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On Unitary Invariant Weakly Complex Berwald Metrics with Vanishing Holomorphic Curvature and Closed Geodesics*

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Abstract In this paper, the authors construct a class of unitary invariant strongly pseudoconvex complex Finsler metrics which are of the form $F = \sqrt{rf(s-t)}$, where $r = \|v\|^2$, $s = \frac{|\langle z,v\rangle|^2}{r}$, $t = \|z\|^2$, $f(w)$ is a real-valued smooth positive function of $w \in \mathbb{R}$, and z is in a unitary invariant domain $M \subset \mathbb{C}^n$. Complex Finsler metrics of this form are unitary invariant. We prove that F is a class of weakly complex Berwald metrics whose holomorphic curvature and Ricci scalar curvature vanish identically and are independent of the choice of the function f. Under initial value conditions on f and its derivative f', we prove that all the real geodesics of $F = \sqrt{rf(s-t)}$ on every Euclidean sphere $\mathbf{S}^{2n-1} \subset M$ are great circles.

Keywords Complex Finsler metrics, Weakly complex Berwald metrics, Closed geodesics
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1 Introduction and Main Results

As is well known, complex Finsler metrics have become a very useful tool in geometric function theory of holomorphic mappings (see [2]). Two most important such metrics are the Carathéodory and Kobayashi metrics, which share in higher dimensions the properties of the Poincaré metric in the unit disc in $\mathbb{C}$. In general, however, these two metrics do not have enough smoothness to allow a differential geometric study. In [8], Lempert proved a fundamental result which states that in smoothly bounded strictly convex domains in $\mathbb{C}^n$ the Kobayashi and Carathéodory metrics agree, and are strongly pseudoconvex complex Finsler metrics in the sense of Abate and Patrizio (see [2]). This fundamental result motivated several authors to investigate the Kobayashi metrics in strictly convex domains from a differential geometric point of view (see [2, 11]). Even in strictly convex domains, however, we do not have the explicit formulae of the Kobayashi and Carathéodory metrics. As was pointed out in [1], "the lack of consideration of explicit examples made the choice of the 'right' notions in the complex setting difficult and sometimes rather artificial $\cdots$, the lack of examples raised the doubt that perhaps

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metrics satisfying such strong conditions occur very infrequently." So we need more explicit examples of complex Finsler metrics.

In the study of differential geometry of complex Finsler metrics, an important class of complex Finsler metrics comes from complex Berwald metrics, which includes Hermitian metrics and complex Minkowski metrics (see [3]). There are also lots of complex Berwald metrics which are neither Hermitian metrics nor complex Minkowski metrics (see [4]).

Let $\langle\cdot,\cdot\rangle$ be the canonical complex Euclidean inner product in $\mathbb{C}^n$, and $\|\cdot\|$ be the norm induced by $\langle\cdot,\cdot\rangle$, that is, for $z=(z^1,\cdots,z^n)$, $v=(v^1,\cdots,v^n)\in\mathbb{C}^n$, $\langle z,v\rangle=\sum_{i=1}^n z^i\overline{v^i}$, $\|z\|=\sqrt{\langle z,z\rangle}$, where in the following, bars and overlines denote conjugations of complex numbers.

In [14], the author introduced the notion of weakly complex Berwald metrics and proved that the complex Wrona metric (see [6, 14] for more details)

$$F(z,v)=\frac{\|v\|^2}{\sqrt{\|z\|^2\|v\|^2-|\langle z,v\rangle|^2}},\quad (z,v)\in\Omega \tag{1.1}$$

in $\mathbb{C}^n$ is a weakly complex Berwald metric, but not a complex Berwald metric. It was proved in [13] that the conformal change of a weakly complex Berwald metric is also a weakly complex Berwald metric. More precisely, if $\sigma(z):M\to\mathbb{R}$ is a real smooth function on M and $F:T^{1,0}M\to[0,+\infty)$ is a weakly complex Berwald metric in the sense of [14], then $\widetilde{F}=\mathrm{e}^{\sigma(z)}F$ is called a conformal change of F, and $\widetilde{F}$ is still a weakly complex Berwald metric.

Note that the complex Wrona metric (1.1) is only smooth on a subset Ω of the slit holomorphic tangent bundle $T^{1,0}\mathbb{C}^n-\{\text{zero section}\}$ of the holomorphic tangent bundle $T^{1,0}\mathbb{C}^n\cong\mathbb{C}^n\times\mathbb{C}^n$ of $\mathbb{C}^n$. More precisely (see [14]),

$$\Omega=\{(z,v)\in\mathbb{C}^n\times\mathbb{C}^n: z\neq\lambda v,\lambda\in\mathbb{C}\}\subset T^{1,0}\mathbb{C}^n-\{\text{zero section}\}\subset T^{1,0}\mathbb{C}^n.$$

Our purpose in this paper is to construct a class of weakly complex Berwald metrics which are smooth on the whole slit holomorphic tangent bundle $T^{1,0}M$-{zero section} for a unitary invariant domain $M\subset\mathbb{C}^n$. More precisely, we shall introduce a class of unitary invariant complex Finsler metrics of the form

$$F=\sqrt{rf(s-t)}, \tag{1.2}$$

where $r=:\|v\|^2$, $s=:\frac{|\langle z,v\rangle|^2}{r}$, $t=:\|z\|^2$, $f(w)$ is a smooth positive function of $w\in\mathbb{R}$, and z is in a domain $M\subset\mathbb{C}^n$, which is unitary invariant. Note that by Cauchy-Schwarz inequality we always have $s\leq t$.

Our consideration of complex Finsler metrics of the form (1.2) is based on the following observation: The complex Wrona metric (1.1) can be rewritten as

$$F(z,v)=\sqrt{\frac{\|v\|^4}{\|z\|^2\|v\|^2-|\langle z,v\rangle|^2}}=\sqrt{rg(s-t)}, \tag{1.3}$$

where

$$g(s-t)=-\frac{1}{s-t}$$

is only smooth on the subset

$$\{(t,s)\in(0,+\infty)\times[0,+\infty):s<t\}\subset\{(t,s)\in[0,+\infty)\times[0,+\infty):s\leq t\},$$

while the later is equivalent to the whole slit holomorphic tangent bundle $T^{1,0}\mathbb{C}^n$-{zero section}. Thus in order to construct a weakly complex Berwald metric which is smooth on the whole slit holomorphic tangent bundle $T^{1,0}M$-{zero section} for some domain $M \subset \mathbb{C}^n$, it is natural to consider the class of complex Finsler metrics of the form (1.2), which is obtained by replacing the function $g(s-t) = -\frac{1}{s-t}$ with smooth positive functions $f(s-t)$ defined on the whole set $\{(t,s) \in [0,+\infty) \times [0,+\infty) : s \le t\}$.

We shall prove that whenever $F = \sqrt{rf(s-t)}$ is a strongly pseudoconvex complex Finsler metric, $F = \sqrt{rf(s-t)}$ is necessarily a weakly complex Berwald metric with vanishing holomorphic curvature and Ricci scalar curvature, which are independent of the concrete choice of the function f. We also prove that there are lots of functions f such that $F = \sqrt{rf(s-t)}$ is strongly pseudoconvex complex Finsler metrics (see Proposition 3.2 and Corollary 3.1). Under some initial-value conditions on f and its derivative f', we prove a surprising result, that is, the real geodesics of the weakly complex Berwald metric $F = \sqrt{rf(s-t)}$ on every Euclidean sphere $\mathbf{S}^{2n-1} \subset M$ are great circles.

Our main results are as follows (see Theorems 4.1–4.3).

Theorem 1.1 *Let $F = \sqrt{rf(s-t)}$ be a strongly pseudoconvex complex Finsler metric on a unitary invariant domain $M \subset \mathbb{C}^n$. Then F is a weakly complex Berwald metric with vanishing holomorphic curvature and Ricci scalar curvature, i.e.,*

$$\breve{K}_F(z,v) \equiv 0, \quad \breve{\mathrm{Ric}}_F(z,v) \equiv 0 \tag{1.4}$$

for any f.

Theorem 1.2 *Let $F = \sqrt{rf(s-t)}$ be a function defined on the slit holomorphic tangent bundle $T^{1,0}M$-{zero section} of a unitary invariant domain $M \subset \mathbb{C}^n$. Then F is a complex Berwald metric if and only if*

$$f(s-t) = a(s-t) + b \tag{1.5}$$

for constants $a, b \in \mathbb{R}$ satisfying $b > 0$ and $b - at > 0$.

Theorem 1.3 *Let $F = \sqrt{rf(s-t)}$ be a strongly pseudoconvex complex Finsler metric on a unitary invariant domain $M \subset \mathbb{C}^n$ such that the Euclidean sphere $\mathbf{S}^{2n-1}(R) \subset M$, and let $\sigma(\tau) = (\sigma^1(\tau), \cdots, \sigma^n(\tau))$ be a real geodesic of F. Then σ satisfies the following system of equations*:

$$\ddot{\sigma}^\alpha = \frac{1}{c_0}\Big[G_{;\overline{\alpha}} - \frac{1}{k}(s-t)f'(\langle\sigma,\dot{\sigma}\rangle, \|\dot{\sigma}\|^2)Y\begin{pmatrix} s_{\overline{\alpha}} \\ t_{;\overline{\alpha}}\end{pmatrix}\Big], \quad \alpha = 1, \cdots, n, \tag{1.6}$$

where c_0 and k are given by (3.5), the 2×2 matrix Y is given by (3.6), and $G_{;\overline{\alpha}}, s_{\overline{\alpha}}, t_{;\overline{\alpha}}$ are given by (2.1).

If moreover, $f(w)$ satisfies $f(-R^2) = 1$, and $f'(-R^2) = \frac{1}{R^2}$, then for any given points $p, q \in \mathbf{S}^{2n-1}(R)$ with $\langle p,q\rangle = 0$, there exists a unique closed geodesic

$$\sigma(\tau) = \frac{1}{2}[(p - \sqrt{-1}q)\mathrm{e}^{\sqrt{-1}\tau} + (p + \sqrt{-1}q)\mathrm{e}^{-\sqrt{-1}\tau}], \quad \tau \in \mathbb{R}, \tag{1.7}$$

on $\mathbf{S}^{2n-1}(R)$ such that $\sigma(0) = p, \dot{\sigma}(0) = q$ and $\sigma, \dot{\sigma} \in \mathbf{S}^{2n-1}(R)$ with $\langle\sigma,\dot{\sigma}\rangle = 0$; furthermore, the arc length $L(\sigma)$ of σ satisfies

$$L(\sigma) = 2\pi R. \tag{1.8}$$

2 Preliminaries

In this section, we shall recall some necessary notations and definitions, which can be found in [2].

Let M be a complex manifold of complex dimension n. Let $z=(z^1,\cdots,z^n)$ be a local coordinate system in M, and $v=(v^1,\cdots,v^n)$ be the local fibre coordinate system defined by the local holomorphic frame field $\{\frac{\partial}{\partial z^1},\cdots,\frac{\partial}{\partial z^n}\}$ on the holomorphic tangent bundle $T^{1,0}M$ of M. Then $(z,v)=(z^1,\cdots,z^n,v^1,\cdots,v^n)$ is a local coordinate system for $T^{1,0}M$. In the following we denote by $\widetilde{M}$ the complement of the zero section in $T^{1,0}M$, i.e., $\widetilde{M}=T^{1,0}M$-{zero section}.

Definition 2.1 (see [2]) *A complex Finsler metric on a complex manifold M is a continuous function $F:T^{1,0}M\to[0,+\infty)$ satisfying*

(i) $G=F^2$ *is smooth on* $\widetilde{M}$;

(ii) $F(z,v)>0$ for all $(z,v)\in\widetilde{M}$;

(iii) $F(z,\lambda v)=|\lambda|F(z,v)$ for all $(z,v)\in T^{1,0}M$ and $\lambda\in\mathbb{C}$.

Definition 2.2 (see [2]) *A complex Finsler metric F is called strongly pseudoconvex if the Levi matrix* $(G_{\alpha\overline{\beta}})=\Big(\frac{\partial^2 G}{\partial v^\alpha\partial\overline{v^\beta}}\Big)$ *is positive definite on* $\widetilde{M}$.

Denote by $U(n)$ the set of all $n\times n$ unitary matrices over the complex number field $\mathbb{C}$.

Definition 2.3 *A complex Finsler metric F on a domain $M\subset\mathbb{C}^n$ is called unitary invariant, if locally $F(zA,vA)=F(z,v)$ for every $z\in M,\ v\in T^{1,0}_zM$ and $A\in U(n)$.*

Note that if F is a unitary invariant complex Finsler metric on M, then locally M is necessarily unitary invariant in the sense that $Az\in M$ whenever $z\in M$ and $A\in U(n)$.

Let $(G^{\overline{\beta}\gamma})$ be the inverse matrix of $(G_{\alpha\overline{\beta}})$. In the following, for functions G,r,t and s defined on $\widetilde{M}$, we denote by indexes like $\alpha,\overline{\beta}$ and so on the derivatives with respect to the v-coordinates; the derivative with respect to the z-coordinates will be indexed after a semicolon; for instance,

$$G_{\alpha\overline{\beta}}=\dot{\partial}_{\overline{\beta}}\dot{\partial}_\alpha G,\quad G_{\alpha;\overline{\mu}}=\partial_{\overline{\mu}}\dot{\partial}_\alpha G,\quad G_{;\overline{\alpha}}=\partial_{\overline{\alpha}}G,\quad s_{\overline{\alpha}}=\dot{\partial}_{\overline{\alpha}}s,\quad s_{\overline{\beta};\mu}=\partial_\mu\dot{\partial}_{\overline{\beta}}s,\quad t_{;\overline{\alpha}}=\partial_{\overline{\alpha}}t,\tag{2.1}$$

where we set

$$\partial_\alpha=\frac{\partial}{\partial z^\alpha},\quad \partial_{\overline{\alpha}}=\frac{\partial}{\partial\overline{z^\alpha}},\quad \dot{\partial}_\alpha=\frac{\partial}{\partial v^\alpha},\quad \dot{\partial}_{\overline{\alpha}}=\frac{\partial}{\partial\overline{v^\alpha}}.$$

In this paper, we also denote the first and second orders of the derivatives of $f(w)$ by f' and f'', respectively.

It is easy to check that

$$r_\alpha=\overline{v^\alpha},\quad r_{\overline{\alpha}}=v^\alpha,\quad r_{;\alpha}=r_{;\overline{\alpha}}=0,\tag{2.2}$$

$$t_{;\alpha}=\overline{z^\alpha},\quad t_{;\overline{\alpha}}=z^\alpha,\quad t_\alpha=t_{\overline{\alpha}}=0,\tag{2.3}$$

$$s_\alpha=\frac{1}{r}[\langle z,v\rangle t_{;\alpha}-sr_\alpha],\quad s_{\overline{\alpha}}=\frac{1}{r}[\overline{\langle z,v\rangle}t_{;\overline{\alpha}}-sr_{\overline{\alpha}}],\tag{2.4}$$

$$s_{;\alpha}=\frac{1}{r}\overline{\langle z,v\rangle}r_\alpha,\quad s_{;\overline{\alpha}}=\frac{1}{r}\langle z,v\rangle r_{\overline{\alpha}},\tag{2.5}$$

$$s_{\alpha\overline{\beta}}=\frac{1}{r}[t_{;\alpha}t_{;\overline{\beta}}-(s_\alpha r_{\overline{\beta}}+s_{\overline{\beta}}r_\alpha)-s\delta_{\alpha\overline{\beta}}],\tag{2.6}$$

$$s_{\overline{\beta};\mu}=\frac{1}{r}[\overline{\langle z,v\rangle}\delta_{\overline{\beta}\mu}-s_{;\mu}r_{\overline{\beta}}],\tag{2.7}$$

$$\sum_{\alpha=1}^n s_\alpha s_{\overline{\alpha}}=\frac{1}{r}s(t-s),\quad s_\alpha z^\alpha=\frac{1}{r}(t-s)\langle z,v\rangle,\quad s_{\overline{\alpha}}\overline{z^\alpha}=\frac{1}{r}(t-s)\overline{\langle z,v\rangle}\tag{2.8}$$

and

$$s_\alpha v^\alpha = s_{\overline{\alpha}} \overline{v^\alpha} = 0, \quad s_{;\alpha} z^\alpha = s_{;\overline{\alpha}} \overline{z^\alpha} = s, \quad t_{;\alpha} z^\alpha = t_{;\overline{\alpha}} \overline{z^\alpha} = t. \tag{2.9}$$

In complex Finsler geometry, there are several well-known complex Finsler connections, for example, the Chern-Finsler connection (see [2]), the complex Rund connection (see [12]) and the complex Berwald connection (see [10]). These connections are suitable for considering different problems in complex Finsler geometry. As we know, given a real Finsler metric F, there is only one nonlinear connection associated to F (see [9]). Given a strongly pseudoconvex complex Finsler metric F, there are two complex nonlinear connections associated to F: The Chern-Finsler nonlinear connection (see [2]) and the complex Berwald nonlinear connection which is also called the Cartan complex nonlinear connection in [10]. Their corresponding complex nonlinear connection coefficients are denoted by $\varGamma^\gamma_{;\mu}$ and $\mathbb{G}^\gamma_{\ \mu}$, respectively.

Let F be a strongly pseudoconvex complex Finsler metric. Then (see [2, 10])

$$\varGamma^\gamma_{;\mu} = G^{\overline{\beta}\gamma} G_{\overline{\beta};\mu}, \quad \mathbb{G}^\gamma_{\ \mu} = \dot{\partial}_\mu (\mathbb{G}^\gamma), \tag{2.10}$$

where $\mathbb{G}^\gamma = \frac{1}{2} \varGamma^\gamma_{;\mu} v^\mu$ and these coefficients satisfy

$$\varGamma^\gamma_{;\mu}(z, \lambda v) = \lambda \varGamma^\gamma_{;\mu}(z, v), \quad \mathbb{G}^\gamma_{\ \mu}(z, \lambda v) = \lambda \mathbb{G}^\gamma_{\ \mu}(z, v), \quad \mathbb{G}^\gamma(z, \lambda v) = \lambda^2 \mathbb{G}^\gamma(z, v) \tag{2.11}$$

for every nonzero complex number $\lambda \in \mathbb{C}$. Differentiating $\varGamma^\gamma_{;\mu}$ and $\mathbb{G}^\gamma_{\ \mu}$ respectively with respect to v^ν, we get the horizontal Chern-Finsler connection coefficients $\varGamma^\gamma_{\nu;\mu}$ and the complex Berwald connection coefficients $\mathbb{G}^\gamma_{\nu\mu}$, respectively, that is,

$$\varGamma^\gamma_{\nu;\mu} = \dot{\partial}_\nu (\varGamma^\gamma_{;\mu}), \quad \mathbb{G}^\gamma_{\nu\mu} = \dot{\partial}_\nu (\mathbb{G}^\gamma_{\ \mu}). \tag{2.12}$$

As we know (see [2, 10]),

$$\varGamma^\gamma_{;\mu} = \varGamma^\gamma_{\nu;\mu} v^\nu, \quad \mathbb{G}^\gamma_{\ \mu} = \mathbb{G}^\gamma_{\nu\mu} v^\nu. \tag{2.13}$$

Therefore, if $\varGamma^\gamma_{\nu;\mu}$ or $\mathbb{G}^\gamma_{\nu\mu}$ are independent of fibre coordinates v, then $\mathbb{G}^\gamma$ are quadratic with respect to the fibre coordinates $v = (v^1, \cdots, v^n)$. It is clear that $\mathbb{G}^\gamma_{\nu\mu} = \mathbb{G}^\gamma_{\mu\nu}$ in general. However, $\varGamma^\gamma_{\nu;\mu} \neq \varGamma^\gamma_{\mu;\nu}$. In [2], F is called a Kähler Finsler metric if $(\varGamma^\gamma_{\mu;\nu} - \varGamma^\gamma_{\nu;\mu}) v^\mu = 0$, which is equivalent to the condition $\varGamma^\gamma_{\mu;\nu} - \varGamma^\gamma_{\nu;\mu} = 0$ because of Chen-Shen's observation (see [5]). In [3], F is called a complex Berwald metric if $\varGamma^\gamma_{\nu;\mu}$ are independent of the fibre coordinates $v = (v^1, \cdots, v^n)$. In [14], F is called a weakly complex Berwald metric if $\mathbb{G}^\gamma_{\nu\mu}$ are independent of the fibre coordinates $v = (v^1, \cdots, v^n)$. Since

$$\mathbb{G}^\gamma_{\nu\mu} = \frac{1}{2} [\varGamma^\gamma_{\nu;\mu} + \varGamma^\gamma_{\mu;\nu} + \dot{\partial}_\nu (\varGamma^\gamma_{\mu;\epsilon}) v^\epsilon],$$

it follows that a complex Berwald metric is necessarily a weakly complex Berwald metric, while the converse is not true (see [14]).

3 Fundamental Tensor and Nonlinear Connection

In this section, we shall derive the fundamental tensor $G_{\alpha\overline{\beta}}$, the Chern-Finsler nonlinear connection coefficients $\varGamma^\gamma_{;\mu}$ and the complex Berwald nonlinear connection coefficients $\mathbb{G}^\gamma_{\ \mu}$ that are associated to a strongly pseudoconvex complex Finsler metric $F = \sqrt{r f(s - t)}$, respectively.

Put

$$B=\begin{pmatrix} s_1 & t_{;1} \\ \vdots & \vdots \\ s_n & t_{;n} \end{pmatrix}, \quad B^*=\begin{pmatrix} s_{\overline{1}} & \cdots & s_{\overline{n}} \\ t_{;\overline{1}} & \cdots & t_{;\overline{n}} \end{pmatrix}. \tag{3.1}$$

Then by (2.8) we have

$$B^*B=\begin{pmatrix} \frac{1}{r}s(t-s) & \frac{1}{r}(t-s)\overline{\langle z,v\rangle} \\ \frac{1}{r}(t-s)\langle z,v\rangle & t \end{pmatrix}. \tag{3.2}$$

Proposition 3.1 *Suppose that $F=\sqrt{rf(s-t)}$ is a strongly pseudoconvex complex Finsler metric on a domain $M\subset\mathbb{C}^n$. Then the fundamental tensor matrix $H=(G_{\alpha\overline{\beta}})$ associated to F and its inverse H^{-1} are given respectively by*

$$H=c_0I_n+BXB^*, \tag{3.3}$$

$$H^{-1}=\frac{1}{c_0}\Big(I_n-\frac{1}{k}BYB^*\Big), \tag{3.4}$$

where

$$c_0=f-sf', \quad k=c_0(c_0+tf')+s(t-s)ff'', \tag{3.5}$$

$$X=\begin{pmatrix} rf'' & 0 \\ 0 & f' \end{pmatrix}, \quad Y=\begin{pmatrix} r(c_0+tf')f'' & -\overline{\langle z,v\rangle}(t-s)f'f'' \\ -\langle z,v\rangle(t-s)f'f'' & c_0f'+s(t-s)f'f'' \end{pmatrix}. \tag{3.6}$$

Proof Differentiating $G=rf(s-t)$ with respect to v^α and $\overline{v^\beta}$ successively, we get

$$\begin{aligned} G_\alpha &= r_\alpha f+rf's_\alpha, \\ G_{\alpha\overline{\beta}} &= f\delta_{\alpha\overline{\beta}}+f'r_\alpha s_{\overline{\beta}}+f's_\alpha r_{\overline{\beta}}+rf''s_\alpha s_{\overline{\beta}}+rf's_{\alpha\overline{\beta}}. \end{aligned} \tag{3.7}$$

Substituting (2.6) into (3.7), we have

$$G_{\alpha\overline{\beta}}=(f-sf')\delta_{\alpha\overline{\beta}}+rf''s_\alpha s_{\overline{\beta}}+f't_{;\alpha}t_{;\overline{\beta}}.$$

Putting c_0 and X as in the first equalities of (3.5)–(3.6), respectively, we obtain

$$G_{\alpha\overline{\beta}}=c_0\delta_{\alpha\overline{\beta}}+(s_\alpha,t_{;\alpha})X\begin{pmatrix} s_{\overline{\beta}} \\ t_{;\overline{\beta}} \end{pmatrix},$$

and (3.3) follows. Note that the fundamental matrix H is a nonsingular matrix since F is strongly pseudoconvex, so by the formula of the inverse of a small-rank adjustment (see [7] p. 19), we can safely suppose that

$$H^{-1}=\frac{1}{c_0}I_n-BZB^* \tag{3.8}$$

for some 2×2 matrix Z to be determined. Then

$$\begin{aligned} I_n &= H^{-1}H \\ &= \Big(\frac{1}{c_0}I_n-BZB^*\Big)(c_0I_n+BXB^*) \\ &= I_n-B\Big(ZB^*BX+c_0Z-\frac{1}{c_0}X\Big)B^*. \end{aligned}$$

Thus in order to determine the 2×2-matrix Z, it suffices to take

$$ZB^*BX + c_0 Z - \frac{1}{c_0} X = 0,$$

or equivalently

$$Z = \frac{1}{c_0} X (B^*BX + c_0 I_2)^{-1}, \tag{3.9}$$

where $(B^*BX + c_0 I_2)^{-1}$ denotes the inverse matrix of the 2×2-matrix $B^*BX + c_0 I_2$. Note that by (3.2) and (3.6), we have

$$\begin{aligned} B^*BX + c_0 I_2 &= \begin{pmatrix} \frac{1}{r} s(t-s) & \frac{1}{r}(t-s)\overline{\langle z, v\rangle} \\ \frac{1}{r}(t-s)\langle z, v\rangle & t \end{pmatrix} \begin{pmatrix} rf'' & 0 \\ 0 & f' \end{pmatrix} + \begin{pmatrix} c_0 & 0 \\ 0 & c_0 \end{pmatrix} \\ &= \begin{pmatrix} s(t-s)f'' + c_0 & \frac{1}{r}(t-s)\overline{\langle z, v\rangle} f' \\ (t-s)\langle z, v\rangle f'' & tf' + c_0 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} &[s(t-s)f'' + c_0] \cdot (tf' + c_0) - [(t-s)\langle z, v\rangle f''] \cdot \left[\frac{1}{r}(t-s)\overline{\langle z, v\rangle} f'\right] \\ &= st(t-s)f'f'' + c_0 s(t-s)f'' + c_0 tf' + c_0^2 - s(t-s)^2 f'f'' \\ &= s^2(t-s)f'f'' + (f - sf')s(t-s)f'' + c_0 tf' + c_0^2 \\ &= s^2(t-s)f'f'' + s(t-s)ff'' - s^2(t-s)f'f'' + c_0 tf' + c_0^2 \\ &= k, \end{aligned}$$

where k is given by the second equality of (3.5). So

$$(B^*BX + c_0 I_2)^{-1} = \frac{1}{k} \begin{pmatrix} tf' + c_0 & -\frac{1}{r}(t-s)\overline{\langle z, v\rangle} f' \\ -(t-s)\langle z, v\rangle f'' & s(t-s)f'' + c_0 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} Z &= \frac{1}{c_0} X (B^*BX + c_0 I_2)^{-1} \\ &= \frac{1}{c_0 k} \begin{pmatrix} rf'' & 0 \\ 0 & f' \end{pmatrix} \begin{pmatrix} tf' + c_0 & -\frac{1}{r}(t-s)\overline{\langle z, v\rangle} f' \\ -(t-s)\langle z, v\rangle f'' & s(t-s)f'' + c_0 \end{pmatrix} \\ &= \frac{1}{c_0 k} \begin{pmatrix} r(tf' + c_0)f'' & -(t-s)\overline{\langle z, v\rangle} f'f'' \\ -(t-s)\langle z, v\rangle f'f'' & [s(t-s)f'' + c_0]f' \end{pmatrix}. \end{aligned}$$

Thus

$$Z = \frac{1}{c_0 k} Y, \tag{3.10}$$

where Y is given by the second equality of (3.6). Substituting (3.10) into (3.8), we get (3.4).

Denote by $M_{n \times m}(\mathbb{C})$ the set of all $n \times m$ matrices over the complex number field $\mathbb{C}$.

Lemma 3.1 *Let $C \in M_{n\times m}(\mathbb{C})$, $D \in M_{m\times n}(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then*

$$\lambda^m|\lambda I_n - CD| = \lambda^n|\lambda I_m - DC|. \tag{3.11}$$

Proof Consider the following blocked matrices:

$$A := \begin{pmatrix} \lambda I_n & C \\ \lambda D & \lambda I_m \end{pmatrix}, \quad P := \begin{pmatrix} I_n & 0 \\ -D & I_m \end{pmatrix}. \tag{3.12}$$

Using the blocked elementary matrix P acting on A from the right-hand side and the left-hand side, respectively, we get

$$\begin{pmatrix} \lambda I_n & C \\ \lambda D & \lambda I_m \end{pmatrix}\begin{pmatrix} I_n & 0 \\ -D & I_m \end{pmatrix} = \begin{pmatrix} \lambda I_n - CD & C \\ 0 & \lambda I_m \end{pmatrix} \tag{3.13}$$

and

$$\begin{pmatrix} I_n & 0 \\ -D & I_m \end{pmatrix}\begin{pmatrix} \lambda I_n & C \\ \lambda D & \lambda I_m \end{pmatrix} = \begin{pmatrix} \lambda I_n & C \\ 0 & \lambda I_m - DC \end{pmatrix}. \tag{3.14}$$

Taking determinants in both sides of (3.13)–(3.14) yields (3.11).

Proposition 3.2 *$F = \sqrt{rf(s-t)}$ is a strongly pseudoconvex complex Finsler metric on a domain $M \subset \mathbb{C}^n$ $(n \geq 3)$ if and only if*

$$f - sf' > 0, \tag{3.15}$$

$$c_0(c_0 + tf') + s(t-s)ff'' > 0 \tag{3.16}$$

for every nonzero vector $v \in T_z^{1,0}M$ and $z \in M$.

Proof By Proposition 3.1, $H = c_0I_n + BXB^*$. Using (3.2), we have

$$B^*BX = \begin{pmatrix} s(t-s)f'' & \frac{1}{r}(t-s)f'\overline{\langle z,v\rangle} \\ \langle z,v\rangle(t-s)f'' & tf' \end{pmatrix}.$$

By Lemma 3.1, we have

$$\begin{aligned}
&|\lambda I_n - H| \\
&= |(\lambda - c_0)I_n - BXB^*| \\
&= (\lambda - c_0)^{n-2}|(\lambda - c_0)I_2 - B^*BX| \\
&= (\lambda - c_0)^{n-2}\begin{vmatrix} (\lambda - c_0) - s(t-s)f'' & -\frac{1}{r}(t-s)f'\overline{\langle z,v\rangle} \\ -\langle z,v\rangle(t-s)f'' & (\lambda - c_0) - tf' \end{vmatrix} \\
&= (\lambda - c_0)^{n-2}\{(\lambda - c_0)^2 - [s(t-s)f'' + tf'](\lambda - c_0) + st(t-s)f'f'' - s(t-s)^2f'f''\} \\
&= (\lambda - c_0)^{n-2}\{\lambda^2 - [c_0 + (c_0 + tf') + s(t-s)f'']\lambda + c_0(c_0 + tf') + s(t-s)ff''\}.
\end{aligned}$$

Since an $n \times n$ Hermitian matrix H has exactly n real eigenvalues, counting multiplicities, we immediately obtain that F is a strongly pseudoconvex complex Finsler metric if and only if H is a positive definite matrix on $\widetilde{M}$; if and only if

$$c_0 > 0, \quad c_0 + (c_0 + tf') + s(t-s)f'' > 0, \quad c_0(c_0 + tf') + s(t-s)ff'' > 0. \tag{3.17}$$

Note that since $c_0 = f - sf'$, it is easy to check that

$$f\big[c_0 + (c_0 + tf') + s(t-s)f''\big] = f^2 + s(t-s)(f')^2 + c_0(c_0 + tf') + s(t-s)ff'',$$

by which the third inequality of (3.17) implies the second one, so F is strongly pseudoconvex if and only if (3.15)–(3.16) hold.

Remark 3.1 It follows from the proof of Proposition 3.2 that if $n = 2$, then F is a strongly pseudoconvex complex Finsler metric if and only if (3.16) holds.

Corollary 3.1 *Suppose that $f(w)$ is a real-valued smooth positive function of $w = s - t \in \mathbb{R}$ satisfying*

$$f - sf' > 0, \quad f' \geq 0, \quad f'' \geq 0 \tag{3.18}$$

on the slit holomorphic tangent bundle $T^{1,0}M$-{zero section} of a domain $M \subset \mathbb{C}^n$. Then $F = \sqrt{rf(s-t)}$ is a strongly pseudoconvex complex Finsler metric on M.

In the following, we denote by $\mathbb{B}^n(R) = \{\|z\|^2 < R^2\}$ the ball in $\mathbb{C}^n$ with center 0 and radius $R > 0$, and its boundary is denoted by $\mathbf{S}^{2n-1}(R)$, i.e., the sphere in $\mathbb{C}^n$ with center 0 and radius $R > 0$.

Example 3.1 Let $f(w) = \mathrm{e}^{\frac{w}{R^2}}$ with $w = s - t$ and $R \in (0, +\infty)$. Then $f' > 0, f'' > 0$ and

$$f - sf' = \Big(1 - \frac{s}{R^2}\Big)f > 0 \quad \text{if and only if} \quad s < R^2. \tag{3.19}$$

Since $s = \frac{|\langle z,v\rangle|^2}{\|v\|^2} \leq \|z\|^2 = t$, $F = \sqrt{r\mathrm{e}^{\frac{s-t}{R^2}}}$ is a strongly pseudoconvex complex Finsler metric on the ball $\mathbb{B}^n(R)$.

Proposition 3.3 *The complex nonlinear connection coefficients $\varGamma^{\gamma}_{;\mu}$ and $\mathbb{G}^{\gamma}_{\ \mu}$ associated to $F = \sqrt{rf(s-t)}$ are given respectively by*

$$\varGamma^{\gamma}_{;\mu} = \frac{1}{c_0}\Big[G_{\overline{\gamma};\mu} - \frac{1}{k}(g_\mu, h_\mu)Y\begin{pmatrix} s_{\overline{\gamma}} \\ t_{;\overline{\gamma}} \end{pmatrix}\Big], \tag{3.20}$$

$$\mathbb{G}^{\gamma}_{\ \mu} = 0, \tag{3.21}$$

where

$$G_{\overline{\gamma};\mu} = (f'r_{\overline{\gamma}} + rf''s_{\overline{\gamma}})(s_{;\mu} - t_{;\mu}) + rf's_{\overline{\gamma};\mu}, \tag{3.22}$$

$$g_\mu = [s(t-s)f'' - sf'](s_{;\mu} - t_{;\mu}), \tag{3.23}$$

$$h_\mu = (t-s)f''\overline{\langle z,v\rangle}(s_{;\mu} - t_{;\mu}) \tag{3.24}$$

and Y is given by (3.6).

Proof The complex nonlinear connection coefficients $\varGamma^{\gamma}_{;\mu}$ and $\mathbb{G}^{\gamma}_{\ \mu}$ associated to F are given respectively by

$$\varGamma^{\gamma}_{;\mu} = G^{\overline{\beta}\gamma}G_{\overline{\beta};\mu}, \quad \mathbb{G}^{\gamma}_{\ \mu} = \frac{\partial \mathbb{G}^\gamma}{\partial v^\mu}, \tag{3.25}$$

where $\mathbb{G}^\gamma = \frac{1}{2}\varGamma^{\gamma}_{;\mu}v^\mu$. By (3.4),

$$G^{\overline{\beta}\gamma} = \frac{1}{c_0}\Big[\delta^{\overline{\beta}\gamma} - \frac{1}{k}(s_\beta, t_{;\beta})Y\begin{pmatrix} s_{\overline{\gamma}} \\ t_{;\overline{\gamma}} \end{pmatrix}\Big]. \tag{3.26}$$

Differentiating $G = rf(s-t)$ with respect to $\overline{v^\beta}$ and z^μ in turn gives

$$G_{\overline{\beta}} = r_{\overline{\beta}}f + rf's_{\overline{\beta}},$$
$$G_{\overline{\beta};\mu} = (f'r_{\overline{\beta}} + rf''s_{\overline{\beta}})(s_{;\mu} - t_{;\mu}) + rf's_{\overline{\beta};\mu}. \tag{3.27}$$

Then by (2.7)–(2.9), we have

$$\sum_{\beta=1}^{n} G_{\overline{\beta};\mu}s_\beta = [s(t-s)f'' - sf'](s_{;\mu} - t_{;\mu}), \tag{3.28}$$

where we used $\overline{\langle z,v\rangle}s_\mu = -s(s_{;\mu} - t_{;\mu})$ in the above equality.

$$\begin{aligned} G_{\overline{\beta};\mu}t_{;\beta} &= [f'\overline{\langle z,v\rangle} + f''\overline{\langle z,v\rangle}(t-s)](s_{;\mu} - t_{;\mu}) + f'[\overline{\langle z,v\rangle}\overline{z^\mu} - \overline{\langle z,v\rangle}s_{;\mu}] \\ &= f''\overline{\langle z,v\rangle}(t-s)(s_{;\mu} - t_{;\mu}). \end{aligned} \tag{3.29}$$

Substituting (3.26)–(3.29) into the first equality of (3.25) yields (3.20).

On the other hand, an easy computation gives

$$(s_{;\mu} - t_{;\mu})v^\mu = s_{\overline{\beta};\mu}v^\mu = 0.$$

So

$$2\mathbb{G}^\gamma = \varGamma^\gamma_{;\mu}v^\mu = G^{\overline{\beta}\gamma}G_{\overline{\beta};\mu}v^\mu = 0. \tag{3.30}$$

Substituting (3.30) into the second equality of (3.25) yields (3.21).

4 Proofs of Main Results

Note that the holomorphic curvature (see [2]) and the Ricci scalar curvature (see [10]) of a strongly pseudoconvex complex Finsler metric F with respect to the Chern-Finsler connection are defined respectively by

$$K_F(z,v) = -\frac{2}{G^2}G_\alpha\delta_{\overline{\nu}}(\varGamma^\alpha_{;\mu})v^\mu\overline{v^\nu}, \quad \mathrm{Ric}_F(z,v) = -v^\alpha\delta_\alpha(\overline{\varGamma^\mu_{;\mu}}), \tag{4.1}$$

where $\delta_\alpha = \partial_\alpha - \varGamma^\mu_{;\alpha}\dot{\partial}_\mu$. The holomorphic curvature and the Ricci scalar curvature of a strongly pseudoconvex complex Finsler metric F with respect to the complex Berwald connection are defined respectively by [14]:

$$\breve{K}_F(z,v) = -\frac{2}{G^2}G_\alpha\breve{\delta}_{\overline{\nu}}(\mathbb{G}^\alpha_{\ \mu})v^\mu\overline{v^\nu}, \quad \breve{\mathrm{Ric}}_F(z,v) = -v^\alpha\breve{\delta}_\alpha(\overline{\mathbb{G}^\mu_{\ \mu}}), \tag{4.2}$$

where $\breve{\delta}_\alpha = \partial_\alpha - \mathbb{G}^\mu_\alpha\dot{\partial}_\mu$. It was proved in [13] that the holomorphic curvature of F is independent of the choice of the Chern-Finsler connection, or the complex Rund connection or the complex Berwald connection. Actually we have $v^\alpha\delta_\alpha = v^\alpha\breve{\delta}_\alpha$ since $v^\alpha\varGamma^\mu_{;\alpha} = v^\alpha\mathbb{G}^\mu_{\ \alpha}$.

Theorem 4.1 *Let $F = \sqrt{rf(s-t)}$ be a strongly pseudoconvex complex Finsler metric on a unitary invariant domain $M \subset \mathbb{C}^n$. Then F is a weakly complex Berwald metric with vanishing holomorphic curvature and Ricci scalar curvature, i.e.,*

$$\breve{K}_F(z,v) \equiv 0, \quad \breve{\mathrm{Ric}}_F(z,v) \equiv 0 \tag{4.3}$$

for any f.

Proof By Proposition 3.3,

$$\mathbb{G}^{\gamma}_{\ \mu} = 0, \tag{4.4}$$

which implies $\mathbb{G}^{\gamma}_{\nu\mu} = 0$. Thus $F = \sqrt{rf(s-t)}$ is a weakly complex Berwald metric. The holomorphic curvature of F is given by

$$\breve{K}_F(z,v) = -\frac{2}{G^2} G_{\alpha} \breve{\delta}_{\overline{\nu}}(\mathbb{G}^{\alpha}_{\ \mu} v^{\mu}) \overline{v^{\nu}} = -\frac{2}{G^2} G_{\alpha} \breve{\delta}_{\overline{\nu}}(2\mathbb{G}^{\alpha}) \overline{v^{\nu}} = 0.$$

Again by Proposition 3.3, we get $\mathbb{G}^{\mu}_{\ \mu} = 0$, which implies that

$$\breve{\mathrm{R}}\mathrm{ic}_F(z,v) = -\overline{v^{\alpha}} \breve{\delta}_{\overline{\alpha}}(\mathbb{G}^{\mu}_{;\mu}) = 0.$$

Remark 4.1 Since the Chern-Finsler connection coefficients satisfy $\varGamma^{\alpha}_{\beta;\mu} = \dot{\partial}_{\beta}(\varGamma^{\alpha}_{;\mu})$, it follows from Proposition 3.3 that in general, $\varGamma^{\alpha}_{;\mu}$ are not linear with respect to the fibre coordinates $v = (v^1, \cdots, v^n)$, so $F = \sqrt{rf(s-t)}$ is in general not a complex Berwald metric.

More precisely, we have the following theorem.

Theorem 4.2 *Let $F = \sqrt{rf(s-t)}$ be a function defined on the slit holomorphic tangent bundle $T^{1,0}M$-{zero section} of a unitary invariant domain $M \subset \mathbb{C}^n$. Then F is a complex Berwald metric if and only if*

$$f(s-t) = a(s-t) + b \tag{4.5}$$

for constants $a, b \in \mathbb{R}$ satisfying $b > 0$ and $b - at > 0$.

Proof By Proposition 3.3,

$$\varGamma^{\gamma}_{;\mu} = \frac{1}{c_0}\Big[G_{\overline{\gamma};\mu} - \frac{1}{k}(g_{\mu}, h_{\mu}) Y \begin{pmatrix} s_{\overline{\gamma}} \\ t_{;\overline{\gamma}} \end{pmatrix} \Big], \tag{4.6}$$

where $G_{\overline{\gamma};\mu}, g_{\mu}$ and h_{μ} are given by (3.22)–(3.24), and Y is given by (3.6). Using (2.2)–(2.9), (4.6) can be written as

$$\begin{aligned}\varGamma^{\gamma}_{;\mu} = {} & \frac{1}{c_0} f' \overline{\langle z, v\rangle} \delta_{\gamma\mu} + \frac{1}{c_0 k}[sf^2 f'' - c_0(c_0 + tf')f'] \overline{z^{\mu}} v^{\gamma} - \frac{1}{c_0 k} f^2 f'' \overline{\langle z, v\rangle} \overline{z^{\mu}} z^{\gamma} \\ & + \frac{1}{c_0 kr} f^2 f'' (\overline{\langle z, v\rangle})^2 \overline{v^{\mu}} z^{\gamma} - \frac{1}{c_0 kr} s(c_0 + tf') f f'' \overline{\langle z, v\rangle} \overline{v^{\mu}} v^{\gamma}. \end{aligned} \tag{4.7}$$

Suppose that F is a complex Berwald metric. Then $\varGamma^{\gamma}_{;\mu}$ are linear with respect to $v = (v^1, \cdots, v^n)$, i.e., $\varGamma^{\gamma}_{;\mu} = \varGamma^{\gamma}_{\nu;\mu}(z) v^{\nu}$. Thus it is necessary that

$$\frac{\partial}{\partial s}\Big(\frac{f'}{c_0}\Big) = 0, \quad \frac{\partial}{\partial s}\Big(\frac{[sf^2 f'' - c_0(c_0 + tf')f']}{c_0 k}\Big) = 0, \quad \frac{\partial}{\partial s}\Big(\frac{f^2 f''}{c_0 k}\Big) = 0, \tag{4.8}$$

$$\frac{f^2 f''}{c_0 k} = 0, \quad \frac{s(c_0 + tf') f f''}{c_0 k} = 0. \tag{4.9}$$

So by (4.9), it is necessary that $f'' = 0$, i.e., $f(s-t) = a(s-t) + b$ for some constants $a, b \in \mathbb{R}$. By Proposition 3.2, it follows that $b > 0$ and $b - at > 0$. In this case, it is easy to check that the equalities in (4.8) hold identically. Conversely, if $f(s-t) = a(s-t) + b$ for some constants $a, b \in \mathbb{R}$ satisfying $b > 0$, $b - at > 0$, then by Proposition 3.1, the fundamental tensor of $F = \sqrt{rf(s-t)}$ is given by $G_{\alpha\overline{\beta}} = c_0\delta_{\alpha\overline{\beta}} + f'\overline{z^\alpha}z^\beta$. Since

$$\frac{\partial c_0}{\partial s} = -sf'' = 0, \quad \frac{\partial f'}{\partial s} = f'' = 0,$$

it follows that $G_{\alpha\overline{\beta}}$ depends only on $z = (z^1, \cdots, z^n)$ and f satisfies the condition in Proposition 3.2, so that F is actually a Hermitian metric.

In [14], the author investigated the real geodesics of the complex Wrona metric on the Euclidean sphere

$$\mathbf{S}^{2n-1}(1) = \{z \in \mathbb{C}^n \mid \|z\|^2 = 1\},$$

and proved that the geodesics on $\mathbf{S}^{2n-1}(1)$ are great circles. In the following we shall prove a surprising result, that is, under some initial value conditions on f and f', the real geodesics of the weakly complex Berwald metrics of the form $F = \sqrt{rf(s-t)}$ on the sphere $\mathbf{S}^{2n-1}(R)$ are great circles, which is independent of the choice of the function f.

Theorem 4.3 *Let $F = \sqrt{rf(s-t)}$ be a strongly pseudoconvex complex Finsler metric on a unitary invariant domain $M \subset \mathbb{C}^n$ such that the Euclidean sphere $\mathbf{S}^{2n-1}(R) \subset M$, and let $\sigma(\tau) = (\sigma^1(\tau), \cdots, \sigma^n(\tau))$ be a real geodesic of F. Then σ satisfies the following system of equations*:

$$\ddot{\sigma}^\alpha = \frac{2}{c_0}\Big[G_{;\overline{\alpha}} - \frac{1}{k}(s-t)f'(\langle\sigma,\dot{\sigma}\rangle, \|\dot{\sigma}\|^2)Y\begin{pmatrix} s_{\overline{\alpha}} \\ t_{;\overline{\alpha}}\end{pmatrix}\Big], \quad \alpha = 1, \cdots, n, \tag{4.10}$$

where c_0 and k are given by (3.5)*, the 2×2 matrix Y is given by* (3.6)*, and $G_{;\overline{\alpha}}, s_{\overline{\alpha}}, t_{;\overline{\alpha}}$ are given by* (2.1)*.*

If moreover, $f(w)$ satisfies $f(-R^2) = 1$ and $f'(-R^2) = \frac{1}{R^2}$, then for any given points $p, q \in \boldsymbol{S}^{2n-1}(R)$ with $\langle p, q\rangle = 0$, there exists a unique closed geodesic

$$\sigma(\tau) = \frac{1}{2}\Big[(p - \sqrt{-1}q)\mathrm{e}^{\sqrt{-1}\tau} + (p + \sqrt{-1}q)\mathrm{e}^{-\sqrt{-1}\tau}\Big], \quad \tau \in \mathbb{R} \tag{4.11}$$

on $\mathbf{S}^{2n-1}(R)$ such that $\sigma(0) = p$, $\dot{\sigma}(0) = q$ and $\sigma, \dot{\sigma} \in \mathbf{S}^{2n-1}(R)$ with $\langle\sigma, \dot{\sigma}\rangle = 0$; furthermore, the arc length $L(\sigma)$ of σ satisfies

$$L(\sigma) = 2\pi R. \tag{4.12}$$

Proof Since σ is a geodesic of F, it follows that locally σ satisfies (see [2, p. 101])

$$\ddot{\sigma}^\alpha + \varGamma^\alpha_{;\mu}\dot{\sigma}^\mu = G^{\overline{\nu}\alpha}G_{\beta\overline{\gamma}}(\overline{\varGamma^\gamma_{\mu;\nu}} - \overline{\varGamma^\gamma_{\nu;\mu}})\dot{\sigma}^\beta\overline{\dot{\sigma}^\mu}. \tag{4.13}$$

By Proposition 3.3, along the curve σ, we have

$$\varGamma^\alpha_{;\mu}\dot{\sigma}^\mu = 2\mathbb{G}^\alpha = 0. \tag{4.14}$$

Thus differentiating (4.14) with respect to $\dot{\sigma}^\nu$ and using (2.12)–(2.13), we get

$$\varGamma^\alpha_{\nu;\mu}\dot{\sigma}^\mu = -\varGamma^\alpha_{;\nu} = -\varGamma^\alpha_{\mu;\nu}\dot{\sigma}^\mu, \tag{4.15}$$

from which we have

$$(\varGamma^\gamma_{\mu;\nu} - \varGamma^\gamma_{\nu;\mu})\dot{\sigma}^\mu = 2\varGamma^\gamma_{;\nu}. \tag{4.16}$$

Since

$$G_{\beta\overline{\gamma}}\dot{\sigma}^{\beta}=G_{\overline{\gamma}},\quad G_{\overline{\gamma}}G^{\overline{\gamma}\beta}=\dot{\sigma}^{\beta},\quad G_{\beta;\overline{\nu}}\dot{\sigma}^{\beta}=G_{;\overline{\nu}},$$

we have

$$G_{\beta\overline{\gamma}}(\overline{\Gamma^{\gamma}_{\mu;\nu}}-\overline{\Gamma^{\gamma}_{\nu;\mu}})\dot{\sigma}^{\beta}\overline{\dot{\sigma}^{\mu}}=2G_{\overline{\gamma}}\overline{\Gamma^{\gamma}_{;\nu}}=2G_{\overline{\gamma}}G^{\overline{\gamma}\beta}G_{\beta;\overline{\nu}}=2G_{;\overline{\nu}}. \tag{4.17}$$

Substituting (4.14) and (4.17) into (4.13), we obtain the geodesic equations

$$\ddot{\sigma}^{\alpha}=2G^{\overline{\nu}\alpha}G_{;\overline{\nu}},\quad \alpha=1,\cdots,n. \tag{4.18}$$

On the other hand, by Proposition 3.1, we have

$$G^{\overline{\nu}\alpha}=\frac{1}{c_0}\Big[\delta_{\alpha\overline{\nu}}-\frac{1}{k}(s_{\nu},t_{;\nu})Y\begin{pmatrix} s_{\overline{\alpha}}\\ t_{;\overline{\alpha}}\end{pmatrix}\Big], \tag{4.19}$$

$$G_{;\overline{\nu}}=r(s_{;\overline{\nu}}-t_{;\overline{\nu}})f'. \tag{4.20}$$

Substituting (4.19)–(4.20) into (4.18) and using (2.2)–(2.9), we get (4.10).

When $\langle\sigma,\dot{\sigma}\rangle=0,\ \|\sigma\|^2=\|\dot{\sigma}\|^2=R^2$, we have

$$r=t=R^2,\quad s=s_{\nu}=s_{;\overline{\nu}}=0,\quad c_0=f(-R^2), \tag{4.21}$$

$$G_{;\overline{\nu}}=-R^2f'(-R^2)\sigma^{\nu},\quad k=f(-R^2)[f(-R^2)+R^2f'(-R^2)], \tag{4.22}$$

$$Y=\begin{pmatrix} R^2[f(-R^2)+R^2f'(-R^2)]f''(-R^2) & 0\\ 0 & f(-R^2)f'(-R^2)\end{pmatrix}. \tag{4.23}$$

Substituting (4.21)–(4.23) into (4.10) and using $f(-R^2)=1,\ f'(-R^2)=\frac{1}{R^2}$, we obtain

$$\ddot{\sigma}^{\alpha}=-\frac{2R^2f'(-R^2)}{f(-R^2)+R^2f'(-R^2)}\sigma^{\alpha}=-\sigma^{\alpha},\quad \alpha=1,\cdots,n. \tag{4.24}$$

It is clear that the general solution of (4.24) is given by

$$\sigma(\tau)=c_1\mathrm{e}^{\sqrt{-1}\tau}+c_2\mathrm{e}^{-\sqrt{-1}\tau}, \tag{4.25}$$

where $c_1,c_2\in\mathbb{C}^n$ are constant vectors to be determined. Substituting the initial conditions $\sigma(0)=p,\ \dot{\sigma}(0)=q$ into (4.25), we obtain

$$c_1=\frac{1}{2}(p-\sqrt{-1}q),\quad c_2=\frac{1}{2}(p+\sqrt{-1}q),$$

which implies (4.11). Differentiating (4.11) with respect to τ, we get

$$\dot{\sigma}(\tau)=\frac{\sqrt{-1}}{2}[(p-\sqrt{-1}q)\mathrm{e}^{\sqrt{-1}\tau}-(p+\sqrt{-1}q)\mathrm{e}^{-\sqrt{-1}\tau}]. \tag{4.26}$$

By the assumption we have

$$\|p\|^2=\|q\|^2=R^2,\quad \langle p,q\rangle=0. \tag{4.27}$$

Thus it follows from (4.11) and (4.26)–(4.27) that the geodesic σ given by (4.11) actually satisfies

$$\|\sigma\|^2=\|\dot{\sigma}\|^2=R^2,\quad \langle\sigma,\dot{\sigma}\rangle=0. \tag{4.28}$$

By the explicit formula (4.11) for the geodesic $\sigma(\tau)$ of F, we see that $\sigma(\tau)$ is actually a periodic function with period 2π. It is clear that σ is a closed geodesic and $\sigma(0)=\sigma(2\pi)=p$. Moreover, by (4.28) we have

$$F(\sigma,\dot\sigma)=\sqrt{\|\dot\sigma\|^2 f\Big(\frac{|\langle\sigma,\dot\sigma\rangle|^2}{\|\dot\sigma\|^2}-\|\sigma\|^2\Big)}=\sqrt{R^2 f(-R^2)}=R.$$

Hence

$$L(\sigma)=\int_0^{2\pi}F(\sigma,\dot\sigma)\mathrm{d}\tau=2\pi R.$$

Example 4.1 Let $f(w)=2\mathrm{e}^{\frac{1}{2}(1+w)}-1$. Then it is easy to check that $F=\sqrt{rf(s-t)}$ is a strongly pseudoconvex complex Finsler metric defined on a domain M such that $\mathbb{B}^n(1)\subset\subset M$ and $f(-1)=f'(-1)=1$. Thus by Theorem 4.3, the geodesics of F on the unit sphere $\mathbf{S}^{2n-1}(1)$ are great circles.

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