19683+140616\,b_{2,0}^4+18032\,b_{2,0}^8)}{108864\,b_{2,0}^2}\pi,$$

which is positive for all real $b_{2,0}$, and hence system (2.5) cannot have an isochronous center at the origin.

By the same method as in the proof of Lemma 3.3, we get the next lemma.

Lemma 3.4 *System* (2.8) *cannot have an isochronous center at the origin.*

Proof of Theorem 2.1 By Lemmas 3.1–3.2 and the coefficient conditions (2.6)–(2.7) and (2.9)–(2.10), we conclude that Theorem 2.1 holds.

Proof of Theorem 2.2 By Lemmas 3.3–3.4 and the coefficient conditions (2.6)–(2.7) and (2.9)–(2.10), we conclude that Theorem 2.2 holds.

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