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Fixed Point Theorems for (p,q)-Quasi-Contraction Mappings in Cone Metric Spaces

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Abstract In this work, the authors introduce the concept of (p,q)-quasi-contraction mapping in a cone metric space. We prove the existence and uniqueness of a fixed point for a (p,q)-quasi-contraction mapping in a complete cone metric space. The results of this paper generalize and unify further fixed point theorems for quasi-contraction, convex contraction mappings and two-sided convex contraction of order 2.

Keywords Fixed points, (p,q)-Quasi-contractions, Cone metric space
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1 Introduction

In 1922, the Banach contraction principle was introduced (see [2]) and it remains a forceful tool in nonlinear analysis (see [11]), which incites many authors to extend it, especially for nonlinear mappings. We cite the contraction type of Kannan [14], Chatterjee [3], Zamfirescu [20], Reich [16] and Ćirić [4], which gives one of the most general contraction conditions, called quasi-contraction.

Inspired by the paper of Huang and Zhang [6], Ilić and Rakočević [7] extended the concept of quasi-contraction mappings to cone metric spaces and provided a generalization of Theorem 1 in [6] to quasi-contraction mappings in complete cone metric spaces. Recently, many authors studied several variants of contraction conditions and proved some fixed point theorems in a cone metric space when the underlying cone is normal or not normal. We cite, for example [12–13, 15, 17–18, 21].

This paper is organized as follows. In Section 2, we give some definitions and preliminaries needed in the sequel. In Section 3, we extend the concept of (p,q)-quasi-contraction mappings (see [5]) in cone metric spaces. These mappings extend Ilić and Rakočević's quasi-contraction

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ones, convex contractive maps of order n (see [1, 8]) and the two-sided convex contraction mappings (see [8]). The main result of this section is that every continuous (p,q)-quasi-contraction mapping in a complete cone metric space has a unique fixed point and the Piccard iteration converges to this point. Moreover, we obtain fixed point theorems for certain classes of discontinuous mappings which generalize many known results.

2 Preliminaries

Let E be a real Banach space. A nonempty subset P of E is said to be a cone if and only if

(i) P is closed and $P \neq \{0\}$,

(ii) for every positive real a, $aP \subset P$, and

(iii) $P + P \subset P$ and $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we can define a partial ordering $\preccurlyeq$ on E with respect to P by $x \preccurlyeq y$ if and only if $(y-x) \in P$. We will indicate by $x \prec y$ that $x \preccurlyeq y$ but $x \neq y$, and by $x \ll y$ that $(y-x) \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called minihedral if $\sup\{x,y\}$ exists for all $x,y \in E$. Recall also that P is called normal, if there is a number $K > 0$ such that for all $x,y \in E$,

$$0 \preccurlyeq x \preccurlyeq y \quad \text{implies} \quad \|x\| \preccurlyeq K\|y\|. \tag{2.1}$$

The least positive number satisfying the above inequality, is called the normal constant of P. In [6], Huang and Zhang introduced the notion of cone metric spaces as a generalization of the metric spaces.

Definition 2.1 (see [6]) *Let P be a cone in a Banach space such that $\operatorname{int} P \neq \emptyset$ and $\preccurlyeq$ is a partial ordering in E with respect to P. A cone metric on a nonempty set X is a function $d : X \times X \to E$ such that, for all $x,y,z \in X$, we have*

(a) *$x = y$ if and only if $d(x,y) = 0$,*

(b) *$0 \preccurlyeq d(x,y) = d(y,x)$, and*

(c) *$d(x,y) \preccurlyeq d(x,z) + d(z,y)$.*

A cone metric space is a pair (X,d) such that X is a nonempty set and d is a cone metric on X.

Now, let's recall some useful definitions needed in the sequel.

Definition 2.2 (see [6]) *Let (X,d) be a cone metric space, and let $\{x_n\}$ be a sequence in X. Then*

(i) *$\{x_n\}$ converges to $x \in X$ if, for every $c \in E$ with $0 \ll c$, there exists a natural number N such that, for all $n \geq N$, we have $d(x_n,x) \ll c$. We denote this convergence by $x_n \to x$ $(n \to \infty)$ or $\lim_{n\to\infty} x_n = x$.*

(ii) *$\{x_n\}$ is a Cauchy sequence if, for every $c \in E$ with $0 \ll c$, there exists a natural number N such that, for all $n,m \geq N$, we have $d(x_n,x_m) \ll c$.*

(iii) *(X,d) is a complete cone metric space if, every Cauchy sequence is convergent.*

In the case where P is a normal cone, we have the following lemmas.

Lemma 2.1 (see [6]) *Let (X, d) be a cone metric space, P be a normal cone with a normal constant K, and $\{x_n\}$ be a sequence in X.*

(i) *If the limit of $\{x_n\}$ exists, then it is unique.*

(ii) *Every convergent sequence in X is a Cauchy sequence.*

(iii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.*

Lemma 2.2 (see [6]) *Let (X, d) be a cone metric space, P be a normal cone with a normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$. Then*

$$d(x_n, y_n) \to d(x, y) \quad (n \to \infty). \tag{2.2}$$

Definition 2.3 (see [6]) *Let (X, d) be a cone metric space. X is called a sequentially compact cone metric space if, for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X.*

Lemma 2.3 *Every sequentially compact cone metric space is a complete cone metric space.*

Proof Let $\{x_n\}$ be a Cauchy sequence in the sequentially compact cone metric space X, and then there exists a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to $x \in X$. By using the following inequality

$$d(x_n, x) \preccurlyeq d(x_n, x_{n_i}) + d(x_{n_i}, x), \tag{2.3}$$

we obtain the desired result.

3 Main Results

In the sequel, we suppose that E is a Banach space, P is a normal cone in E with $\operatorname{int} P \neq \emptyset$, K is the normal constant of P, and $\preccurlyeq$ is a partial ordering in E with respect to P. In this section, we generalize the Fisher's quasi-contraction mapping on a cone metric space. We notice that such a mapping is a generalization of Ilić and Rakočević's quasi-contraction mappings acting on cone metric spaces (see [7]).

For this purpose, we introduce the definition of (p, q)-quasi-contraction mappings in the cone metric spaces as follows.

Definition 3.1 *Let (X, d) be a cone metric space and p, q be two natural numbers such that $0 < p \leq q$. The mapping $T : X \to X$ is said to be a (p, q)-quasi-contraction, if there exists a number $c \in [0, 1)$ such that for every $x, y \in X$, there is $u \in C_{p,q}(x, y)$, such that*

$$d(T^p x, T^q y) \preccurlyeq cu, \tag{3.1}$$

where

$$C_{p,q}(x, y) = \{d(T^r x, T^s y),\ d(T^r x, T^{r'} x),\ d(T^s y, T^{s'} y) : 0 \leq r,\ r' \leq p \text{ and } 0 \leq s, s' \leq q\}.$$

Next, let $n \in \mathbb{N}$, $x \in X$ and $O(x : n) = \{x, Tx, T^2 x, \cdots, T^n x\}$. The following subset of X

$$O(x : \infty) = \{x, Tx, T^2 x, \cdots\}$$

is called the orbit of T at x. The partial cone metric space (X,d) is said to be T-orbitally complete if, every Cauchy sequence contained in an orbit of T converges in X. Obviously, every complete cone metric space is T-orbitally complete, but the converse does not hold (see [14, Example 3.1]).

Let (X,d) be a cone metric space. The mapping $T : X \to X$ is said to be an orbitally (p,q)-quasi-contraction, if T is (p,q)-quasi-contractive on any orbit of X.

Example 3.1 Let's consider the following cone metric space stated in [6]:

$$E = \mathbb{R}^2, \quad P = \{(x,y) \in E : x, y \geq 0\}, \quad X = \mathbb{R},$$

and its cone metric $d : X \times X \to E$ is defined by

$$d(x,y) = (|y-x|, \alpha|y-x|), \quad \text{where } \alpha \geq 0 \text{ is a constant.}$$

Notice that the mapping T defined on the cone metric space (X,d) by $Tx = x$ is an orbitally (p,q)-quasi-contraction, but it is not a (p,q)-quasi-contraction.

Now, let's first consider the following subset of E defined by

$$\Delta(x,p,n) = \{d(a,b) : a, b \in \{T^i x,\ p \leq i \leq n\}\}.$$

In particular cases, and for simplicity, we denote

$$\Delta(x,n) = \Delta(x,0,n) \quad \text{and} \quad \Delta(x:\infty) = \{d(a,b) : a, b \in O(x:\infty)\}.$$

Finally, for any subset $F \subset E$, we denote $\delta(F) = \sup\{\|x\|, x \in F\}$.

The following lemma is crucial for the main results.

Lemma 3.1 *Let (X,d) be a cone metric space, and let $T : X \to X$ be an orbitally (p,q)-quasi-contraction mapping. Then, $\delta(\Delta(x:\infty))$ is finite.*

Proof Let $x \in X$. If $Tx = x$, then it is obvious that $\Delta(x:\infty)$ is bounded. Now, suppose that $Tx \neq x$ and consider $n_0 \in \mathbb{N}$ such that $\max\{c^{n_0}K, c^{n_0}K^2\} < 1$. Choose i, j, and $n \in \mathbb{N}$ such that

$$n_0 q \leq i < j \leq n. \tag{3.2}$$

Since T is a (p,q)-quasi-contraction, we deduce that

$$d(T^i x, T^j x) \preccurlyeq c u_1,$$

where $u_1 \in \Delta(x, i-q, n)$. By the same argument, it follows that there exists $u_2 \in \Delta(x, i-2q, n)$ such that $u_1 \preccurlyeq c u_2$. Hence

$$d(T^i x, T^j x) \preccurlyeq c^2 u_2.$$

After n_0 iterations, we conclude that there exists $u_{n_0} \in \Delta(x, i-n_0 q, n)$ such that

$$d(T^i x, T^j x) \preccurlyeq c^{n_0} u_{n_0}. \tag{3.3}$$

By using (3.2) and the fact that $\Delta(x, i-n_0q, n) \subset \Delta(x,0,n) = \Delta(x,n)$, we infer that $u_{n_0} \in \Delta(x,n)$. Now, since P is normal and $c^{n_0}K < 1$, it follows that

$$\|d(T^ix, T^jx)\| < \delta(\Delta(x,n)). \tag{3.4}$$

We conclude that

$$\delta(\Delta(x,n)) = \max\{\|d(T^ix,T^lx)\|, \|d(T^ix,T^jx)\| : 0 \le i,j < n_0q \le l \le n\}. \tag{3.5}$$

From (3.5), there are two possible cases.

Case 1 Suppose that $\delta(\Delta(x,n)) = \|d(T^ix,T^lx)\|$ for some $0 \le i < n_0q \le l \le n$. Since d is a cone metric, by the triangular inequality, it follows that

$$d(T^ix,T^lx) \preccurlyeq d(T^ix,T^{n_0q}x) + d(T^{n_0q}x,T^lx).$$

Taking into account that P is normal, and by using (3.3), we deduce that

$$\|d(T^ix,T^lx)\| \le K\|d(T^ix,T^{n_0q}x)\| + c^{n_0}K^2\delta(\Delta(x,n)).$$

Hence

$$\delta(\Delta(x,n)) \le \frac{K}{1-c^{n_0}K^2}\delta(\Delta(x,n_0q)). \tag{3.6}$$

Case 2 Suppose that $\delta(\Delta(x,n)) = \|d(T^ix,T^jx)\|$ for some $1 \le i,j \le n_0$. Since $d(T^ix,T^jx) \in \Delta(x,n_0)$, it follows that

$$\delta(\Delta(x,n)) \le \delta(\Delta(x,n_0q)). \tag{3.7}$$

Inequalities (3.6)–(3.7) imply that

$$\delta(\Delta(x,n)) \le \max\left\{1, \frac{K}{1-c^{n_0}K}\right\}\delta(\Delta(x,n_0q)). \tag{3.8}$$

Since $\delta(\Delta(x,\infty)) = \sup\{\delta(\Delta(x,n)) : n \in \mathbb{N}\}$, we conclude that $\delta(\Delta(x,\infty))$ is finite.

Theorem 3.1 *Let (X,d) be a cone metric space, and let $T : X \to X$ be a continuous and T-orbitally (p,q)-quasi-contraction mapping. If X is T-orbitally complete, then the sequence $\{T^nx\}$ converges to a fixed point for every $x \in X$. Moreover, if T is a (p,q)-quasi-contraction, then for any $x \in X$, the fixed point is unique.*

Proof If $Tx = x$, then the result holds. In the rest of the proof, we will suppose that $Tx \ne x$ and we will prove that $\{T^nx\}$ is a Cauchy sequence.

For this purpose, let $\epsilon > 0$, and choose N such that $Kc^N\delta(\Delta(x,\infty)) < \epsilon$. For every two natural numbers n,m such that $m \ge n \ge Nq$, there exists $u_1 \in \Delta(x,n-q,m)$ such that

$$d(T^nx,T^mx) \preccurlyeq cu_1.$$

Since T is a T-orbitally (p,q)-quasi-contraction mapping, every $v_1 \in \Delta(x, n-q, m)$ satisfies $v_1 \preccurlyeq cv_2$, where $v_2 \in \Delta(x, n-2q, m)$, and after N steps, we deduce that there exists $u_N \in \Delta(x,m)$ such that

$$d(T^n x, T^m x) \preccurlyeq c^N u_N.$$

By applying Lemma 3.1, we obtain that

$$\|d(T^n x, T^m x)\| \leq kc^N \delta(\Delta(x:\infty)) < \epsilon,$$

and then $\{T^n x\}$ is a Cauchy sequence in (X,d). Since (X,d) is orbitally complete, there exists $y \in X$ such that $\{T^n x\}$ converges to y. The continuity of T shows that y is a fixed point of T.

Now, if T is a (p,q)-quasi-contraction on X, and if we suppose that there exists another $z \in X$ such that $Tz = z$, then

$$d(z,y) = d(T^p z, T^q y) \preccurlyeq cd(z,y). \tag{3.9}$$

Since $c < 1$, we have $d(z,y) = 0$. Hence, the fixed point of T is unique.

Example 3.2 Let $P = \{(x,y) \in \mathbb{R}^2 \text{ such that } x, y \geq 0\}$ be a normal cone of the Banach space $\mathbb{R}^2$ and let

$$X = \{(x,0) \in \mathbb{R}^2 \text{ such that } 0 \leq x \leq 1\}.$$

For every $(x,0), (y,0) \in X$, we define the metric d as follows:

$$d((x,0),(y,0)) = (|x-y|, \gamma|x-y|), \quad \text{where } \gamma \in [0,1].$$

Clearly, (X,d) is a complete cone metric space. Now, let $T : X \to X$ be a mapping such that $T((x,0)) = \left(\frac{x^3}{3}, 0\right)$. Notice that T is a continuous mapping which has $(0,0)$ as a unique fixed point.

Moreover, we have

$$d(T(x,0), T(y,0)) = \frac{x^2+y^2+xy}{3} d((x,0),(y,0)),$$

which implies that T is not a Banach contraction mapping. However, since

$$\begin{aligned}
d(T^2(x,0), T^2(y,0)) &= d\Big(T\Big(\frac{x^3}{3},0\Big), T\Big(\frac{y^3}{3},0\Big)\Big) \\
&= \Big(\Big|\frac{x^9-y^9}{81}\Big|, \gamma\Big|\frac{x^9-y^9}{81}\Big|\Big) \\
&= \frac{x^6+x^3y^3+y^6}{27}\Big(\Big|\frac{x^3-y^3}{3}\Big|, \gamma\Big|\frac{x^3-y^3}{3}\Big|\Big) \\
&\leq \frac{1}{9} d(T(x,0), T(y,0)),
\end{aligned}$$

we infer that T is a $(2,2)$-quasi-contraction.

As a corollary of Theorem 3.1, when X is a metric space, we obtain the main result of Fisher [5, Theorem 2].

Corollary 3.1 *Let T be (p, q)-quasi-contraction on a complete metric space X into itself and assume that T is the continuous. Then, T has a unique fixed point in X.*

Notice that in Theorem 3.1, the continuity of T, when $p = 1$, is not needed. In this case, we prove the following theorem.

Theorem 3.2 *Let (X, d) be a cone metric space, and let $T : X \to X$ be continuous such that T-orbitally is the $(1, q)$-quasi-contraction mapping. If X is T-orbitally complete, then the sequence $\{T^n x\}$ converges to a fixed point for every $x \in X$. Moreover, if T is $(1, q)$-quasi-contractive, then for any $x \in X$, the fixed point is unique and T is continuous in such a point.*

Proof By using Theorem 3.1, the sequence $\{T^n x\}$ converges to some $y \in X$. Let $n \in \mathbb{N}$ be large enough. Then, we have

$$\begin{aligned} d(y, Ty) &\preccurlyeq d(y, T^n y) + d(T^n y, Ty) \\ &\preccurlyeq d(y, T^n y) + cu, \end{aligned}$$

where

$$u \in \{d(T^{n-i} y, T^{n-j} y),\ d(Ty, T^{n-j} y),\ d(y, T^{n-j} y),\ d(y, Ty) : 0 \le i, j \le q\}.$$

Taking into account that P is a normal cone with a constant K, we deduce that

$$\|d(y, Ty)\| \le K\|d(y, T^n y)\| + Kc\|u\|. \tag{3.10}$$

Since the sequence $\{T^n x\}$ converges to y, it follows that $d(y, Ty) = 0$ and hence, y is a fixed point of T.

Now, assume besides that T is a $(1, q)$-quasi-contraction mapping. The uniqueness of the fixed point may be checked in a similar way as in Theorem 3.1. It remains to prove that T is continuous in the fixed point y. To show this, let $\{y_n\}$ be a sequence of points in the cone metric space X which converges to y. Then, there exists $u \in C_{1,q}(y_n, y)$ such that

$$d(Ty_n, y) = d(Ty_n, T^q y) \preccurlyeq cu \preccurlyeq c[d(y_n, y) + d(y, Ty_n)],$$

and it follows that

$$d(Ty_n, y) \preccurlyeq \frac{c}{1-c} d(y_n, y).$$

Hence

$$\|d(Ty_n, y)\| \le \frac{cK}{1-c} \|d(y_n, y)\|, \tag{3.11}$$

so we conclude that $\lim_{n\to\infty} \|d(Ty_n, y)\| = 0$, which completes the proof.

As a corollary, when $p = q = 1$, we have the main result of Ilić and Rakočević [7].

Corollary 3.2 *Let (X, d) be a complete cone metric space, and let $T : X \to X$ satisfy the following inequality*

$$d(Tx, Ty) \preccurlyeq cu \tag{3.12}$$

for some $u \in \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$. *Then* T *has a unique fixed point in* X*, and for every* $x \in X$*, the sequence* $\{T^n x\}$ *converges to the fixed point.*

Notice that, when X is a metric space, if we take $p = 1$, we obtain Theorem 3 in [5].

Corollary 3.3 *Let* T *be a* $(1,q)$*-quasi-contraction on a complete metric space* X *into itself. Then* T *has a unique fixed point in* X*.*

Remark 3.1 It should be noticed that if P is minihedral and $p = q = 2$, then we obtain, as particular cases, Istratescu's fixed point theorem for convex contraction mappings of order 2 (see [8, Theorem 1.2]), the Istratescu's fixed point theorem for two-sided convex contraction of order 2 (see [8, Theorem 2.3]) in complete metric spaces, and their generalizations to the cone metric spaces obtained by Alghamdi et al. [1].

Theorem 3.3 *Let* X *be a sequentially compact minihedral cone metric space. Let* T *be a continuous mapping on* X *which satisfies that for every* $x, y \in X$*, there exists* $u \in C_{p,q}(x,y)$ *such that*

$$d(T^p x, T^q y) < u. \tag{3.13}$$

Then T *has a unique fixed point in* X*.*

Proof We suppose that T is not a (p,q)-quasi-contraction. Since X is a minihedral cone metric space, there exists an increasing sequence $\{c_n\}$ of numbers converging to 1 and two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$c_n u_n \prec d(T^p x_n, T^q y_n)$$

for any $u_n \in C_{p,q}(x_n, y_n)$. Since X is a sequentially compact cone metric space, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ which converges to x and y, respectively. It follows that, for any $u_{n_i} \in C_{p,q}(x_{n_i}, y_{n_i})$, $i \in \mathbb{N}^*$, we have

$$c_{n_i} u_{n_i} \prec d(T^p x_{n_i}, T^q y_{n_i}).$$

Since T is continuous, letting i tend to infinity, we infer that, for any $u \in C_{p,q}(x,y)$,

$$u \preccurlyeq d(T^p x, T^q y). \tag{3.14}$$

Inequalities (3.13)–(3.14) give a contradiction, so then T is a (p,q)-quasi-contraction and the conclusion follows by Theorem 3.1.

We notice that if $p = q = 1$, we have the following corollary.

Corollary 3.4 *Let* X *be a sequentially compact minihedral cone metric space. Let* T *be a continuous mapping on* X *satisfying the inequality*

$$d(Tx, Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for every $x, y \in X$*, for which the right-hand side of the inequality is non-zero. Then* T *has a unique fixed point in* X*.*

Next, we prove that a (p,q)-quasi-contraction mapping satisfies a property $(\mathcal{P})$. We say that the mapping T has the property $(\mathcal{P})$ if, $\mathrm{Fix}(T) = \mathrm{Fix}(T^n)$ for all $n \geq 1$, i.e., every periodic point is a fixed point. Such a property was introduced by Rhoades in his works (see [9–10, 18]). Recently, it was generalized for the quasi-contraction mappings on the cone metric spaces by Kadelburg et al. [13], and for the convex contraction mappings on cone metric spaces by Alghamdi et al. [1].

Theorem 3.4 *Let (X,d) be a complete cone metric space, and let $T : X \to X$ be a continuous and (p,q)-quasi-contraction mapping. Then T has the property $(\mathcal{P})$.*

Proof Let $n \in \mathbb{N}^*$. It is clear that $\mathrm{Fix}(T) \subset \mathrm{Fix}(T^n)$. It remains to prove the inverse inclusion. For this purpose, let's choose $y \in \mathrm{Fix}(T^n)$, and then $T^n(Ty) = T(T^n y) = Ty$, so $O(y:\infty) \subset \mathrm{Fix}(T^n)$. Arguing as for Theorem 3.1, we have that $\{T^n y\}$ is a Cauchy sequence, and then

$$d(y,Ty) = d(T^{nm}y, T^{nm+1}y) \to 0,$$

so $d(y,Ty) = 0$, which ends the proof.

Corollary 3.5 *Let T be a continuous (p,q)-quasi-contraction mapping on a complete cone metric space (X,d). Then, T and T^n have a unique common fixed point for every natural number n.*

Remark 3.2 It should be noticed that Corollary 3.5 is an extension of Corollary 1 and Corollary 2 obtained by Alghamdi et al. [1].

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