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A Note on the Maximal Functions on Weighted Harmonic AN Groups

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Abstract In this paper, the authors point out that the methods used by Li (2004, 2005, 2007) can be applied to study maximal functions on weighted harmonic AN groups.

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1 Introduction

Let (H, d) be a metric space and ϱ is a Borel measure on H. Denote by $B(x, r)$ the open ball with center $x \in H$ and of radius $r > 0$.

For a locally integrable function f, the centered Hardy-Littlewood maximal function of f is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{\varrho(B(x,r))} \int_{B(x,r)} |f(y)| \mathrm{d}\varrho(y), \quad x \in H.$$

Similarly, the uncentered maximal function of f is defined as

$$M_* f(x) = \sup_{\substack{x \in B(z,r) \\ r>0}} \frac{1}{\varrho(B(z,r))} \int_{B(z,r)} |f(y)| \mathrm{d}\varrho(y), \quad x \in H.$$

Notice that $Mf(x) \le M_* f(x)$ and $\|M_* f\|_{L^\infty} \le \|f\|_{L^\infty}$.

If the measure ϱ satisfies the doubling condition, then M_* is of the weak type $(1, 1)$ (see [4]).

It becomes complicated if the measure ρ does not satisfy the doubling condition, for example, when H is a space of exponential growth. In this case, M and M_* generally have different properties.

A typical example is the non-compact symmetric space. In 1974, Clerc and Stein [3] obtained the L^p $(p > 1)$ boundedness for the centered maximal function M. Subsequently, Strömberg [14] proved that M is of the weak type $(1, 1)$. For non-compact symmetric spaces of real rank 1, Ionescu proved in [12] that the uncentered maximal function M_* is bounded from $L^{2,u}$ to $L^{2,v}$ if and only if $u = 1$, $v = \infty$. On the other hand, he obtained in [11] that M_* is bounded on L^p in the sharp range $p \in (2, \infty]$ on symmetric spaces of arbitrary real rank.

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Another example is the cuspidal manifold $\mathrm{Cusp}(\mathbb{X}) = \mathbb{R}^+ \times \mathbb{X}$ (see [5]). In [6–7], Li studied the centered maximal function M and the uncentered maximal function M_* on $\mathrm{Cusp}(\mathbb{X})$ and its weighted cases. Precisely, given $N \geq 0$, denote by $d_{\mathbb{X}}$ the distance of $\mathbb{X}$ and $\mathrm{d}\mu_{\mathbb{X}}$ the induced measure respectively, then the geodesic distance of $\mathrm{Cusp}(\mathbb{X})$ (see [5]) is given by

$$d(Y, Y') = \operatorname{arc\,cosh} \frac{y^2 + y'^2 + \mathrm{d}_{\mathbb{X}}^2(x, x')}{2yy'}, \quad \forall Y = (y, x),\ Y' = (y', x') \in \mathrm{Cusp}(\mathbb{X}). \tag{1.1}$$

Consider the measure $\mathrm{d}\mu(y, x) = y^{-N-1}\mathrm{d}y\mathrm{d}\mu_{\mathbb{X}}(x)$ and suppose that there exist $\omega_1 > 0,\ \omega_2 \geq 0$ such that the volume $|B(x, r)|$ of the ball with center $x \in \mathbb{X}$ and of radius r satisfies

$$|B(x, r)| \sim r^{\omega_1}\chi_{r\leq 1} + r^{\omega_2}\chi_{r>1}, \quad \forall r > 0,\ x \in \mathbb{X}. \tag{1.2}$$

Set $\omega = \max\{\omega_1, \omega_2\}$, $p_1 = \frac{N}{2N-\omega}$ $(N < \omega < 2N)$, $p_0 = \frac{2N}{2N-\omega}$ $(\omega < 2N)$. Then the following results can be obtained.

Theorem A (1) *If $\omega \geq 2N$, then M is not bounded on L^p for any $1 \leq p < +\infty$.*

(2) *If $\omega \leq N$, then M is bounded from L^1 to $L^{1,\infty}$, and thus bounded on L^p for any $1 < p \leq +\infty$.*

(3) *If $N < \omega < 2N$, then M is bounded from $L^{p_1,1}$ to $L^{p_1,\infty}$, and bounded on L^p for any $p_1 < p < +\infty$, but M is not bounded on L^p for any $1 \leq p < p_1$.*

Theorem B (1) *If $\omega < 2N$, then for any $p_0 < p \leq +\infty$, M_* is bounded on L^p and is bounded from $L^{p_0,1}$ to $L^{p_0,\infty}$, but M_* is not bounded on L^p for any $1 \leq p \leq p_0$.*

(2) *If $\omega \geq 2N$, then M_* is not bounded on L^p for any $1 \leq p < +\infty$.*

Theorem C *If $\omega_1 = \omega < 2N$, then for any $\alpha > 1$, M_* is not bounded from $L^{p_0,\alpha}$ to $L^{p_0,\infty}$, and therefore M_* is not bounded on L^p for any $1 \leq p \leq p_0$.*

Furthermore, in [8], Li considered a more general class of non-doubling measure $\mathrm{d}\mu_\beta = \beta(y)\mathrm{d}y\mathrm{d}\mu_{\mathbb{X}}(x)$, where $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying

$$\frac{\beta(s)}{\beta(t)} \leq C, \quad \forall 0 < \frac{t}{2} \leq s \leq 2t. \tag{1.3}$$

As a consequence of (1.3), there exist two constants $A > 0$ and $\alpha \in \mathbb{R}$ such that

$$\frac{\beta(s)}{\beta(t)} \leq A\Big(\frac{s}{t}\Big)^{\alpha}, \quad \forall 0 < s \leq 2t. \tag{1.4}$$

Then on the weighted manifold $H = (\mathrm{Cusp}(\mathbb{X}), \mathrm{d}\mu_\beta)$ where β satisfies (1.4) with $\alpha > -1$, the following result has been established in [8] for the centered maximal function M_{μ_β} and uncentered maximal function M_{*,μ_β} associated with $\mathrm{d}\mu_\beta$:

Theorem D *M_{μ_β} is bounded from $L^1(\mathrm{d}\mu_\beta)$ to $L^{1,\infty}(\mathrm{d}\mu_\beta)$ and therefore is bounded on $L^p(\mathrm{d}\mu_\beta)$ for $p > 1$; M_{*,μ_β} is bounded on L^p for $p > 1$.*

For more results about maximal functions in the setting of exponential growth (see for example [1–2, 9–10, 13] and references therein). In this article, we study the maximal functions on weighted harmonic AN groups.

This paper is organized as follows. In Section 2 we will present some facts about harmonic AN groups and in Section 3 we will state our main results and give an explanation.

Throughout this paper, C will denote various constants which depend only on the dimension. $A \lesssim B$ means $A \le CB$ with such a C, and $A \sim B$ stands for $A \le CB$ and $B \le CA$.

2 Main Results and the Interpretation

In this section, we consider the maximal functions on the harmonic AN groups $S = \mathbb{R}^+ \times \mathbb{H}(2n, m)$. We will point out that the methods used in [6–8] can be applied to this case.

In what follows, we consider the measure $\mathrm{d}\mu = a^{\sigma - Q - 1}\mathrm{d}a\mathrm{d}x\mathrm{d}\rho$ and $\mathrm{d}\mu_\beta = \beta(a)\mathrm{d}a\mathrm{d}x\mathrm{d}\rho$ on S, where β satisfies (1.4) with $\alpha > -1$.

Denote by M and M_* the centered and uncentered maximal functions associated with the measure $\mathrm{d}\mu$, respectively, and by M_{μ_β} and M_{*,μ_β} the centered and uncentered maximal function associated with the measure $\mathrm{d}\mu_\beta$ respectively.

Set

$$\begin{aligned} p_0 &= \frac{Q-\sigma}{Q-2\sigma}, \quad 0 < \sigma < \frac{Q}{2}, \\ p_1 &= \frac{2(Q-\sigma)}{Q-2\sigma}, \quad \sigma < \frac{Q}{2}. \end{aligned}$$

Then we can obtain the following theorems, of which we omitt the detailed proofs.

Theorem 2.1 (1) *If $\frac{Q}{2} \le \sigma \le Q$, then M is not bounded on L^p for any $1 < p < \infty$.*

(2) *If $\sigma < 0$ or $\sigma > Q$, then M is bounded on L^p for any $1 < p \le \infty$ and is bounded from L^1 to $L^{1,\infty}$.*

(3) *If $0 < \sigma < \frac{Q}{2}$, then M is not bounded on L^p for any $1 \le p \le p_0$, but it is bounded from $L^{p_0,1}$ to $L^{p_0,\infty}$ and is bounded on L^p for any $p_0 < p \le \infty$.*

Theorem 2.2 (1) *If $\frac{Q}{2} \le \sigma \le Q$, then M_* is not bounded on L^p for any $1 < p < \infty$.*

(2) *If $\sigma > Q$, then M_* is bounded on L^p for any $p > 1$.*

(3) *If $\sigma < \frac{Q}{2}$, then M_* is not bounded on L^p for any $1 \le p \le p_1$, but it is bounded from $L^{p_1,1}$ to $L^{p_1,\infty}$ and is bounded on L^p for any $p_1 < p \le \infty$.*

Theorem 2.3 *If $\sigma < \frac{Q}{2}$, then for any $\gamma > 1$, M_* is not bounded from $L^{p_1,\gamma}$ to $L^{p_1,\infty}$.*

Theorem 2.4 (1) *M_{μ_β} is bounded from $L^1(\mathrm{d}\mu_\beta)$ to $L^{1,\infty}(\mathrm{d}\mu_\beta)$ and therefore is bounded on $L^p(\mathrm{d}\mu_\beta)$ for any $p > 1$.*

(2) *M_{*,μ_β} is bounded on $L^p(\mathrm{d}\mu_\beta)$ for any $p > 1$.*

To prove the above theorems, we need the following volume estimates of balls in $\mathbb{R}^+ \times \mathbb{H}(2n, m)$.

Lemma 2.1 *Given a ball $B((a_0, (x_0, \rho_0)), r)$ centered at $(a_0, (x_0, \rho_0)) \in \mathbb{R}^+ \times \mathbb{H}(2n, m)$ and* *of radius r, then we have*

$$\mu(B((a_0,(x_0,\rho_0)),r)) \sim \begin{cases} a_0^\sigma r^{2n+m+1}, & 0<r\le 1,\\ a_0^\sigma r \mathrm{e}^{\frac{r}{2}(m+n)}, & \sigma=\dfrac{m+n}{2},\ r>1,\\ a_0^\sigma \mathrm{e}^{r\sigma}, & \sigma>\dfrac{m+n}{2},\ r>1,\\ a_0^\sigma \mathrm{e}^{r(m+n-\sigma)}, & \sigma<\dfrac{m+n}{2},\ r>1 \end{cases} \tag{2.1}$$

and

$$\mu_\beta(B((a_0,(x_0,\rho_0)),r)) \sim \begin{cases} a_0^{m+n+1}\mathrm{e}^{r(m+n+1)}\beta(a_0\cosh r), & r>1,\\ a_0^{m+n+1}(\sinh r)^{m+2n+1}\beta(a_0\cosh r), & 0<r\le 1. \end{cases}$$

We point out that the above volume estimates of $r>1$ can be obtained by Proposition 4.1 of [7] and Corollary 4.2 of [8] since d and d_* (see the following Remark 2.2) are equivalent for large d.

Remark 2.1 When $\sigma=0$ in Theorem 2.1, the weak type (1,1) boundedness of M has been obtained in [1].

Remark 2.2 The above theorems can be interpreted by the models in [6–8].

In fact, set $d_{*,\mathbb{H}}=d_{\mathbb{H}}^2$ and define d_* in $\mathbb{R}^+\times\mathbb{H}(2n,m)$ as

$$\cosh d_* = \frac{d_{*,\mathbb{H}}^2((x,\rho),(x',\rho'))+a^2+a'^2}{2aa'}. \tag{2.2}$$

Define the ball $B(x,r)$ associated with $d_{*,\mathbb{H}}$ as

$$B(x,r)=\{y\in\mathbb{H}(2n,m): d_{*,\mathbb{H}}(y,x)<r\},$$

and then the volume of $B(x,r)$ is $r^Q|B_{\mathbb{H}}(e_{\mathbb{H}},1)|$ (see Section 2). The condition (1.2) here then becomes

$$|B(x,r)|\sim r^Q, \quad \forall r>0,\ x\in\mathbb{H}(2n,m), \tag{2.3}$$

so $\omega=\omega_1=\omega_2=Q$. If we set $\mathbb{X}=\mathbb{H}(2n,m)$ and $\mathrm{Cusp}(\mathbb{X})=\mathbb{R}^+\times\mathbb{H}(2n,m)$, then the models in [6–8] remain valid and similar results can be obtained for the maximal functions associated with d_*.

Thanks to the inequality $a+b\ge 2\sqrt{ab}$ $(a,b\ge 0)$, we can prove that the distances d and d_* are in some sense equivalent for large d_*, that is, there exists a constant $C>0$ such that for $\forall\mathcal{Y},\mathcal{Y}'\in\mathbb{R}^+\times\mathbb{H}(2n,m)$:

$$d_*(\mathcal{Y},\mathcal{Y}')-C\le d(\mathcal{Y},\mathcal{Y}')\le d_*(\mathcal{Y},\mathcal{Y}')+C, \quad \forall d_*(\mathcal{Y},\mathcal{Y}')\gg C.$$

Therefore, we can use the methods in [6–8] when d is large. On the other hand, the local maximal functions are always of the weak type (1,1) according to [4] since the measures $\mathrm{d}\mu$ and $\mathrm{d}\mu_\beta$ satisfy the local doubling property by Lemma 2.1. Thus we obtain Theorems 2.1–2.4 for the centered and uncentered maximal functions associated with d.

Remark 2.3 Li pointed out in [8] that harmonic AN groups are a typical example of the spaces of quasi-hyperbolic type, that is, there exists a constant $C > 1$ such that

$$C^{-1}d_*(\mathcal{Y},\mathcal{Y}') \le d(\mathcal{Y},\mathcal{Y}') \le Cd_*(\mathcal{Y},\mathcal{Y}'), \quad \mathcal{Y},\mathcal{Y}' \in \mathbb{R}^+ \times \mathbb{H}(2n,m), \tag{2.4}$$

$$d_*(\mathcal{Y},\mathcal{Y}') - C \le d(\mathcal{Y},\mathcal{Y}') \le d_*(\mathcal{Y},\mathcal{Y}') + C, \quad \forall\ d_*(\mathcal{Y},\mathcal{Y}') \gg C. \tag{2.5}$$

However, we find that (2.4) fails if d_* is small enough. In fact, denote by o the identity element of $\mathbb{R}^+ \times \mathbb{H}(2n,m)$, then for any $g = (a,(x,\rho)) \in \mathbb{R}^+ \times \mathbb{H}(2n,m)$, we have

$$\begin{aligned} d(g,o) &= \operatorname{arc\,cosh} \frac{\frac{|x|^4}{16} + |\rho|^2 + a^2 + 1 + \frac{|x|}{2}(a+1)}{2a} \\ &= \operatorname{arc\,cosh}\Big[1 + \frac{\frac{|x|^4}{16} + |\rho|^2 + (a-1)^2 + \frac{|x|^2}{2}(a+1)}{2a}\Big] \\ &\sim \Big[\frac{\frac{|x|^4}{16} + |\rho|^2 + (a-1)^2 + \frac{|x|}{2}(a+1)}{a}\Big]^{\frac{1}{2}}, \quad a \to 1,\ (x,\rho) \to 0. \end{aligned}$$

Similarly, we have

$$d_*(g,o) \sim \Big[\frac{\frac{|x|^4}{16} + |\rho|^2 + (a-1)^2}{a}\Big]^{\frac{1}{2}}, \quad a \to 1,\ (x,\rho) \to 0.$$

If $d(g,o) \lesssim d_*(g,o)$ holds, then

$$\frac{|x|^2}{2}(a+1) \lesssim \frac{|x|^4}{16} + |\rho|^2 + (a-1)^2, \quad a \to 1,\ (x,\rho) \to 0,$$

which is not right. Thus (2.4) does not hold for small d_*.

In spite of this, (2.5) is still enough to get Theorems 2.1–2.4.

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