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Gröbner-Shirshov Basis for Degenerate Ringel-Hall Algebras of Type F_4*

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Abstract In this paper, by using the Frobenius morphism and the multiplication formulas of the generic extension monoid algebra, the authors first give a presentation of the degenerate Ringel-Hall algebra, and then construct the Gröbner-Shirshov basis for degenerate Ringel-Hall algebras of type F_4.

Keywords Gröbner-Shirshov basis, Frobenius map, Degenerate Ringel-Hall algebras, Multiplication formulas
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1 Introduction

Through the works of Buchberger [1], Bergman [2] and Shirshov [3], the Gröbner-Shirshov basis theory has become a powerful tool for the solution of the reduction problem in algebra and provides a computational approach for the study of structures of algebras. The degenerate Ringel-Hall algebra is the specialization of the Ringel-Hall algebra at $q = 0$ and in [4] Reineke gave a remarkable basis which closes under multiplication.

In this paper, we first give a presentation of the degenerate Ringel-Hall algebra $\mathfrak{H}_0(F_4)$ by using the method of Frobenius morphism (see [5]) and the idea of monoid algebra (see [4]). Then, by using the relations which are computed to give this presentation, we construct a Gröbner-Shirshov basis for the degenerate Ringel-Hall algebra $\mathfrak{H}_0(F_4)$.

2 Some Preliminaries

First, we recall some relevant notions and results about the Gröbner-Shirshov basis theory from [6].

Let S be a linearly ordered set, k be a field and $k\langle S\rangle$ the free associative algebra generated by S over k. Let S^* be the free monoid generated by S. Order S^* by the deg-lex order " $<$ ". Then any polynomial $f \in k\langle S\rangle$ has the leading word $\overline{f}$. We call f monic if the coefficient of $\overline{f}$ is 1.

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Let $f, g \in k\langle S\rangle$ be two monic polynomials and $\omega \in S^*$. If $\omega = \overline{f}b = a\overline{g}$ for some $a, b \in S^*$ such that $\deg(\overline{f})+\deg(\overline{g}) < \deg(\omega)$, then $(f, g)_\omega = fb-ag$ is called the intersection composition of f and g relative to ω. If $\omega = \overline{f} = a\overline{g}b$ for some $a, b \in S^*$, then $(f, g)_\omega = f - agb$ is called the inclusion composition of f and g relative to ω.

Let $R \subset k\langle S\rangle$ be a monic set. A composition $(f, g)_\omega$ is called trivial modulo (R, ω) if $(f, g)_\omega = \sum \alpha_i a_i t_i b_i$, where each $\alpha_i \in k,\ t_i \in R,\ a_i, b_i \in S^*$ and $a_i\overline{t}_i b_i < \omega$.

R is called a Gröbner-Shirshov basis if any composition of polynomials from R is trivial modulo R.

A well order " $<$ " on S^* is monomial if for any $u, v \in S^*$, we have

$$u > v \Longrightarrow \omega_1 u\omega_2 > \omega_1 v\omega_2 \quad \text{for all} \quad \omega_1, \omega_2 \in S^*.$$

A standard result about the Gröbner-Shirshov basis theory is the following lemma.

Lemma 2.1 (see [6]) (Composition-Diamond Lemma) *Let k be a field, $A = k\langle S \mid R\rangle = k\langle S\rangle/Id(R)$ and " $<$ " be a monomial order on S^*, where $Id(R)$ is the ideal of $k\langle S\rangle$ generated by R. Then the following statements are equivalent:*

(a) *R is a Gröbner-Shirshov basis*;

(b) *$f \in Id(R) \Longrightarrow \overline{f} = a\overline{t}b$ for some $t \in R$ and $a, b \in S^*$;*

(c) *$\mathrm{Irr}(R) = \{u \in S^* \mid u \neq a\overline{t}b, t \in R, a, b \in S^*\}$ is a k-linear basis of the algebra A.*

Next, we recall some relevant notions and results about the Frobenius morphism method from [5].

Let (Q, σ) be a quiver Q with the automorphism σ. The associated valued quiver $\Gamma = \Gamma(Q, \sigma)$ is defined as follows. Its vertex set Γ_0 and arrow set Γ_1 are simply the sets of σ-orbits in Q_0 and Q_1, respectively. For $\rho \in Q_1$, its tail (resp., head) is the σ-orbit of tails (resp., heads) of arrows in ρ. The valuation of Γ is given by

$$d_i = |\{\text{vertices in } \sigma\text{-orbit } i\}| \quad \text{for } i \in \Gamma_0,$$
$$m_\rho = |\{\text{arrows in } \sigma\text{-orbit } \rho\}| \quad \text{for } \rho \in \Gamma_1.$$

Let $\mathbb{F}_q$ be the finite field of q elements and $\mathcal{K} = \overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$.

Definition 2.1 (see [5, 7]) *Let M be a vector-space over $\mathcal{K}$. An $\mathbb{F}_q$-linear isomorphism $F : M \longrightarrow M$ is called a Frobenius map if it satisfies:*

(a) *$F(\lambda m) = \lambda^q F(m)$ for all $m \in M$ and $\lambda \in \mathcal{K}$;*

(b) *For any $m \in M$, $F^n(m) = m$ for some $n > 0$.*

Let C be a $\mathcal{K}$-algebra with identity 1. We do not assume generally that C is finite-dimensional. A map $F_C : C \longrightarrow C$ is called a Frobenius morphism on C if it is a Frobenius map on the $\mathcal{K}$-space C, and it is also an $\mathbb{F}_q$-algebra isomorphism sending 1 to 1.

Let $A := \mathcal{K}Q$ be the path algebra of Q over $\mathcal{K}$. Then σ induces a Frobenius morphism $F = F_{Q,\sigma} = F_{Q,\sigma,q} : A \longrightarrow A$ given by $\sum\limits_s x_s p_s \longmapsto \sum\limits_s x_s^q \sigma(p_s)$, where $\sum\limits_s x_s p_s$ is a $\mathcal{K}$-linear combination of paths p_s and $\sigma(p_s) = \sigma(\rho_t)\cdots\sigma(\rho_1)$ if $p_s = \rho_t \cdots \rho_1$ for arrows $\rho_1, \cdots, \rho_t \in Q_1$. Then the fixed-point algebra

$$A(q) = \mathfrak{A}(Q, \sigma; q) := A^F = \{a \in A \mid F(a) = a\}$$

is an $\mathbb{F}_q$-algebra associated with (Q, σ).

Definition 2.2 (see [5]) *Let (Q,σ) be a quiver with the automorphism σ. A representation $V=(V_i,\phi_\rho)$ of Q is called F-stable (or equivalently, an F-stable A-module) if there is a Frobenius map $F_V:\bigoplus_{i\in Q_0}V_i\longrightarrow\bigoplus_{i\in Q_0}V_i$ satisfying $F_V(V_i)=V_{\sigma_i}$ for all $i\in Q_0$ such that $F_V\phi_\rho=\phi_{\sigma(\rho)}F_V$ for each arrow $\rho\in Q_1$*

For an F-stable representation $V=(V_i,\phi_\rho)$, let $\mathbf{dim}V=\sum_{i\in\Gamma_0}(\dim V_i)i\in\mathbb{N}\Gamma_0$ and $\dim V=\sum_{i\in\Gamma_0}\dim V_i$ denote the dimension vector and the dimension of V, respectively. An F-stable representation is called indecomposable if it is nonzero and not isomorphic to a direct sum of two nonzero F-stable representations.

Lemma 2.2 (see [8]) *There is a one-to-one correspondence between isoclasses of indecomposable $A(q)$-modules and isoclasses of indecomposable F-stable A-modules.*

Then, we recall some relevant notions and results about the degenerate Ringel-Hall algebra from [8].

From now on, we assume that (Q,σ) is a Dynkin quiver Q with the automorphism σ. Dlab and Ringel [9–10] have shown that there is a bijection from the isoclasses of indecomposable $A(q)$-modules to the set $\Phi^+=\Phi^+(Q,\sigma)$ of positive roots in the root system associated with the valued quiver $\Gamma=\Gamma(Q,\sigma)$. For each $\alpha\in\Phi^+$, let $M_q(\alpha)$ denote the corresponding indecomposable $A(q)$-module, so $\mathbf{dim}M_q(\alpha)=\alpha$. By the Krull-Schmidt theorem, every $A(q)$-module M is isomorphic to

$$M_q(\lambda):=\bigoplus_{\alpha\in\Phi^+}\lambda(\alpha)M_q(\alpha)$$

for some function $\lambda:\Phi^+\longrightarrow\mathbb{N}$. Thus, the isoclasses of $A(q)$-modules are indexed by the set

$$\mathfrak{B}=\mathfrak{B}(Q,\sigma)=:\{\lambda\mid\lambda:\Phi^+\longrightarrow\mathbb{N}\}=\mathbb{N}^{\Phi^+},$$

which is independent of q. By Lemma 2.2, the isoclasses of F-stable $\mathcal{K}Q$-modules are also indexed by $\mathfrak{B}$. Clearly, for each $i\in\Gamma_0$, there is a complete simple $A(q)$-module S_i corresponding to i.

For $M,N_1,\cdots,N_t\in A(q)$-mod, let $F^M_{N_1,\cdots,N_t}$ be the number of filtrations

$$M=M_0\supseteq M_1\supseteq\cdots\supseteq M_{t-1}\supseteq M_t=0$$

such that $M_{i-1}/M_i\cong N_i$ for all $1\le i\le t$ is finite. By [11], $F^M_{N_1,\cdots,N_t}$ is a polynomial in q when q varies. More precisely, for $\lambda,\mu,\nu\in\mathfrak{B}=\mathfrak{B}(Q,\sigma)$, there exists a polynomial $\varphi^\lambda_{\mu,\nu}(\mathrm{q})\in\mathbb{Z}[\mathrm{q}]$ (the polynomial ring over $\mathbb{Z}$ in one indeterminate q) such that $\varphi^\lambda_{\mu,\nu}(q_k)=F^{M_{q_k}(\lambda)}_{M_{q_k}(\mu),M_{q_k}(\nu)}$ holds for any finite field k with q_k elements.

The generic Ringel-Hall algebra $\mathfrak{H}=\mathfrak{H}_{\mathrm{q}}(Q,\sigma)$ is the free module over $\mathbb{Z}[\mathrm{q}]$ with the basis $\{u_\lambda\mid\lambda\in\mathfrak{B}\}$ and the multiplication defined by

$$u_\mu u_\nu=\sum_{\lambda\in\mathfrak{B}}\varphi^\lambda_{\mu,\nu}(\mathrm{q})u_\lambda.$$

It is an $\mathbb{N}^{|\Gamma_0|}$-graded algebra

$$\mathfrak{H}=\bigoplus_{e\in\mathbb{N}^{|\Gamma_0|}}\mathfrak{H}_e,$$

where $\mathfrak{H}_e$ is spanned by all $\mu_\alpha,\ \alpha \in \mathfrak{B}_e := \{\beta \in \mathfrak{B} \mid \mathbf{dim} M_q(\beta) = e\}$.

For each $\lambda \in \mathfrak{B}$, set $M_q(\lambda)_{\mathcal{K}} := M_q(\lambda) \otimes_{\mathbb{F}_q} \mathcal{K}$ which is the F-stable $\mathcal{K}Q$-module corresponding to λ.

Now by specializing q to 0, we obtain the $\mathbb{Z}$-algebra $\mathfrak{H}_0(Q,\sigma)$, called the degenerate Ringel-Hall algebra associated with $\Gamma = \Gamma(Q,\sigma)$. In other words, $\mathfrak{H}_0(Q,\sigma) = \mathfrak{H}_{\mathrm{q}}(Q,\sigma) \otimes_{\mathbb{Z}[\mathrm{q}]} \mathbb{Z}$, where $\mathbb{Z}$ is viewed as a $\mathbb{Z}[\mathrm{q}]$-module with the action of q by zero. By abuse of notations, we also write $u_\lambda = u_\lambda \otimes 1$. Thus, the set $\{u_\lambda \mid \lambda \in \mathfrak{B}\}$ is a $\mathbb{Z}$-basis of $\mathfrak{H}_0(Q,\sigma)$. Let $u_i = u_{[S_i]} \otimes 1$ in $\mathfrak{H}_0(Q,\sigma)$ for $i \in \Gamma_0$.

Lemma 2.3 (see [12]) *As a $\mathbb{Z}$-algebra, $\mathfrak{H}_0(Q,\sigma)$ is generated by u_i, $i \in \Gamma_0$.*

Finally, we recall some relevant notions and results about monoid algebras from [4].

For $\mathcal{K}Q$-modules M and N, the generic extension $M * N$ of M by N was defined in [13] as the unique (up to isomorphism) element in $\mathbf{Ext}^1_{\mathcal{K}Q}(M,N)$ having an endomorphism algebra of the minimal dimension. As shown in [4], the star operation $*$ defines the structure of a monoid on the set $\mathcal{M}_Q = \mathcal{M}_{Q,\mathcal{K}}$ of isoclasses of $\mathcal{K}Q$-modules.

Proposition 2.1 (see [7]) *If M and N are two F-stable $\mathcal{K}Q$-modules, then $M * N$ is also F-stable.*

By this proposition, the set of isoclasses $[M]$ of F-stable $\mathcal{K}Q$-modules, together with the operation $[M] * [N] = [M * N]$, defines a submonoid $\mathcal{M}_{Q,\sigma}$ of $\mathcal{M}_Q$ with the unit element $[0]$.

Since all the indecomposable $A(q)$-modules are indexed by the set $\mathfrak{B}$, we give an enumeration on Φ^+ defined by $\beta_1, \beta_2, \cdots, \beta_N$ such that for all prime powers q,

$$\mathbf{Hom}_{A(q)}(M_q(\beta_s), M_q(\beta_t)) \neq 0 \quad \text{implies} \quad s \leq t.$$

Moreover, in this case, $\mathbf{Ext}^1_{A(q)}(M_q(\beta_s), M_q(\beta_t)) \neq 0$ implies $s > t$. Thus, we give an enumeration on indecomposable $A(q)$-modules and set $M_q(\beta_1) \prec M_q(\beta_2) \prec \cdots M_q(\beta_N)$.

By the definition of the generic extension, if $\mathbf{Ext}^1_{A(q)}(M,N) = 0$, then $M * N \cong M \oplus N$. Consequently, we have the following known result.

Lemma 2.4 *Each element $[M_q(\lambda)_{\mathcal{K}}]$ in $\mathcal{M}_{Q,\sigma}$ with $\lambda \in \mathfrak{B}$ can be written as*

$$[M_q(\lambda)_{\mathcal{K}}] = [M_q(\beta_1)_{\mathcal{K}}]^{*\lambda_{\beta_1}} * \cdots * [M_q(\beta_N)_{\mathcal{K}}]^{*\lambda_{\beta_N}}.$$

Moreover, these elements form a $\mathbb{Z}$-basis of $\mathbb{Z}\mathcal{M}_{Q,\sigma}$.

For a dimension vector $d = \sum_{i \in \Gamma_0} d_i i \in \mathbb{N}\Gamma_0$, we consider the affine space

$$R_d = \prod_{\alpha: i \to j} \mathbf{Hom}_{\mathcal{K}}(\mathcal{K}^{d_i}, \mathcal{K}^{d_j}).$$

Then the group $G_d := \prod_{i \in \Gamma_0} \mathrm{GL}_{d_i}(\mathcal{K})$ acts on R_d by conjugation, i.e., by

$$(g_i) \cdot (x_\rho)_\rho = (g_j x_\rho g_i^{-1})_{\rho: i \to j}.$$

The orbits of G_d correspond bijectively to the isoclasses of representations of Γ of the dimension vector d. Denote by $\mathcal{O}_M$ the orbit corresponding to the isoclass $[M]$. Since there are only finitely many G_d-orbits in R_d, there exists a dense one, whose corresponding representation is denoted by E_d.

Lemma 2.5 (see [14]) *Let $i_1, \cdots, i_n$ be an enumeration of Γ_0 such that $k < l$ if there is an arrow from i_k to i_l. Then for all $d = \sum_{k=1}^{n} d_k \alpha_{i_k} \in \Phi^+$, we have $[E_d] = [S_{i_1}]^{*d_i} * \cdots * [S_{i_n}]^{*d_n}$ in $\mathcal{M}_{Q,\,\sigma}$.*

Like the Ringel-Hall algebras, there is a natural grading on the monoid algebra $\mathbb{Z}\mathcal{M}_{Q,\sigma}$ in terms of dimension vectors:

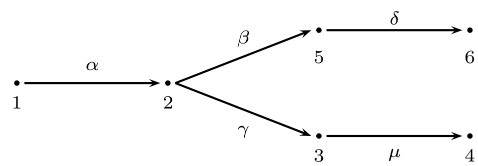
$$\mathbb{Z}\mathcal{M}_{Q,\sigma} = \bigoplus_{e \in \mathbb{N}^{|\Gamma_0|}} \mathbb{Z}\mathcal{M}_e,$$

where $\mathbb{Z}\mathcal{M}_e$ is spanned by all $[M_q(\alpha)_{\mathcal{K}}],\ \alpha \in \mathfrak{B}_e$.

3 Presentation of Degenerate Ringel-Hall Algebra $\mathfrak{H}_0(F_4)$

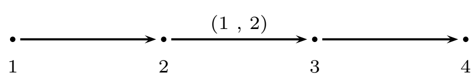
In the following, we consider the quiver:

(I) E_6 :



where σ is the automorphism of E_6 such that $\sigma(1) = 1,\ \sigma(2) = 2,\ \sigma(3) = 5,\ \sigma(4) = 6,\ \sigma(5) = 3,\ \sigma(6) = 4,\ \sigma(\alpha) = \alpha,\ \sigma(\beta) = \gamma,\ \sigma(\gamma) = \beta,\ \sigma(\delta) = \mu,\ \sigma(\mu) = \delta$. Then the associated valued quiver F_4 with the valuation $\varepsilon_1 = 1,\ \varepsilon_2 = 1,\ \varepsilon_3 = 2,\ \varepsilon_4 = 2$ has the form:

(II)



Then we have $\mathfrak{H}_0(F_4) = \mathfrak{H}_0(E_6, \sigma)$. It is easy to see that the following relations hold in $\mathfrak{H}_0(F_4)$ (for $1 \le i, j \le 4$):

(F1) $u_i u_j = u_j u_i$ for $|i - j| \ge 2$, (F2) $u_i^2 u_{i+1} = u_i u_{i+1} u_i$ for $i \in \{1, 3\}$,
(F3) $u_i u_{i+1}^2 = u_{i+1} u_i u_{i+1}$ for $i \in \{1, 3\}$, (F4) $u_3 u_2 u_3 = u_2 u_3^2$,
(F5) $u_2 u_3 u_2^2 u_3 = u_2^3 u_3^2$, (F6) $u_2^2 u_3 u_2 = u_2^3 u_3$.

Now consider the corresponding monoid algebra $\mathbb{Z}\mathcal{M}_{E_6,\sigma}$. By [14], the following relations hold in $\mathbb{Z}\mathcal{M}_{E_6,\sigma}$:

($\mathcal{F}$1) $[S_i] * [S_j] = [S_j] * [S_i]$ for $|i - j| \ge 2$,
($\mathcal{F}$2) $[S_i]^{*2}[S_{i+1}] = [S_i] * [S_{i+1}] * [S_i]$ for $i \in \{1, 3\}$,
($\mathcal{F}$3) $[S_i][S_{i+1}]^{*2} = [S_{i+1}] * [S_i] * [S_{i+1}]$ for $i \in \{1, 3\}$,
($\mathcal{F}$4) $[S_3] * [S_2] * [S_3] = [S_2] * [S_3]^{*2}$,
($\mathcal{F}$5) $[S_2] * [S_3] * [S_2]^{*2} * [S_3] = [S_2]^{*3} * [S_3]^{*2}$,
($\mathcal{F}$6) $[S_2]^{*2} * [S_3] * [S_2] = [S_2]^{*3} * [S_3]$.

In the following, we prove that the set $\{[S_1], [S_2], [S_3], [S_4]\}$ and the relations ($\mathcal{F}$1)–($\mathcal{F}$6) between them give a presentation of the monoid algebra $\mathbb{Z}\mathcal{M}_{E_6,\sigma}$.

Proposition 3.1 *The monoid algebra $\mathbb{Z}\mathcal{M}_{E_6,\sigma}$ has a presentation with generators $[S_i]$ $(1 \le i \le 4)$ and relations $(\mathcal{F}1)$–$(\mathcal{F}6)$.*

Proof For convenience, set $\mathbb{Z}\mathcal{M} = \mathbb{Z}\mathcal{M}_{E_6,\sigma}$. Let $\mathcal{S}$ be the free $\mathbb{Z}$-algebra with generators s_i $(1 \le i \le 4)$. Consider the ideal $\mathfrak{J}$ generated by the following elements for $1 \le i, j \le 4$,

$$(F'1)\ s_is_j - s_js_i \quad \text{for } |i-j| \ge 2, \qquad (F'2)\ s_i^2 s_{i+1} - s_is_{i+1}s_i \quad \text{for } i \in \{1,3\},$$
$$(F'3)\ s_is_{i+1}^2 - s_{i+1}s_is_{i+1} \quad \text{for } i \in \{1,3\}, \qquad (F'4)\ s_3s_2s_3 - s_2s_3^2,$$
$$(F'5)\ s_2s_3s_2^2s_3 - s_2^3s_3^2, \qquad (F'6)\ s_2^2s_3s_2 - s_2^3s_3.$$

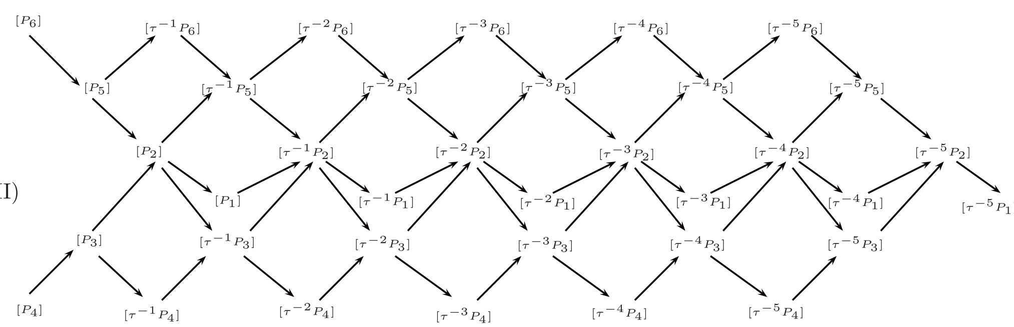
Then, there is a surjective monoid algebra homomorphism $\eta : \mathcal{S} \longrightarrow \mathbb{Z}\mathcal{M}$ given by $s_i \longmapsto [S_i]$ with $1 \le i \le 4$. Because we have $\mathcal{F}i = 0$ $(1 \le i \le 6)$ in $\mathbb{Z}\mathcal{M}$, the map η induces a surjective algebra homomorphism $\overline{\eta} : \mathcal{S}/\mathfrak{J} \longrightarrow \mathbb{Z}\mathcal{M}$ given by $s_i + \mathfrak{J} \longmapsto [S_i]$ $(1 \le i \le 4)$. To complete the proof, it suffices to show that $\overline{\eta}$ is injective.

Set $f_i = s_i + \mathfrak{J}$ $(1 \le i \le 4)$. Given a $\mathcal{K}F_4$-module M with the dimension vector $\mathbf{dim}M := (a,b,c,d)$, we define a monomial in $\mathcal{S}/\mathfrak{J}$ by

$$\mathfrak{n}(M) = f_1^a f_2^b f_3^c f_4^d.$$

It is known that the Auslander-Reiten quiver for $\mathcal{K}E_6$ is as follows:

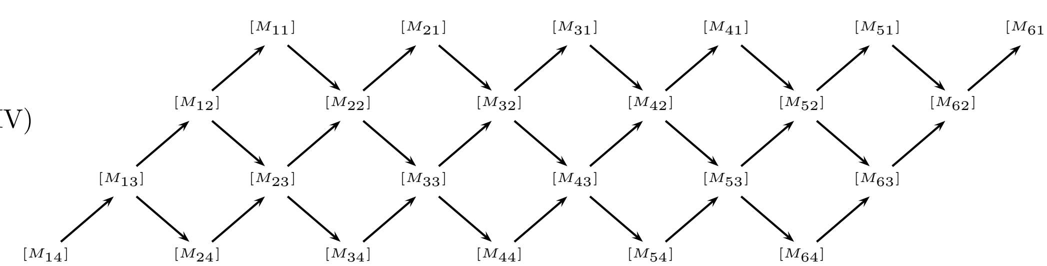
(III)



where each P_i $(1 \le i \le 6)$ is the indecomposable projective $\mathcal{K}E_6$-module corresponding to the vertex i and τ is the Auslander-Reiten translation.

Using the Frobenius morphism $F = F_{E_6,\sigma} = F_{E_6,\sigma,q}$ introduced in Section 3, it is easy to see that P_1 and P_2 are F-stable and all other P_i have F-period 2 with $P_3^{[1]} = P_5, P_4^{[1]} = P_6$. By folding the Auslander-Reiten quiver of $\mathcal{K}E_6$, we obtain the Auslander-Reiten quiver of $A(q) = (\mathcal{K}E_6)^F \cong \mathcal{K}F_4$:

(IV)



where M_{ij} denotes the indecomposable $\mathcal{K}F_4$-modules, $1 \leq i \leq 6$ and $1 \leq j \leq 4$. Here $M_{11} = P_1^F, M_{12} = P_2^F, M_{13} = (P_3 \oplus P_5)^F, M_{14} = (P_4 \oplus P_6)^F$ and $\tau = \tau_{A^F}$ is the Auslander-Reiten translation of $A(q)$ (see [7] for details). Moreover, the dimension vectors of M_{ij} $(1 \leq i \leq 6, 1 \leq j \leq 4)$ and the associated monomials in $\mathcal{S}/\mathfrak{J}$ are given by

$$\begin{aligned}
&\mathbf{dim}M_{14} = (0,0,0,1) \text{ and } \mathfrak{n}(M_{14}) = f_4,\\
&\mathbf{dim}M_{13} = (0,0,1,1) \text{ and } \mathfrak{n}(M_{13}) = f_3f_4,\\
&\mathbf{dim}M_{12} = (0,1,1,1) \text{ and } \mathfrak{n}(M_{12}) = f_2f_3f_4,\\
&\mathbf{dim}M_{11} = (1,1,1,1) \text{ and } \mathfrak{n}(M_{11}) = f_1f_2f_3f_4,\\
&\mathbf{dim}M_{21} = (0,1,1,0) \text{ and } \mathfrak{n}(M_{21}) = f_2f_3\\
&\mathbf{dim}M_{22} = (1,2,2,1) \text{ and } \mathfrak{n}(M_{22}) = f_1f_2^2f_3^2f_4,\\
&\mathbf{dim}M_{23} = (0,2,2,1) \text{ and } \mathfrak{n}(M_{23}) = f_2^2f_3^2f_4,\\
&\mathbf{dim}M_{24} = (0,0,1,0) \text{ and } \mathfrak{n}(M_{24}) = f_3,\\
&\mathbf{dim}M_{31} = (1,2,1,1) \text{ and } \mathfrak{n}(M_{31}) = f_1f_2^2f_3f_4,\\
&\mathbf{dim}M_{32} = (1,3,2,1) \text{ and } \mathfrak{n}(M_{32}) = f_1f_2^3f_3^2f_4,\\
&\mathbf{dim}M_{33} = (2,4,3,2) \text{ and } \mathfrak{n}(M_{33}) = f_1^2f_2^4f_3^3f_4^2,\\
&\mathbf{dim}M_{34} = (0,2,1,1) \text{ and } \mathfrak{n}(M_{34}) = f_2^2f_3f_4,\\
&\mathbf{dim}M_{41} = (1,1,1,0) \text{ and } \mathfrak{n}(M_{41}) = f_1f_2f_3,\\
&\mathbf{dim}M_{42} = (2,3,2,1) \text{ and } \mathfrak{n}(M_{42}) = f_1^2f_2^3f_3^2f_4,\\
&\mathbf{dim}M_{43} = (2,4,3,1) \text{ and } \mathfrak{n}(M_{43}) = f_1^2f_2^4f_3^3f_4,\\
&\mathbf{dim}M_{44} = (2,2,2,1) \text{ and } \mathfrak{n}(M_{44}) = f_1^2f_2^2f_3^2f_4,\\
&\mathbf{dim}M_{51} = (0,1,0,0) \text{ and } \mathfrak{n}(M_{51}) = f_2,\\
&\mathbf{dim}M_{52} = (1,2,1,0) \text{ and } \mathfrak{n}(M_{52}) = f_1f_2^2f_3,\\
&\mathbf{dim}M_{53} = (2,4,2,1) \text{ and } \mathfrak{n}(M_{53}) = f_1^2f_2^4f_3^2f_4,\\
&\mathbf{dim}M_{54} = (0,2,1,0) \text{ and } \mathfrak{n}(M_{54}) = f_2^2f_3,\\
&\mathbf{dim}M_{61} = (1,0,0,0) \text{ and } \mathfrak{n}(M_{61}) = f_1,\\
&\mathbf{dim}M_{62} = (1,1,0,0) \text{ and } \mathfrak{n}(M_{62}) = f_1f_2,\\
&\mathbf{dim}M_{63} = (2,2,1,0) \text{ and } \mathfrak{n}(M_{63}) = f_1^2f_2^2f_3,\\
&\mathbf{dim}M_{64} = (2,2,1,1) \text{ and } \mathfrak{n}(M_{64}) = f_1^2f_2^2f_3f_4.
\end{aligned}$$

Now we give an enumeration of the indecomposable $A(q)$-modules in figure (IV):

$$(*) \qquad M_{i4} \prec M_{i3} \prec M_{i2} \prec M_{i1} \prec M_{i+1,4} \prec M_{i+1,3} \prec M_{i+1,2} \prec M_{i+1,1}.$$

Now, by using the relations $(F'1)$–$(F'6)$, we compute the relations between $\mathfrak{n}(M_{ij})$ $(1 \leq i \leq 6, 1 \leq j \leq 4)$ in $\mathcal{S}/\mathfrak{J}$:

$$\begin{aligned}
&\mathfrak{n}(M_{13})\mathfrak{n}(M_{14}) = f_3f_4f_4 = f_4f_3f_4 = \mathfrak{n}(M_{14})\mathfrak{n}(M_{13});\\
&\mathfrak{n}(M_{21})\mathfrak{n}(M_{14}) = f_2f_3f_4 = \mathfrak{n}(M_{12});\\
&\mathfrak{n}(M_{22})\mathfrak{n}(M_{13}) = f_1f_2^2f_3^2f_4f_3f_4
\end{aligned}$$

$$
\begin{aligned}
&= f_1f_2^2f_3f_4f_3f_3f_4 \quad (\text{by } (F'2)) \\
&= f_1f_2^2f_3f_4f_3f_4f_3 \quad (\text{by } (F'2)) \\
&= f_2f_1f_2f_3f_4f_3f_4f_3 \quad (\text{by } (F'3)) \\
&= f_2f_1f_2f_3^2f_4f_4f_3 \quad (\text{by } (F'2)) \\
&= f_2f_1f_3f_2f_3f_4^2f_3 \quad (\text{by } (F'4)) \\
&= f_2f_3f_1f_2f_3f_4^2f_3 \quad (\text{by } (F'1)) \\
&= f_2f_3f_1f_2f_4f_3f_4f_3 \quad (\text{by } (F'3)) \\
&= f_2f_3f_4f_1f_2f_3f_4f_3 \quad (\text{by } (F'1)) \\
&= \mathfrak{n}(M_{12})\mathfrak{n}(M_{11})\mathfrak{n}(M_{24}).
\end{aligned}
$$

In this way, we get the following set B of relations:

1. $\mathfrak{n}(M_{i+1,1})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i2})$,
2. $\mathfrak{n}(M_{i+1,4})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i3})$,
3. $\mathfrak{n}(M_{i+3,1})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i1})$,
4. $\mathfrak{n}(M_{i+3,3})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i+2,3})$,
5. $\mathfrak{n}(M_{i+4,2})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i+2,1})$,
6. $\mathfrak{n}(M_{i+4,4})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i+2,4})$,
7. $\mathfrak{n}(M_{63})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{64})$,
8. $\mathfrak{n}(M_{i+4,1})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i2})$,
9. $\mathfrak{n}(M_{i+4,2})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i+1,2})$,
10. $\mathfrak{n}(M_{i+4,3})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i+2,3})$,
11. $\mathfrak{n}(M_{i+4,4})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i+1,3})$,
12. $\mathfrak{n}(M_{62})\mathfrak{n}(M_{13}) = \mathfrak{n}(M_{11})$,
13. $\mathfrak{n}(M_{63})\mathfrak{n}(M_{13}) = \mathfrak{n}(M_{44})$,
14. $\mathfrak{n}(M_{i+1,1})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+1,3})$,
15. $\mathfrak{n}(M_{i+3,1})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+1,2})$,
16. $\mathfrak{n}(M_{i+3,2})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+2,3})$,
17. $\mathfrak{n}(M_{i+4,1})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+2,4})$,
18. $\mathfrak{n}(M_{i+4,2})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+2,2})$,
19. $\mathfrak{n}(M_{61})\mathfrak{n}(M_{12}) = \mathfrak{n}(M_{11})$,
20. $\mathfrak{n}(M_{62})\mathfrak{n}(M_{12}) = \mathfrak{n}(M_{31})$,
21. $\mathfrak{n}(M_{i+1,1})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+1,2})$,
22. $\mathfrak{n}(M_{63})\mathfrak{n}(M_{12}) = \mathfrak{n}(M_{42})$,
23. $\mathfrak{n}(M_{i+2,2})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+2,3})$,
24. $\mathfrak{n}(M_{i+3,1})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+3,4})$,
25. $\mathfrak{n}(M_{i+4,1})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+2,1})$,
26. $\mathfrak{n}(M_{i+4,2})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+3,2})$,
27. $\mathfrak{n}(M_{i+4,4})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+2,2})$,
28. $\mathfrak{n}(M_{62})\mathfrak{n}(M_{11}) = \mathfrak{n}(M_{64})$,
29. $\mathfrak{n}(M_{i3})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i4})\mathfrak{n}(M_{i3})$,
30. $\mathfrak{n}(M_{i2})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i4})\mathfrak{n}(M_{i2})$,
31. $\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i4})\mathfrak{n}(M_{i1})$,
32. $\mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i1})$,
33. $\mathfrak{n}(M_{i+1,3})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i2})$,
34. $\mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i4})\mathfrak{n}(M_{i+2,1})$,
35. $\mathfrak{n}(M_{i+2,2})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,4})$,
36. $\mathfrak{n}(M_{i+2,4})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i4})\mathfrak{n}(M_{i+2,4})$,
37. $\mathfrak{n}(M_{i+3,2})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,1})$,
38. $\mathfrak{n}(M_{i+3,4})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i1})$,
39. $\mathfrak{n}(M_{i+4,1})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i4})\mathfrak{n}(M_{i+4,1})$,
40. $\mathfrak{n}(M_{i+4,3})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i+2,1})$,
41. $\mathfrak{n}(M_{61})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{14})\mathfrak{n}(M_{61})$,
42. $\mathfrak{n}(M_{62})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{14})\mathfrak{n}(M_{62})$,
43. $\mathfrak{n}(M_{64})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{14})\mathfrak{n}(M_{64})$,
44. $\mathfrak{n}(M_{i2})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i3})\mathfrak{n}(M_{i2})$,
45. $\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i3})\mathfrak{n}(M_{i1})$,
46. $\mathfrak{n}(M_{i+1,1})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i+1,4})$,
47. $\mathfrak{n}(M_{i+1,4})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i3})\mathfrak{n}(M_{i+1,4})$,
48. $\mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i1})$,
49. $\mathfrak{n}(M_{i+2,2})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,3})$,
50. $\mathfrak{n}(M_{i+2,4})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i2})$,
51. $\mathfrak{n}(M_{i+3,1})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,4})$,
52. $\mathfrak{n}(M_{i+3,2})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,2})$,

53. $\mathfrak{n}(M_{i+3,3})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i+1,2})$,
54. $\mathfrak{n}(M_{61})\mathfrak{n}(M_{13}) = \mathfrak{n}(M_{13})\mathfrak{n}(M_{61})$,
55. $\mathfrak{n}(M_{64})\mathfrak{n}(M_{13}) = \mathfrak{n}(M_{11})\mathfrak{n}(M_{11})$,
56. $\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i1})$,
57. $\mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,3})$,
58. $\mathfrak{n}(M_{i+1,3})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i+1,3})$,
59. $\mathfrak{n}(M_{i+1,4})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i+1,4})$,
60. $\mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,4})$,
61. $\mathfrak{n}(M_{i+2,2})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i+2,4})$,
62. $\mathfrak{n}(M_{i+2,4})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i+2,4})$,
63. $\mathfrak{n}(M_{i+3,3})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+1,1})\mathfrak{n}(M_{i+2,3})$,
64. $\mathfrak{n}(M_{i+3,4})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,2})$,
65. $\mathfrak{n}(M_{i+4,3})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+2,2})\mathfrak{n}(M_{i+2,1})$,
66. $\mathfrak{n}(M_{i+4,4})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i+1,1})\mathfrak{n}(M_{i+2,4})$,
67. $\mathfrak{n}(M_{64})\mathfrak{n}(M_{12}) = \mathfrak{n}(M_{11})\mathfrak{n}(M_{31})$,
68. $\mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,2})$,
69. $\mathfrak{n}(M_{i+1,3})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,3})$,
70. $\mathfrak{n}(M_{i+1,4})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,4})$,
71. $\mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,1})$,
72. $\mathfrak{n}(M_{i+2,3})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,3})$,
73. $\mathfrak{n}(M_{i+2,4})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,4})$,
74. $\mathfrak{n}(M_{i+3,2})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i+3,4})$,
75. $\mathfrak{n}(M_{i+3,3})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+2,2})\mathfrak{n}(M_{i+3,4})$,
76. $\mathfrak{n}(M_{i+3,4})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+3,4})$,
77. $\mathfrak{n}(M_{i+4,3})\mathfrak{n}(M_{i1}) = \mathfrak{n}(M_{i+2,1})\mathfrak{n}(M_{i+3,2})$,
78. $\mathfrak{n}(M_{61})\mathfrak{n}(M_{11}) = \mathfrak{n}(M_{11})\mathfrak{n}(M_{61})$,
79. $\mathfrak{n}(M_{63})\mathfrak{n}(M_{11}) = \mathfrak{n}(M_{41})\mathfrak{n}(M_{64})$,
80. $\mathfrak{n}(M_{64})\mathfrak{n}(M_{11}) = \mathfrak{n}(M_{11})\mathfrak{n}(M_{64})$,
81. $\mathfrak{n}(M_{i+2,3})\mathfrak{n}(M_{i4}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+2,4})$,
82. $\mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,4})$,
83. $\mathfrak{n}(M_{i+1,3})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i2})\mathfrak{n}(M_{i2})\mathfrak{n}(M_{i+1,4})$,
84. $\mathfrak{n}(M_{i+2,3})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,3})$,
85. $\mathfrak{n}(M_{i+3,4})\mathfrak{n}(M_{i3}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,4})$,
86. $\mathfrak{n}(M_{i+2,3})\mathfrak{n}(M_{i2}) = \mathfrak{n}(M_{i1})\mathfrak{n}(M_{i+1,2})\mathfrak{n}(M_{i+2,4})$,

where each first subscript belongs to the set $\{1, 2, 3, 4, 5, 6\}$.

Remark 3.1 By comparing the set B with the minimal Gröbner-Shirshov basis given in [15], we found that the right-hand side of each one in B is just the minimal term (we forget the coefficient) of the right-hand side of the corresponding one in the minimal Gröbner-Shirshov basis in [15]. But at the moment, we do not know the reason.

Now we are ready to prove the injectivity of

$$\overline{\eta}:\ \mathcal{S}/\mathfrak{J} \longrightarrow \mathbb{Z}\mathcal{M}, \quad s_i + \mathfrak{J} \longmapsto [S_i], \quad 1 \le i \le 4.$$

For convenience, we set

$$V_1 = M_{14}, V_2 = M_{13}, V_3 = M_{12}, V_4 = M_{11}, V_5 = M_{24}, V_6 = M_{23}, V_7 = M_{22}, V_8 = M_{21},$$
$$V_9 = M_{34}, V_{10} = M_{33}, V_{11} = M_{32}, V_{12} = M_{31}, V_{13} = M_{44}, V_{14} = M_{43}, V_{15} = M_{42}, V_{16} = M_{41},$$
$$V_{17} = M_{54}, V_{18} = M_{53}, V_{19} = M_{52}, V_{20} = M_{51}, V_{21} = M_{64}, V_{22} = M_{63}, V_{23} = M_{62}, V_{24} = M_{61}.$$

Then by the order $(*)$, we have $V_1 \prec \cdots \prec V_{24}$. Given a monomial $\omega = f_{i_1} \cdots f_{i_m}$ $(1 \le i_1 \le i_m \le 4)$, we have

$$\omega = f_{i_1} \cdots f_{i_m} = \mathfrak{n}(S_{i_1}) \cdots \mathfrak{n}(S_{i_m}).$$

Applying the relations in B repeatedly, we can get $\omega = \mathfrak{n}(V_1)^{n_1} \cdots \mathfrak{n}(V_\mu)^{n_{24}}$ for some $n_1, \cdots, n_{24} \ge 0$. Hence, all the monomials $\mathfrak{n}(V_1)^{n_1} \cdots \mathfrak{n}(V_\mu)^{n_{24}}$ with $n_1, \cdots, n_{24} \ge 0$ span $\mathcal{S}/\mathfrak{J}$.

On the other hand, Lemma 2.5 implies that for $n_1, \cdots, n_{24} \geq 0$,

$$\overline{\eta}(\mathfrak{n}(V_1)^{n_1} \cdots \mathfrak{n}(V_{24})^{n_{24}}) = [V_1]^{*n_1} * \cdots * [V_{24}]^{*n_{24}}.$$

By Lemma 2.4, the elements $[V_1]^{*n_1} * \cdots * [V_{24}]^{*n_{24}}$ with $n_1, \cdots, n_{24} \geq 0$ form a basis of $\mathbb{Z}\mathcal{M}_{E_6,\sigma}$. Consequently, the morphism $\overline{\eta}$ is injective.

Hence we have following result.

Proposition 3.2 *There are graded $\mathbb{Z}$-algebra isomorphisms*

$$\Phi : \mathbb{Z}\mathcal{M}_{E_6,\sigma} \longrightarrow \mathfrak{H}_0(F_4), \quad [S_i] \longmapsto u_i, \quad 1 \leq i \leq 4.$$

Proof By Lemma 2.3 and Proposition 3.1, there is a surjective $\mathbb{Z}$-algebra homomorphism $\Phi : \mathbb{Z}\mathcal{M}_{E_6,\sigma} \longrightarrow \mathfrak{H}_0(F_4)$ given by $[S_i] \longmapsto u_i$ with $1 \leq i \leq 4$. Since $\{[M_q(\lambda)_{\mathcal{K}}] \mid \lambda \in \mathfrak{B}\}$ and $\{u_\lambda \mid \lambda \in \mathfrak{B}\}$ are bases for $\mathbb{Z}\mathcal{M}_{E_6,\sigma}$ and $\mathfrak{H}_0(F_4)$, respectively, we know that Φ is an isomorphism.

So we have the following theorem.

Theorem 3.1 *The generators u_i $(1 \leq i \leq 4)$ and the relations* (F1)–(F6) *give a presentation of $\mathfrak{H}_0(F_4)$.*

4 Gröbner-Shirshov Basis for $\mathfrak{H}_0(F_4)$

For any monomial $u \in \mathfrak{H}_0(F_4)$, we define the length $l(u)$ of u to be the number of the $u_i \in C$ occuring in u. Now, we define a degree lexicographic order $\prec$ on the monomials in $\mathfrak{H}_0(F_4)$ as follows:

$$u \prec v \quad \text{if and only if } l(u) < l(v) \text{ or } l(u) = l(v) \text{ and } u < v,$$

and then it is a monomial order (see [16]).

We have already shown that $\mathfrak{H}_0(F_4)$ is an associative algebra over $\mathbb{Z}$ generated by $C = \{u_1, u_2, u_3, u_4\}$ with the generating relations

$$\mathcal{F}' = \begin{cases} u_1u_3 = u_3u_1, & u_1u_4 = u_4u_1, & u_2u_4 = u_4u_2, \\ u_1u_2^2 = u_2u_1u_2, & u_1^2u_2 = u_1u_2u_1, & u_3u_4^2 = u_4u_3u_4, \\ u_3^2u_4 = u_3u_4u_3, & u_3u_2u_3 = u_2u_3^2, & u_2u_3u_2^2u_3 = u_2^3u_3^2, \\ u_2^2u_3u_2 = u_2^3u_3. \end{cases}$$

In the following, we apply the algebra isomorphism $\Phi \circ \overline{\eta}$ to the relations 1, 29 and 81 in B.

(1) We apply $\Phi \circ \overline{\eta}$ to the relations 1:

$$\mathfrak{n}(M_{21})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{12}), \quad \mathfrak{n}(M_{31})\mathfrak{n}(M_{24}) = \mathfrak{n}(M_{22}), \quad \mathfrak{n}(M_{41})\mathfrak{n}(M_{34}) = \mathfrak{n}(M_{32}),$$
$$\mathfrak{n}(M_{51})\mathfrak{n}(M_{44}) = \mathfrak{n}(M_{42}), \quad \mathfrak{n}(M_{61})\mathfrak{n}(M_{54}) = \mathfrak{n}(M_{52}),$$

So then, we have 3 relations (two identical relations are omitted):

$$u_1u_2^2u_3^2u_4 = u_1u_2^2u_3u_4u_3, \quad u_1u_2^3u_3^2u_4 = u_1u_2u_3u_2^2u_3u_4, \quad u_1^2u_2^3u_3^2u_4 = u_2u_1^2u_2^2u_3^2u_4.$$

(2) We apply $\Phi \circ \overline{\eta}$ to relations 29:

$$\mathfrak{n}(M_{13})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{14})\mathfrak{n}(M_{13}), \qquad \mathfrak{n}(M_{23})\mathfrak{n}(M_{24}) = \mathfrak{n}(M_{24})\mathfrak{n}(M_{23}),$$
$$\mathfrak{n}(M_{33})\mathfrak{n}(M_{34}) = \mathfrak{n}(M_{34})\mathfrak{n}(M_{33}), \qquad \mathfrak{n}(M_{43})\mathfrak{n}(M_{44}) = \mathfrak{n}(M_{44})\mathfrak{n}(M_{43}),$$
$$\mathfrak{n}(M_{53})\mathfrak{n}(M_{54}) = \mathfrak{n}(M_{54})\mathfrak{n}(M_{53}), \qquad \mathfrak{n}(M_{63})\mathfrak{n}(M_{64}) = \mathfrak{n}(M_{64})\mathfrak{n}(M_{63}),$$

and we have 6 relations:

$$u_3u_4^2 = u_4u_3u_4, \qquad u_2^2u_3^2u_4u_3 = u_3u_2^2u_3^2u_4,$$
$$u_1^2u_2^4u_3^3u_4^2u_2^2u_3u_4 = u_2^2u_3u_4u_1^2u_2^4u_3^3u_4^2, \qquad u_1^2u_2^4u_3^3u_4u_1^2u_2^2u_3^2u_4 = u_1^2u_2^2u_3^2u_4u_1^2u_2^4u_3^3u_4,$$
$$u_1^2u_2^4u_3^2u_4u_2^2u_3 = u_2^2u_3u_1^2u_2^4u_3^2u_4, \qquad u_1^2u_2^2u_3u_1^2u_2^2u_3u_4 = u_1^2u_2^2u_3u_4u_1^2u_2^2u_3.$$

(3) We apply $\Phi \circ \overline{\eta}$ to the relations 81:

$$\mathfrak{n}(M_{33})\mathfrak{n}(M_{14}) = \mathfrak{n}(M_{11})\mathfrak{n}(M_{11})\mathfrak{n}(M_{34}), \qquad \mathfrak{n}(M_{43})\mathfrak{n}(M_{24}) = \mathfrak{n}(M_{21})\mathfrak{n}(M_{21})\mathfrak{n}(M_{44}),$$
$$\mathfrak{n}(M_{53})\mathfrak{n}(M_{34}) = \mathfrak{n}(M_{31})\mathfrak{n}(M_{31})\mathfrak{n}(M_{54}), \qquad \mathfrak{n}(M_{63})\mathfrak{n}(M_{44}) = \mathfrak{n}(M_{41})\mathfrak{n}(M_{41})\mathfrak{n}(M_{64}),$$

and we have 4 relations:

$$u_1^2u_2^4u_3^3u_4^3 = u_1u_2u_3u_4u_1u_2u_3u_4u_2^2u_3u_4, \qquad u_1^2u_2^4u_3^3u_4u_3 = u_2u_3u_2u_3u_1^2u_2^2u_3^2u_4,$$
$$u_1^2u_2^4u_3^2u_4u_2^2u_3u_4 = u_1u_2^2u_3u_4u_1u_2^2u_3u_4u_2^2u_3, \qquad u_1^2u_2^2u_3u_1^2u_2^2u_3^2u_4 = u_1u_2u_3u_1u_2u_3u_1^2u_2^2u_3u_4.$$

By applying the algebra isomorphism $\Phi \circ \overline{\eta}$ to all the relations in B, we get a new set $\mathcal{F}''$ of the relations (since there are 247 relations in $\mathcal{F}''$, to save space, we do not write them all here).

By computing all possible compositions between the elements of $\mathcal{F}' \cup \mathcal{F}''$, we get the following non-trivial compositions, that is, the new set $\mathcal{F}'''$ of the relations in $\mathfrak{H}_0(F_4)$:

$$u_1u_2u_3u_2^2u_3u_4 = u_2u_1u_2u_3u_4u_2u_3, \qquad u_1u_2u_3u_4u_2u_1u_2u_3u_4 = u_2u_1u_2u_3u_4u_1u_2u_3u_4,$$
$$u_2u_3u_1u_2u_3u_4u_2u_3 = u_2u_3u_2u_3u_1u_2u_3u_4, \qquad u_1u_2u_3u_4u_2u_3u_1u_2u_3u_4 = u_2u_3u_1u_2u_3u_4u_1u_2u_3u_4,$$
$$u_2u_1u_2u_3u_2^2u_3 = u_2^2u_3u_2u_1u_2u_3, \qquad u_1u_2u_3u_4u_2u_3u_2^2u_3u_4 = u_2^2u_3u_4u_1u_2u_3u_4u_2u_3,$$
$$u_2u_1u_2^2u_3 = u_2u_1u_2u_3u_2, \qquad u_2u_1u_2u_3u_1u_2u_3u_4u_2^2u_3 = u_2^2u_3u_2u_1u_2u_3u_1u_2u_3u_4,$$
$$u_1u_2u_1u_2u_3u_4u_2 = u_2u_1u_2u_1u_2u_3u_4, \qquad u_1u_2u_1u_2u_3u_4u_1u_2u_3u_4 = u_1u_2u_3u_4u_1u_2u_1u_2u_3u_4,$$
$$u_1u_2u_1u_2u_3u_2 = u_2u_1u_2u_1u_2u_3, \qquad u_1u_2u_3u_2^2u_3u_4u_2^2u_3u_4 = u_2^2u_3u_4u_1u_2u_3u_2^2u_3u_4,$$
$$u_1u_2u_1u_2u_3u_1u_2u_3 = u_1u_2u_3u_1u_2u_1u_2u_3, \qquad u_1u_2u_3u_1u_2u_3u_1u_2u_3u_4 = u_1u_2u_3u_1u_2u_3u_4u_1u_2u_3,$$
$$u_1u_2u_1u_2u_1u_2u_3 = u_1u_2u_1u_2u_3u_1u_2, \qquad u_1u_2u_3u_1u_2u_3u_4u_2u_3 = u_2u_3u_1u_2u_3u_1u_2u_3u_4,$$
$$u_2u_3u_2u_3u_2u_3u_4 = u_2u_3u_2u_3u_4u_2u_3, \qquad u_1u_2u_1u_2u_3u_4u_1u_2u_3 = u_1u_2u_3u_1u_2u_1u_2u_3u_4,$$
$$u_2^2u_3u_2u_1u_2u_3u_4 = u_2u_1u_2u_3u_4u_2^2u_3,$$
$$u_1u_2u_3u_2u_3u_4u_1u_2u_3u_4 = u_1u_2u_3u_4u_1u_2u_3u_2u_3u_4,$$
$$u_1u_2u_3u_2u_1u_2u_3 = u_2u_1u_2u_3u_1u_2u_3,$$
$$u_2u_3u_2u_3u_4u_2u_3u_4 = u_2u_3u_4u_2u_3u_2u_3u_4,$$
$$u_1u_2u_3u_4u_2u_3u_2u_3u_4 = u_2u_3u_2u_3u_4u_1u_2u_3u_4,$$
$$u_1u_2u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_4 = u_1u_2^2u_3u_4u_1u_2u_3u_1u_2^2u_3u_4,$$
$$u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_4u_2u_3 = u_2u_3u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_4,$$
$$u_1u_2u_3u_1u_2u_3u_4u_1u_2u_3u_4 = u_1u_2u_3u_4u_1u_2u_3u_1u_2u_3u_4,$$
$$u_1u_2u_3u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_4 = u_1u_2u_3u_1u_2u_3u_4u_1u_2u_3u_2^2u_3u_4,$$
$$u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_4 = u_1u_2u_3u_2^2u_3u_4u_1u_2u_3u_4u_1u_2u_3u_2^2u_3u_4.$$

We set $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{F}'''$. Then by the construction of the set $\mathcal{F}$ of the relations in $\mathfrak{H}_0(F_4)$, we get our main result in this paper.

Theorem 4.1 *With the notations above, $\mathcal{F}$ is a Gröbner-Shirshov basis for $\mathfrak{H}_0(F_4)$.*

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