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Approximate Representation of Bergman Submodules*

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Abstract In the present paper, the author shows that if a homogeneous submodule $\mathcal{M}$ of the Bergman module $L_a^2(B_d)$ satisfies

$$P_{\mathcal{M}} - \sum_i M_{z^i} P_{\mathcal{M}} M_{z^i}^* \leq \frac{c}{N+1} P_{\mathcal{M}}$$

for some number $c > 0$, then there is a sequence $\{f_j\}$ of multipliers and a positive number c' such that $c' P_{\mathcal{M}} \leq \sum\limits_j M_{f_j} M_{f_j}^* \leq P_{\mathcal{M}}$, i.e., $\mathcal{M}$ is approximately representable. The author also proves that approximately representable homogeneous submodules are p-essentially normal for $p > d$.

Keywords Approximate representation, Essential normality, Bergman submodule
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1 Introduction

In his papers [2–3], Arveson raised the conjecture whether or not every homogeneous submodule of the Drury-Arveson module is essentially normal. More precisely, let $\mathcal{M}$ be a homogeneous submodule of the Drury-Arveson module H_d^2 over the unit ball B_d, and $R_i := M_{z_i}|_{\mathcal{M}}$ be the restriction of the coordinate operator on $\mathcal{M}$, and then the conjecture asks whether the commutators $[R_i^*, R_i]$ should be compact for $i = 1, \cdots, d$. If the answer is yes and moreover, these commutators are in the Schatten-von Neumann class $\mathcal{L}^p$, then $\mathcal{M}$ is called p-essentially normal. As can be seen in [2–3], this problem has deeply linked to C^*-extension theory, index theory, algebraic geometry and other branches of mathematics.

There is much literature on this topic. Guo [16] proved that in the case $d = 2$, each homogeneous submodule is p-essentially normal for $p > 2$. In their remarkable paper, Guo and Wang [17] gave the proof of p-essential normality of principal homogeneous submodules for $p > d$, and as a consequence, they proved p-essential normality of homogeneous submodules for $p > 3$ in the case $d = 3$. They also proved that quasi-homogeneous submodules of the Bergman module $L_a^2(B_2)$ are p-essentially normal for $p > 2$ (see [18]). Shalit [24] proved that submodules possessing the stable division property are essentially normal. Douglas and Wang [6] proved the p-essential normality of submodules of the Bergman module generated by a single polynomial for $p > d$. Fang and Xia [14] extended the approach of Douglas and Wang and proved the

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p-essential normality of submodules generated by a single polynomial of the Hardy module and beyond, when $p > d$. Guo and Zhao [19] proved p-essential normality of principal quasi-homogeneous submodules for $p > d$, and that of quasi-homogeneous submodules in the case $d = 3$. Douglas and Wang [7] and Kennedy [20] made discussions on essential normality of sums of essentially normal submodules. Recently, Engliš and Eschmeier [11] proved essential normality of homogeneous submodules spanned by a radical ideal of good zero variety. Other related discussions on this topic can be found in Eschmeier [12], Douglas and Sarkar [5] and Kennedy and Shalit [21].

In [1], Arveson found that for submodules $\mathcal{M}$ of the Drury-Arveson module H_d^2, the projection onto $\mathcal{M}$ can be represented as

$$P_{\mathcal{M}} = (\mathrm{SOT}) \sum_k M_{\varphi_k} M_{\varphi_k}^*, \tag{1.1}$$

where each φ_k is an analytic multiplier of H_d^2. McCullough [22] generalized this result to submodules of Hilbert modules determined by complete Nevanlinna-Pick kernels, and proved that for submodules $\mathcal{M} \subset \mathcal{H}$, the projections onto $\mathcal{M}$ have the form (1.1). For a polynomial $q \in \mathbb{C}[z_1, \cdots, z_d]$, we use T_q to denote the analytic Toeplitz operator of symbol q. The author and Yu [25] proved that for bounded operators $T \in B(L_a^2(B_d))$ of the form (1.1), the commutator $[T, T_{z_i}]$ belongs to the Schatten-von Neumann p-class $\mathcal{L}^p$ for $p > d$ and $i = 1, \cdots, d$. Therefore, if the projection onto a submodule $\mathcal{M} \subset L_a^2(B_d)$ can be represented by the form (1.1), then $\mathcal{M}$ is p-essentially normal for $p > d$. The following question arises.

Question 1 For which Hilbert modules, the projection onto every submodule can be represented as an SOT limit like (1.1)?

Engliš [8, 10] proved that the affirmative answer to this question can only be given to Hilbert modules of complete Nevanlinna-Pick kernels. For Bergman modules, we prove in Lemma 2.3 that (1.1) does not hold for nontrivial submodules. Therefore, we lower our expectation and ask the following question.

Question 2 Given a submodule $\mathcal{M} \subset L_a^2(B_d)$, is there a sequence of multipliers $\{\varphi_k : k = 0, 1, \cdots\}$ such that there are positive numbers c_1, c_2 and Schatten-von Neumann p-class operators K_1, K_2 relevant to $\mathcal{M}$, making

$$c_1 P_{\mathcal{M}} + K_1 \leq (\mathrm{SOT}) \sum_k T_{\varphi_k} T_{\varphi_k}^* \leq c_2 P_{\mathcal{M}} + K_2? \tag{1.2}$$

When this happens, we say that $\mathcal{M}$ has a p-approximate representation by multipliers $\{\varphi_k : k = 0, 1, \cdots\}$.

If a submodule $\mathcal{M} \subset L_a^2(B_d)$ has a p $(> d)$-approximate representation, then we can prove that $\mathcal{M}$ is p-essentially normal. Details can be found in Lemma 3.3 and Proposition 3.4. Actually by a counterexample of the non-essentially-normal submodule given by Gleason, Richter and Sundberg [15], not all submodules of $L_a^2(B_d)$ have p-approximate representations. However, for homogeneous submodules, the answer to Question 2 seems to be affirmative.

Engliš [9–10] tried to answer Question 2 and proved that projections onto each submodule $\mathcal{M} \subset L_a^2(B_d)$ can be written as

$$P_{\mathcal{M}} = T_1 - T_2,$$

where T_1 and T_2 are operators of the form (1.1), which are not assumed to be bounded.

As in [17], we use N to denote the number operator that maps homogeneous polynomials f to $\deg(f)f$, and then we can talk about its functional calculus.

In all the computable examples, when I is generated by monomials or by a single homogeneous polynomial, or when $d = 3$, we find that the following condition holds.

Condition 1.1 There is a positive number c relevant to $\mathcal{M}$ such that

$$P_{\mathcal{M}} - \sum_{i=1}^{d} M_{z_i} P_{\mathcal{M}} M_{z_i}^* \leq \frac{c}{N+1} P_{\mathcal{M}}.$$

This is a sufficient condition for p ($> d$)-essential normality of homogeneous submodules of $L_a^2(B_d)$, and it is nearly necessary in the sense that it holds for all the known examples. We prove in the present paper that when a homogeneous Bergman submodule $\mathcal{M}$ satisfies Condition 1.1, it does have p ($> d$)-approximate representations. Therefore, p ($> d$)-approximate representability can be seen as nearly equivalent to p-essential normality of homogeneous Bergman submodules.

In Section 2, we introduce some terminologies and notations, and make some discussions on the relation between defect operators and projections onto submodules.

In Section 3, we discuss the relation between p-essential normality and p-approximate representability.

2 Preliminaries

Given a multi-index $\alpha \in \mathbb{Z}_+^d$ and $z \in \mathbb{C}^d$, we write

$$\begin{aligned} \alpha! &= \alpha_1! \cdots \alpha_d!, \\ |\alpha| &= \alpha_1 + \cdots + \alpha_d, \\ z^\alpha &= z_1^{\alpha_1} \cdots z_d^{\alpha_d} \end{aligned}$$

for abbreviation.

The Drury-Arveson space H_d^2 is defined as the Hilbert space of analytic functions over the unit ball $B_d \subset \mathbb{C}^d$, generated by the reproducing kernel

$$K_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle}, \quad \lambda \in B_d.$$

In other words, H_d^2 is the completion of the polynomial ring $\mathbb{C}[z_1, \cdots, z_d]$ with respect to the inner product defined by

$$\langle z^\alpha, z^\alpha \rangle = \frac{\alpha!}{|\alpha|!} \tag{2.1}$$

for $\alpha \in \mathbb{Z}_+^n$, and $\langle z^\alpha, z^\beta \rangle = 0$ whenever $\alpha \neq \beta$. H_d^2 equipped with the natural $\mathbb{C}[z_1, \cdots, z_d]$-module structure defined by multiplication by polynomials is called the Drury-Arveson module, or the d-shift Hilbert module.

Let $\mathrm{d}\nu$ denote the normalized Lebesgue measure on B_d. The Bergman space $L_a^2(B_d)$ is the completion of $\mathbb{C}[z_1, \cdots, z_d]$ with respect to the inner product $\langle f, g \rangle = \int_{B_d} f(z)\overline{g}(z)\mathrm{d}\nu(z)$. One can compute that $\langle z^\alpha, z^\alpha\rangle = \frac{\alpha!d!}{(|\alpha|+d)!}$ when $\alpha \in \mathbb{Z}_+^d$, and $z^\alpha \perp z^\beta$ when $\alpha \neq \beta$, for which the details can be seen in [23]. $L_a^2(B_d)$ has the reproducing kernel

$$K_\lambda(z) = \frac{1}{(1-\langle z,\lambda\rangle)^{d+1}}, \quad \lambda \in B_d.$$

$L_a^2(B_d)$ also has the natural $\mathbb{C}[z_1,\cdots,z_d]$-module structure defined by multiplication by polynomials.

Given $q \in \mathbb{C}[z_1,\cdots,z_d]$, we use M_q to denote the multiplication operator of symbol q on H_d^2, and T_q to denote the analytic Toeplitz operator of symbol q on $L_a^2(B_d)$, respectively. A submodule is defined as a closed subspace which is invariant under multiplication by polynomials. Submodules generated by homogeneous polynomials are called homogeneous.

Let $\mathcal{M}$ be a submodule of H_d^2, and then the quotient module $H_d^2/\mathcal{M}$ is isometrically isomorphic to $\mathcal{M}^\perp$, on which the action by polynomial q is defined as $S_q = P_{\mathcal{M}^\perp} M_q|_{\mathcal{M}^\perp}$. When all the commutators $[S_{z_i}^*, S_{z_i}]$ $(i=1,\cdots,d)$ are compact, the module $\mathcal{M}^\perp$ is said to be essentially normal.

On H_d^2, one can compute $M_{z_i}^* z^\alpha = \frac{\alpha_i}{|\alpha|} z^{\alpha-e_i}$, where e_i denotes the multi-index with 1 on the i-th coordinate and 0 elsewhere. On the other hand, on $L_a^2(B_d)$ we have $T_{z_i}^* z^\alpha = \frac{\alpha_i}{|\alpha|+d} z^{\alpha-e_i}$.

The following lemma from [2–3] and [4] provides us with the basic viewpoint of p-essentially normal submodules.

Lemma 2.1 *Let $\mathcal{M}$ be a submodule of $H_d^2 \otimes \mathbb{C}^r$. Then the following statements are equivalent for $p > d$:*

(1) $\mathcal{M}$ *is p-essentially normal;*

(2) $\mathcal{M}^\perp$ *is p-essentially normal;*

(3) $[P_{\mathcal{M}}, M_{z_i}] = P_{\mathcal{M}} M_{z_i} - M_{z_i} P_{\mathcal{M}}$ *are in $\mathcal{L}^{2p}$ for $1 \leq i \leq d$.*

This lemma is also valid for the Hardy modules, the Bergman modules, etc.

Assume that $\mathcal{M}$ is a submodule of H_d^2. By [1],

$$P_{\mathcal{M}} - \sum_{i=1}^d M_{z_i} P_{\mathcal{M}} M_{z_i}^* \geq 0,$$

and therefore one can define the defect operator of $\mathcal{M}$ by

$$\Delta(\mathcal{M}) := \Big(P_{\mathcal{M}} - \sum_{i=1}^d M_{z_i} P_{\mathcal{M}} M_{z_i}^*\Big)^{\frac{1}{2}}.$$

Arveson [1] also proved that there is a sequence $\{\varphi_j\}$ of multipliers such that

$$P_{\mathcal{M}} = \sum_j M_{\varphi_j} M_{\varphi_j}^*,$$

and each of such sequences satisfies that $\sum |\varphi_j(\lambda)|^2$ tends to 1 as λ tends non-tangentially to z for almost every $z \in \partial B_d$.

When $\mathcal{M}$ is homogeneous, $\Delta^2(\mathcal{M})$ keeps the degree of polynomials, and hence is diagonalizable. Therefore $\Delta^2(\mathcal{M})$ can be written as $\sum\limits_j f_j \otimes f_j$ where $\{f_j\}$ is a sequence of pairwise orthogonal eigenvectors, each of which is homogeneous. As a direct observation, the following lemma reveals the relationship between $P_{\mathcal{M}}$ and $\Delta^2(\mathcal{M})$, which is a key tool in the study of Drury-Arveson submodules. Since we did not see a formal statement of it, we write it down here and give a proof.

Lemma 2.2 *Let $\mathcal{M} \subset H_d^2$ be a homogeneous submodule, and $\{f_j\}$ be a sequence of polynomials such that $\Delta^2(\mathcal{M}) = (\mathrm{SOT}) \sum f_j \otimes f_j$, and then we have $P_{\mathcal{M}} = (\mathrm{SOT}) \sum M_{f_j} M_{f_j}^*$. In particular, we can choose $\{f_j\}$ to be pairwise orthogonal eigenvectors of $\Delta^2(\mathcal{M})$, each of which is homogeneous.*

Proof Given an operator $B \in B(H_d^2)$, we define $\sigma(B) = \sum\limits_{i=1}^{d} M_{z^i} B M_{z^i}^*$, and then σ is positive. Since

$$\Delta^2(\mathcal{M}) = P_{\mathcal{M}} - \sigma(P_{\mathcal{M}}),$$

we have

$$\sigma^n(\Delta^2(\mathcal{M})) = \sigma^n(P_{\mathcal{M}}) - \sigma^{n+1}(P_{\mathcal{M}})$$

for $n = 0, 1, \cdots$. Therefore

$$(\mathrm{SOT}) \sum_{n=0}^{\infty} \sigma^n(\Delta^2(\mathcal{M})) = P_{\mathcal{M}} - (\mathrm{SOT}) \lim_n \sigma^{n+1}(P_{\mathcal{M}}) = P_{\mathcal{M}}.$$

If we define

$$A_N := \sum_{j=1}^{N} M_{f_j} M_{f_j}^*$$

for each $N \in \mathbb{N}$, then

$$\begin{aligned} A_N - \sigma(A_N) &= \sum_{j=1}^{N} M_{f_j} \Big(I - \sum_{i=1}^{d} M_{z_j} M_{z_j}^* \Big) M_{f_j}^* \\ &= \sum_{j=1}^{N} M_{f_j} (1 \otimes 1) M_{f_j}^* \\ &= \sum_{j=1}^{N} f_j \otimes f_j \\ &\leq \Delta^2(\mathcal{M}). \end{aligned}$$

This implies

$$\begin{aligned} A_N &= (\mathrm{SOT}) \sum_{n=0}^{\infty} \sigma^n(A_N - \sigma(A_N)) \\ &\leq (\mathrm{SOT}) \sum_{n=0}^{\infty} \sigma^n(\Delta^2(\mathcal{M})) \\ &= P_{\mathcal{M}} \end{aligned}$$

since σ is positive. Hence

$$A := (\mathrm{SOT}) \sum_{j=1}^{\infty} M_{f_j} M_{f_j}^* = (\mathrm{SOT}) \lim_N A_N \leq P_{\mathcal{M}},$$

and then we have

$$\begin{aligned} A-\sigma(A) &= (\mathrm{SOT})\sum_j M_{f_j}M_{f_j}^* - \sum_{i=1}^d M_{z_j}\Big((\mathrm{SOT})\sum_j M_{f_j}M_{f_j}^*\Big)M_{z_j}^* \\ &= (\mathrm{SOT})\sum_j M_{f_j}\Big(I-\sum_{i=1}^d M_{z_j}M_{z_j}^*\Big)M_{f_j}^* \\ &= (\mathrm{SOT})\sum_j M_{f_j}(1\otimes 1)M_{f_j}^* \\ &= (\mathrm{SOT})\sum_j f_j\otimes f_j \\ &= \Delta^2(\mathcal{M}). \end{aligned}$$

Therefore

$$P_{\mathcal{M}} = (\mathrm{SOT})\sum_{n=0}^\infty \sigma^n(\Delta^2(\mathcal{M})) = (\mathrm{SOT})\sum_{n=0}^\infty \sigma^n(A-\sigma(A)) = A,$$

which completes the proof.

Next we prove that, of the Bergman module $L_a^2(B_d)$, only the trivial submodules can be represented by the form (1.1).

Lemma 2.3 *Let $\mathcal{M}\subset L_a^2(B_d)$ be a submodule, and there is a sequence of analytic multipliers $\{\varphi_k\}$ such that*

$$P_{\mathcal{M}} = (\mathrm{SOT})\sum_k T_{\varphi_k}T_{\varphi_k}^*,$$

so then $\mathcal{M}=\{0\}$ or $\mathcal{M}=L_a^2(B_d)$.

Proof Let $k_z=\frac{K_z}{\|K_z\|}, z\in B_d$ be the normalized reproducing kernel, and then it holds for each $z\in B_d$ that

$$\sum_k |\varphi_k(z)|^2 = \sum_k \langle T_{\varphi_k}^* k_z, T_{\varphi_k}^* k_z\rangle = \langle P_{\mathcal{M}}k_z, k_z\rangle \le 1.$$

Take a nonzero $f\in\mathcal{M}$, then we have

$$\begin{aligned} &\int_{B_d}\Big(\sum_k |\varphi_k(z)|^2\Big)|f(z)|^2\mathrm{d}\nu(z) \\ &= \sum_k \int_{B_d}|\varphi_k(z)f(z)|^2\mathrm{d}\nu(z) \\ &= \sum_k \|T_{\varphi_k}f\|^2 \\ &\ge \sum_k \|T_{\varphi_k}^* f\|^2 \\ &= \langle P_{\mathcal{M}}f, f\rangle \\ &= \int |f(z)|^2\mathrm{d}\nu(z). \end{aligned}$$

Hence $\sum\limits_k |\varphi_k(z)|^2\equiv 1$ for $z\in B_d$. Since each φ_k is analytic, $|\varphi_k|^2$ must be subharmonic, and so is their summation. Since $\sum\limits_k |\varphi_k|^2$ achieves its maximum at the origin, so must be every $|\varphi_k|^2$.

By the maximum principle, $|\varphi_k|^2$ should be constant in B_d. Take the second-order partial derivative of $|\varphi_k(z)|^2$ with respect to ∂z_i and $\partial \overline{z}_i$ and we get $\frac{\partial \varphi_k(z)}{\partial z_i}\frac{\partial \overline{\varphi}_k(z)}{\partial \overline{z}_i} \equiv 0$. Therefore for each k and i, we have $\frac{\partial \varphi_k(z)}{\partial z_i} \equiv 0$, which induces that each φ_k is a constant. This implies $P_{\mathcal{M}} = 0$ or 1, which completes the proof.

3 *p*-Essential Normality and Approximate Representability

We use H_d^2 to denote the d-dimensional Drury-Arveson module generated by the polynomial ring $\mathbb{C}[z_1, \cdots, z_d]$, and H_{d+1}^2 to denote the $(d+1)$-dimensional Drury-Arveson module generated by $\mathbb{C}[z_0, z_1, \cdots, z_d]$. From the idea of Fang and Xia [13], for integers $n \geq 0$, write H_n for the closed subspace of H_{d+1}^2 spanned by $z_0^n\mathbb{C}[z_1, \cdots, z_d]$, and then we have $H_{d+1}^2 = \bigoplus\limits_{n\geq 0} H_n$. Obviously, H_d^2 is isometrically isomorphic to H_0, the mapping $f \mapsto z_0^{d-1}f$ maps the Hardy module $H^2(\partial B_d)$ isometrically isomorphic to H_{d-1}, and the mapping $f \mapsto z_0^d f$ maps the Bergman module $L_a^2(B_d)$ isometrically isomorphic to H_d, etc.

Take a homogeneous ideal $I \subset \mathbb{C}[z_1, \cdots, z_d]$, and let $M_n \subset H_n$ be the closed subspace of H_{d+1}^2 spanned by $z_0^n I$ $(n \geq 0)$. Then $M = \bigoplus\limits_{n\geq 0} M_n$ is the submodule of H_{d+1}^2 spanned by $I \cup z_0 I \cup z_0^2 I \cup \cdots$. It can be seen that $M^\perp = \bigoplus\limits_n (H_n \ominus M_n)$.

Lemma 3.1 *M is a reduced subspace for M_{z_0}.*

Proof Obviously M is invariant for M_{z_0}, so we only need to prove that $M^\perp$ is also invariant for M_{z_0}. To see this, let $f \in \mathbb{C}[z_1, \cdots, z_d]$ be homogeneous, such that $z_0^n f \in H_n \ominus M_n$. For every $g \in I$, it holds that

$$\begin{aligned}\langle z_0^{n+1} f, z_0^{n+1} g\rangle &= \langle M_{z_0}^* M_{z_0}(z_0^n f), z_0^n g\rangle \\ &= \frac{n+1}{n+1+\deg(f)}\langle z_0^n f, z_0^n g\rangle \\ &= 0.\end{aligned}$$

Hence $z_0^{n+1} f \in H_{n+1} \ominus M_{n+1}$. Since $H_n \ominus M_n$ is homogeneous with respect to $z_1, \cdots, z_d$, we have $M_{z_0}(H_n \ominus M_n) \subset H_{n+1} \ominus M_{n+1}$. By $M^\perp = \bigoplus\limits_n (H_n \ominus M_n)$, we conclude that $M^\perp$ is invariant for M_{z_0}.

Proposition 3.1 *Let $I \subset \mathbb{C}[z_1, \cdots, z_d]$ and $M \subset H_{d+1}^2$ be as aforementioned. Then there is a sequence $\{f_j\} \subset I$ of homogeneous polynomials, such that*

$$P_M = (\mathrm{SOT})\sum_{n,j} \frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!} M_{z_0^n f_j} M_{z_0^n f_j}^*.$$

Proof Let $\Delta^2(M) = P_M - \sum\limits_{i=1}^d M_{z_i} P_M M_{z_i}^* - M_{z_0} P_M M_{z_0}^*$ be the square of the defect operator of M. Then each H_n reduces $\Delta^2(M)$, and $\Delta^2(M)|_{H_0}$ is actually the square of the defect operator of M_0 in the Drury-Arveson module H_0. Assume that the homogeneous polynomial $g \in I$ is an eigenvector for $\Delta^2(M)$ corresponding to eigenvalue λ, and we claim that $z_0^n g$ is also an eigenvector for $\Delta^2(M)$ corresponding to eigenvalue $\frac{\deg g}{n+\deg g}\lambda$.

Denoting $m=\deg(g)$ and supposing $g=\sum\limits_{|\alpha|=m} c_\alpha z^\alpha$, then we have

$$\sum_{i=1}^{d} M_{z_i} P_M M_{z_i}^* g = P_M g - \Delta^2(M) g = (1-\lambda) g,$$

and for $1 \le i \le d$,

$$\begin{aligned} M_{z_i}^*(z_0^n g) &= \sum_\alpha c_\alpha \frac{\alpha_i}{m+n} z_0^n z^{\alpha-e_i} \\ &= \frac{m}{m+n} z_0^n \sum_\alpha c_\alpha \frac{\alpha_i}{m} z^{\alpha-e_i} \\ &= \frac{m}{m+n} z_0^n M_{z_i}^* g. \end{aligned}$$

Therefore

$$\begin{aligned} \Delta^2(M)(z_0^n g) &= P_M(z_0^n g) - \sum_{i=1}^{d} M_{z_i} P_M M_{z_i}^*(z_0^n g) - M_{z_0} P_M M_{z_0}^*(z_0^n g) \\ &= z_0^n g - \sum_{i=1}^{d} M_{z_i} P_M \Big[\frac{m}{m+n} z_0^n M_{z_i}^* g\Big] - \frac{n}{m+n} z_0^n g \\ &= \frac{m}{m+n} z_0^n g - \frac{m}{m+n} \sum_{i=1}^{d} M_{z_i} P_M M_{z_0^n} M_{z_i}^* g \\ &= \frac{m}{m+n} z_0^n g - \frac{m}{m+n} M_{z_0^n} \sum_{i=1}^{d} M_{z_i} P_M M_{z_i}^* g \\ &= \frac{m}{m+n} z_0^n g - \frac{m}{m+n} z_0^n (1-\lambda) g \\ &= \frac{m}{m+n} \lambda z_0^n g \quad \text{(by lemma 3.1)}, \end{aligned}$$

and the claim is proved.

Suppose $\Delta^2(M)|_{H_0} = \sum f_j \otimes f_j$, where the homogeneous polynomials $\{f_j\}$ form a sequence of pairwise orthogonal eigenvectors for $\Delta^2(M)$, corresponding to the eigenvalues $\lambda_j := \|f_j\|^2$. Therefore by the claim we have

$$\begin{aligned} \Delta^2(M) &= (\mathrm{SOT}) \sum_{j,n} \frac{\deg(f_j)}{n+\deg(f_j)} \frac{(n+\deg(f_j))!}{n!\deg(f_j)!} z_0^n f_j \otimes z_0^n f_j \\ &= (\mathrm{SOT}) \sum_{j,n} \frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!} z_0^n f_j \otimes z_0^n f_j, \end{aligned}$$

and the proof of the proposition can be completed by Lemma 2.2.

As an application, we have the following result.

Corollary 3.1 *Let $I \subset \mathbb{C}[z_1, \cdots, z_d]$ be a homogeneous ideal such that the submodule $\mathcal{M}_0 \subset H_d^2$ generated by I satisfies*

$$\sum_{i=1}^{d} [(M_{z_i} P_{\mathcal{M}_0})^*, M_{z_i} P_{\mathcal{M}_0}] \le \frac{c}{N+1} P_{\mathcal{M}_0}$$

for some number $c > 0$. Then the submodule $\mathcal{M} \subset H^2_{d+1}$ generated by $I \cup z_0 I \cup \cdots$ satisfies

$$\sum_{i=1}^{d+1}[(M_{z_i}P_{\mathcal{M}})^*, M_{z_i}P_{\mathcal{M}}] \leq \frac{c+1}{N+1}P_{\mathcal{M}},$$

and therefore is p-essentially normal for $p > d+1$.

Proof By the hypothesis we have

$$\begin{aligned}
&\sum_{i=1}^{d}[(M_{z_i}P_{\mathcal{M}_0})^*, M_{z_i}P_{\mathcal{M}_0}] \\
=& \sum_{i=1}^{d}(P_{\mathcal{M}_0}M_{z_i}^*M_{z_i}P_{\mathcal{M}_0} - M_{z_i}P_{\mathcal{M}_0}M_{z_i}^*) \\
=& \frac{N+d}{N+1}P_{\mathcal{M}_0} - \sum_{i=1}^{d}M_{z_i}P_{\mathcal{M}_0}M_{z_i}^* \\
=& \frac{d-1}{N+1}P_{\mathcal{M}_0} + \Delta^2(\mathcal{M}_0) \\
\leq& \frac{c}{N+1}P_{\mathcal{M}_0}.
\end{aligned} \tag{3.1}$$

Therefore $c \geq d-1$ and $\Delta^2(\mathcal{M}_0) \leq \frac{c-d+1}{N+1}P_{\mathcal{M}_0}$. Let homogeneous polynomials $\{f_j\}$ form a sequence of pairwise orthogonal eigenvectors of $\Delta^2(\mathcal{M}_0)$, such that $\Delta^2(\mathcal{M}_0) = \sum_j f_j \otimes f_j$. Then we have $\|f_j\|^2 \leq \frac{c-d+1}{\deg(f_j)+1}$. Hence

$$\begin{aligned}
&\left\|\frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!}z_0^n f_j \otimes z_0^n f_j\right\| \\
=& \frac{\deg(f_j)}{n+\deg(f_j)}\|f_j\|^2 \\
\leq& \frac{\deg(f_j)}{n+\deg(f_j)} \cdot \frac{c-d+1}{\deg(f_j)+1} \\
\leq& \frac{c-d+1}{n+\deg(f_j)+1},
\end{aligned}$$

which gives

$$\Delta^2(\mathcal{M}) \leq \frac{c-d+1}{N+1}P_{\mathcal{M}} \tag{3.2}$$

by the proof of Proposition 3.1.

Similar to the computation in (3.1) and by (3.2), we have

$$\sum_{i=1}^{d+1}[(M_{z_i}P_{\mathcal{M}})^*, M_{z_i}P_{\mathcal{M}}] = \frac{d}{N+1}P_{\mathcal{M}} + \Delta^2(\mathcal{M}) \leq \frac{c+1}{N+1}P_{\mathcal{M}}.$$

By [17], homogeneous ideals of $\mathbb{C}[z_1, z_2, z_3]$ satisfy the hypothesis of this corollary, and therefore we have the following corollary by induction.

Corollary 3.2 *If a homogeneous submodule $\mathcal{M} \subset H_d^2$ is generated by polynomials that depend on at most* 3 *variables, then $\mathcal{M}$ is p-essentially normal for $p > d$.*

Next we prove that Condition 1.1 is sufficient for p $(> d)$-approximate representability of homogeneous submodules $\mathcal{M} \subset L_a^2(B_d)$. We need the following lemma, which is believed to be well known.

Lemma 3.2 *Let I be a homogeneous ideal of $\mathbb{C}[z_1, \cdots, z_d]$. Denote by $\mathcal{M}', \mathcal{M}$ the submodules of $H_d^2, L_a^2(B_d)$ generated by I, respectively. Then $\mathcal{M}'$ satisfies Condition* 1.1 *if and only if $\mathcal{M}$ does.*

A generalized version of this lemma can be found in the Ph. D. thesis of K. Wang. We put a shorter proof in our special case here for completion of the reasoning.

Proof of Lemma 3.2 As before, let M denote the submodule of H_{d+1}^2 generated by $I, z_0 I, \cdots$. Suppose

$$P_{M_0} - \sum_{i=1}^{d} M_{z_i} P_{M_0} M_{z_i}^* = \sum_j \lambda_j e_j \otimes e_j,$$

where the homogeneous polynomials $\{e_j \in I\}$ form an orthonormal basis of H_0, and $\{\lambda_j\}$ are the corresponding eigenvalues. For each j, We have

$$\sum_{i=1}^{d} M_{z_i} P_{M_0} M_{z_i}^*(e_j) = (1 - \lambda_j) e_j,$$

and thus

$$\begin{aligned}\sum_{i=1}^{d} M_{z_i} P_{M_d} M_{z_i}^*(z_0^d e_j) &= \frac{\deg(e_j)}{\deg(e_j) + d} \sum_{i=1}^{d} M_{z_i} P_{M_d} M_{z_0^d} M_{z_i}^* e_j \\ &= \frac{\deg(e_j)}{\deg(e_j) + d}(1 - \lambda_j) z_0^d e_j,\end{aligned}$$

which implies

$$\begin{aligned}\Big(P_{M_d} - \sum_{i=1}^{d} M_{z_i} P_{M_d} M_{z_i}^*\Big) z_0^d e_j &= z_0^d e_j - \frac{\deg(e_j)}{\deg(e_j) + d}(1 - \lambda_j) z_0^d e_j \\ &= \Big(\lambda_j + \frac{d}{\deg(e_j) + d}(1 - \lambda_j)\Big) z_0^d e_j.\end{aligned}$$

Therefore $z_0^d e_j$ is an eigenvector of $P_{M_d} - \sum\limits_{i=1}^{d} M_{z_i} P_{M_d} M_{z_i}^*$, corresponding to the eigenvalue $\lambda_j + \frac{d}{\deg(e_j)+d}(1 - \lambda_j)$. Obviously,

$$\begin{aligned}\lambda_j &\le \lambda_j + \frac{d}{\deg(e_j) + d}(1 - \lambda_j) \\ &\le \lambda_j + \frac{d}{\deg(e_j) + d} \\ &\le \lambda_j + \frac{d}{\deg(e_j) + 1}.\end{aligned}$$

From this we find that $\lambda_j \leq \frac{c_1}{\deg(e_j)+1}, \forall j$ for some $c_1 > 0$ if and only if $\lambda_j + \frac{d}{\deg(e_j)+d}(1-\lambda_j) \leq \frac{c_2}{\deg(e_j)+1}, \forall j$ for some $c_2 > 0$. By the isomorphic isomorphisms $H_d^2 \to H_0$ and $L_a^2(B_d) \to H_d$, the lemma is proved.

Proposition 3.2 *Let $I \subset \mathbb{C}[z_1, \cdots, z_d]$ be a nontrivial homogeneous ideal, and $\mathcal{M} \subset L_a^2(B_d)$ be the submodule generated by I. Assume that Condition* 1.1 *holds for $\mathcal{M}$, and then it can be p-approximately represented by homogeneous multipliers for each $p > d$.*

Proof As before, let $M \subset H_{d+1}^2$ be the submodule generated by $I \cup z_0 I \cup \cdots$. Let homogeneous polynomials $\{f_j\} \subset I$ be a sequence of pairwise orthogonal eigenvectors of $\Delta^2(M)$, such that $\Delta^2(M)|_{H_0} = \sum f_j \otimes f_j$, and then by assumption there is a number $c > 0$ making $\|f_j\|^2 \leq \frac{c}{\deg(f_j)+1}$. By Proposition 3.1, we have

$$P_M = \sum_{n,j} \frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!} M_{z_0^n f_j} M_{z_0^n f_j}^*.$$

On the other hand, for $n \in \mathbb{N}$, denote $\mathcal{M}_n := \bigoplus_{m \geq n} M_m$, and then we can compute

$$P_{\mathcal{M}_{n+1}} \Delta^2(\mathcal{M}_n) P_{\mathcal{M}_{n+1}} = P_{\mathcal{M}_{n+1}} \Delta^2(M) P_{\mathcal{M}_{n+1}}$$

and

$$\begin{aligned} P_{M_n} \Delta^2(\mathcal{M}_n) P_{M_n} &= P_{M_n} - \sum_{i=1}^{d} P_{M_n} M_{z_i} P_{M_n} M_{z_i}^* P_{M_n} \\ &= P_{M_n} \Delta^2(M) P_{M_n} + P_{M_n} M_{z_0} P_{M_{n-1}} M_{z_0}^* P_{M_n} \\ &= P_{M_n} \Delta^2(M) P_{M_n} + \frac{n}{N} P_{M_n}. \end{aligned}$$

Therefore we have

$$P_{\mathcal{M}_n} \Delta^2(\mathcal{M}_n) P_{\mathcal{M}_n} = P_{\mathcal{M}_n} \Delta^2(M) P_{\mathcal{M}_n} + \frac{n}{N} P_{M_n},$$

and consequently if we let $\{e_k\} \subset \mathbb{C}[z_1, \cdots, z_d]$ be any sequence of homogeneous polynomials that form an orthonormal basis of M_0, then

$$\begin{aligned} \Delta^2(\mathcal{M}_n) = &\sum_k \frac{(n+\deg(e_k)-1)!}{(n-1)!\deg(e_k)!} z_0^n e_k \otimes z_0^n e_k \\ &+ \sum_{m \geq n} \sum_j \frac{(m+\deg(f_j)-1)!}{m!(\deg(f_j)-1)!} z_0^m f_j \otimes z_0^m f_j. \end{aligned}$$

Therefore by Lemma 2.2,

$$P_{\mathcal{M}_n} = \sum_k \frac{(n+\deg(e_k)-1)!}{(n-1)!\deg(e_k)!} M_{z_0^n e_k} M_{z_0^n e_k}^* + \sum_{m \geq n} \sum_j \frac{(m+\deg(f_j)-1)!}{m!(\deg(f_j)-1)!} M_{z_0^m f_j} M_{z_0^m f_j}^*.$$

For integers $m \geq 0$, define

$$T_m := \sum_j \frac{(m+\deg(f_j)-1)!}{m!(\deg(f_j)-1)!} M_{z_0^m f_j} M_{z_0^m f_j}^*,$$

and then we have

$$\sum_{m=0}^{n-1} T_m|_{\mathcal{M}_n} = P_{\mathcal{M}_n} - \sum_{m\geq n} T_m|_{\mathcal{M}_n}$$
$$= \sum_k \frac{(n+\deg(e_k)-1)!}{(n-1)!\deg(e_k)!} M_{z_0^n e_k} M^*_{z_0^n e_k}.$$

By the hypothesis,

$$\begin{aligned}&\sum_j \frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!} z_0^n f_j \otimes z_0^n f_j \\ =& \sum_j \frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!} \|f_j\|^2 \Big(z_0^n \frac{f_j}{\|f_j\|} \otimes z_0^n \frac{f_j}{\|f_j\|}\Big) \\ \leq& \sum_j \frac{(n+\deg(f_j)-1)!}{n!(\deg(f_j)-1)!} \frac{c}{\deg(f_j)+1} \Big(z_0^n \frac{f_j}{\|f_j\|} \otimes z_0^n \frac{f_j}{\|f_j\|}\Big) \\ \leq& c\sum_j \frac{(n+\deg(f_j)-1)!}{n!\deg(f_j)!} \Big(z_0^n \frac{f_j}{\|f_j\|} \otimes z_0^n \frac{f_j}{\|f_j\|}\Big) \\ \leq& \frac{c}{n}\sum_k \frac{(n+\deg(e_k)-1)!}{(n-1)!\deg(e_k)!} z_0^n e_k \otimes z_0^n e_k,\end{aligned}$$

which implies

$$\begin{aligned} T_n &\leq \frac{c}{n}\sum_k \frac{(n+\deg(e_k)-1)!}{(n-1)!\deg(e_k)!} M_{z_0^n e_k} M^*_{z_0^n e_k} \\ &\leq \frac{c}{n}\sum_{m=0}^{n-1} T_m. \end{aligned}$$

Thus by induction we can find a number $C > 0$ such that $\sum\limits_{m=0}^{d} T_m \leq CT_0$, and hence $T_0|_{H_d} \geq C^{-1}P_{M_d}$. This proves the proposition by the isometric isomorphism between $L_a^2(B_d)$ and H_d.

Remark 3.1 Up to now, all known examples of homogeneous submodules $\mathcal{M} \subset L_a^2(B_d)$ on which Arveson's conjecture hold satisfy Condition 1.1, including submodules generated by monomials (see [2, 12]), principal homogeneous submodules, and homogeneous submodules of $L_a^2(B_3)$ (see [17]). By Proposition 3.2, these submodules are p $(> d)$-approximately representable. In this sense, we can see p $(> d)$-approximate representability as a nearly necessary condition of Arveson's conjecture. It is reasonable to conjecture that, every homogeneous submodule of $L_a^2(B_d)$ or $H^2(\partial B_d)$ should be p $(> d)$-approximately representable.

In fact, p $(> d)$-approximate representability is also sufficient for Arveson's conjecture, and the remaining part of this section is devoted to proving this. The proof is based on a result of Zhao and Yu.

Proposition 3.3 (see [25]) *Let* $T \overset{\text{SOT}}{=} \sum\limits_{k=1}^{\infty} T_{\varphi_k} T^*_{\varphi_k}$ *be a bounded operator on the Bergman module or the Hardy module over* B_d*, where*

$$\{\varphi_k \in H^\infty(B_d) : k = 1, 2, \cdots\}$$

is a sequence of multipliers. Then the commutator $[T, T_{z_i}]$ *belongs to the Schatten-von Neumann class* $\mathcal{L}^{2p}$ *for* $p > d$*, and there is a constant* C *depending only on* p *and* d *such that*

$$\|[T, T_{z_i}]\|_{2p} \leq C\|T\|.$$

To derive p $(> d)$-essential normality of submodules from this proposition, we need the following lemma.

Lemma 3.3 *If* T *is a normal operator on the Hilbert space* H *with closed range, and* $C \in B(H)$*, then* $[T, C] \in \mathcal{L}^p$ *implies* $[P_{\operatorname{ran} T}, C] \in \mathcal{L}^p$*, where* $P_{\operatorname{ran} T}$ *is the projection onto* $\operatorname{ran} T$.

Proof Set $K = \operatorname{ran} T$, and $K^\perp = \ker T^* = \ker T$. Since T is normal, we have $K = \operatorname{ran} T = (\ker T)^\perp$. The operator $T' = T|_K \colon K \to K$ is invertible by the inverse mapping theorem. With respect to the decomposition $H = K \oplus K^\perp$, T and C can be written as

$$T = \begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

Since

$$[T, C] = \begin{bmatrix} T'C_1 - C_1T' & T'C_2 \\ -C_3T' & 0 \end{bmatrix} \in \mathcal{L}^p,$$

both $T'C_2$ and C_3T' are in $\mathcal{L}^p$. Since T' is invertible, the operators C_2 and C_3 are also in $\mathcal{L}^p$. Then the desired result follows from the equality

$$[P_{\operatorname{ran} T}, C] = \begin{bmatrix} 0 & C_2 \\ -C_3 & 0 \end{bmatrix}.$$

As a consequence of Proposition 3.3 and Lemma 3.3, we have the following result.

Proposition 3.4 *Let* $\mathcal{M} \subset L_a^2(B_d)$ *be a homogeneous submodule that can be* p $(> d)$*-approximately represented by homogeneous multipliers, and then* $\mathcal{M}$ *is* p*-essentially normal.*

Combining Propositions 3.2 and 3.4, we immediately get the following proposition.

Proposition 3.5 *Let* $I \in \mathbb{C}[z_1, \cdots, z_d]$ *be a homogeneous ideal,* $\mathcal{M}$ *be the submodule of* $L_a^2(B_d)$ *generated by* I*, which can be* p $(> d)$*-approximately represented by homogeneous multipliers, then the submodule* $\mathcal{M}' \subset H_d^2$ *generated by* I *is* p*-essentially normal.*

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