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On the Number of Limit Cycles in Small Perturbations of a Piecewise Linear Hamiltonian System with a Heteroclinic Loop*

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Abstract In this paper, the authors consider limit cycle bifurcations for a kind of non-smooth polynomial differential systems by perturbing a piecewise linear Hamiltonian system with a center at the origin and a heteroclinic loop around the origin. When the degree of perturbing polynomial terms is n $(n \geq 1)$, it is obtained that n limit cycles can appear near the origin and the heteroclinic loop respectively by using the first Melnikov function of piecewise near-Hamiltonian systems, and that there are at most $n + [\frac{n+1}{2}]$ limit cycles bifurcating from the periodic annulus between the center and the heteroclinic loop up to the first order in ε. Especially, for $n = 1, 2, 3$ and 4, a precise result on the maximal number of zeros of the first Melnikov function is derived.

Keywords Limit cycle, Heteroclinic loop, Melnikov function, Chebyshev system, Bifurcation, Piecewise smooth system
2000 MR Subject Classification 34C05, 34C07, 37G15

1 Introduction and Main Results

A large number of problems in mechanics, electrical engineering and the theory of automatic control are described by non-smooth systems (see [1–3] and the references therein). The basic methods of the qualitative theory were established or developed by Filippov in [4]. Due to the variety of the forms of non-smoothness, piecewise smooth systems exhibit not only all kinds of bifurcations that occur in smooth systems, but also complicated nonstandard bifurcation phenomenon that are unique to piecewise smooth ones including grazing (see [5–6]), sliding effects (see [7]), border collision (see [8]), etc. For limit cycle bifurcations of piecewise smooth planar systems with two regions, Han and Zhang [9] proved that linear systems can have two limit cycles near a focus of either the focus-focus, focus-parabolic or the parabolic-parabolic type (see [10] for the definition). In [11–12], the authors constructed two different classes of piecewise smooth quadratic planar systems with a focus of the focus-focus type, and showed that 5 and 9 limit cycles can respectively appear in Hopf bifurcation. In [13], Liu and Han

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considered a piecewise polynomial system of the form

$$(\dot{x}, \dot{y}) = \begin{cases} (b^+ y + \varepsilon p_n^+(x, y, \delta), -b^+ x + \varepsilon q_n^+(x, y, \delta)), & x \geq 0, \\ (b^- y + \varepsilon p_n^-(x, y, \delta), -b^- x + \varepsilon q_n^-(x, y, \delta)), & x < 0, \end{cases}$$

where $b^{\pm} > 0$, and $p_n^{\pm}$ and $q_n^{\pm}$ are arbitrary polynomials of degree n ($n \geq 1$). It was proved that the maximal number of limit cycles of the above system is n up to the first order in ε. Recently, in [14] we studied the following system:

$$\begin{cases} \dot{x} = -y + \varepsilon p_n^+(x, y), \\ \dot{y} = 1 - x + \varepsilon q_n^+(x, y), \end{cases} \quad x \geq 0, \qquad \begin{cases} \dot{x} = -y + \varepsilon p_n^-(x, y), \\ \dot{y} = x + \varepsilon q_n^-(x, y), \end{cases} \quad x < 0, \tag{1.1}$$

where $p_n^{\pm}$ and $q_n^{\pm}$ are also arbitrary polynomials of degree n ($n \geq 1$). For the unperturbed system of the above, there exists a family of the periodic orbits between the origin which is an elementary center of parabolic-focus type and a compound homoclinic loop around the origin having a saddle at $(1, 0)$. We obtained that there exist systems of the form (1.1) having at least $n + [\frac{n+1}{2}]$ limit cycles for ε small, and gained an upper bound $2n + [\frac{n+1}{2}]$ of the number of limit cycles for this system up to the first order in ε.

However, so far there have been few papers studying heteroclinic bifurcations inside the class of piecewise polynomial differential systems in the literature obtained. In this paper, we study this problem by using the first Melnikov function of piecewise near-Hamiltonian systems.

First, we recall the first-order Melnikov function for piecewise smooth near-Hamiltonian systems deduced in [13]. Consider a general form of a piecewise near-Hamiltonian system on the plane

$$\begin{cases} \dot{x} = H_y + \varepsilon p(x, y, \delta), \\ \dot{y} = -H_x + \varepsilon q(x, y, \delta), \end{cases} \tag{1.2}$$

where

$$H(x, y) = \begin{cases} H^+(x, y), & x \geq 0, \\ H^-(x, y), & x < 0, \end{cases}$$

$$p(x, y, \delta) = \begin{cases} p^+(x, y, \delta), & x \geq 0, \\ p^-(x, y, \delta), & x < 0, \end{cases}$$

$$q(x, y, \delta) = \begin{cases} q^+(x, y, \delta), & x \geq 0, \\ q^-(x, y, \delta), & x < 0, \end{cases}$$

$H^{\pm}$, $p^{\pm}$ and $q^{\pm}$ are C^{∞}, $\varepsilon > 0$ is small, and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact. We know that this system has two subsystems which are called the right subsystem and the left subsystem respectively, i.e.,

$$\begin{cases} \dot{x} = H_y^+ + \varepsilon p^+(x, y, \delta), \\ \dot{y} = -H_x^+ + \varepsilon q^+(x, y, \delta) \end{cases} \tag{1.2a}$$

and

$$\begin{cases} \dot{x} = H_y^- + \varepsilon p^-(x, y, \delta), \\ \dot{y} = -H_x^- + \varepsilon q^-(x, y, \delta). \end{cases} \tag{1.2b}$$

Suppose that $(1.2)|_{\varepsilon=0}$ has a family of periodic orbits around the origin and satisfies the following two assumptions.

Assumption (I): There exists an interval $J = (\alpha, \beta)$ and two points $A(h) = (0, a(h))$ and $A_1(h) = (0, a_1(h))$ such that for $h \in J$,

$$H^+(A(h)) = H^+(A_1(h)) = h, \quad H^-(A(h)) = H^-(A_1(h)) = \widetilde{h}, \quad a(h) < a_1(h).$$

Assumption (II): The subsystem $(1.2a)|_{\varepsilon=0}$ has an orbital arc L_h^+ starting from $A(h)$ and ending at $A_1(h)$ defined by $H^+(x,y) = h,\ x \geq 0$; the subsystem $(1.2b)|_{\varepsilon=0}$ has an orbital arc L_h^- starting from $A_1(h)$ and ending at $A(h)$ defined by $H^-(x,y) = H^-(A_1(h)),\ x < 0$.

Under assumptions (I) and (II), $(1.2)|_{\varepsilon=0}$ has a family of piecewise smooth periodic orbits L_h orientated anticlockwise with $L_h = L_h^+ \cup L_h^-,\ h \in J$ (see Figure 1).

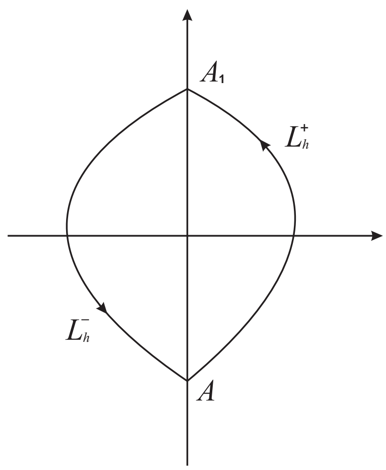

Figure 1 The closed orbits of $(1.2)|_{\varepsilon=0}$.

Lemma 1.1 (see [13]) *Under the assumptions* (I) *and* (II), *for the first order Melnikov function of system* (1.2), *we have*

$$M(h,\delta) = \frac{H_y^+(A)}{H_y^-(A)}\Big[\frac{H_y^-(A_1)}{H_y^+(A_1)}\int_{L_h^+} q^+\mathrm{d}x - p^+\mathrm{d}y + \int_{L_h^-} q^-\mathrm{d}x - p^-\mathrm{d}y\Big], \quad h \in J. \tag{1.3}$$

Further, if $M(h_0) = 0$ and $M'(h_0) \neq 0$ for some $h_0 \in J$, then for $\varepsilon > 0$ small, (1.2) *has a unique limit cycle near L_{h_0}. If h_0 is a zero of $M(h)$ having an odd multiplicity, then for $\varepsilon > 0$ small,* (1.2) *has at least one limit cycle near L_{h_0}. Also, if $M(h)$ has at most k zeros in h on the interval J, then* (1.2) *has at most k limit cycles bifurcating from the open annulus $\bigcup\limits_{\alpha<h<\beta} L_h$.*

In this paper, we take

$$H^+(x,y) = \frac{1}{2}((x-1)^2 - y^2), \quad x \geq 0, \tag{1.4}$$

$$H^-(x,y) = \frac{1}{2}((x+1)^2 - y^2), \quad x < 0, \tag{1.5}$$

and suppose

$$p(x,y)=\begin{cases}p^+(x,y)=\sum\limits_{i+j=0}^{n}a_{ij}^+x^iy^j, & x\geq 0,\\ p^-(x,y)=\sum\limits_{i+j=0}^{n}a_{ij}^-x^iy^j, & x<0,\end{cases}$$
$$q(x,y)=\begin{cases}q^+(x,y)=\sum\limits_{i+j=0}^{n}b_{ij}^+x^iy^j, & x\geq 0,\\ q^-(x,y)=\sum\limits_{i+j=0}^{n}b_{ij}^-x^iy^j, & x<0.\end{cases} \tag{1.6}$$

When (1.4) and (1.5) hold, the system (1.2) has the form

$$\begin{cases}\dot{x}=-y+\varepsilon p^+(x,y),\\ \dot{y}=1-x+\varepsilon q^+(x,y),\end{cases} x\geq 0, \qquad \begin{cases}\dot{x}=-y+\varepsilon p^-(x,y),\\ \dot{y}=-(1+x)+\varepsilon q^-(x,y),\end{cases} x<0. \tag{1.7}$$

For $\varepsilon=0$, this system has a family of piecewise periodic orbits given by

$$L_h=\Big\{(x,y)\mid H^+(x,y)=\frac{h}{2},\ x\geq 0\Big\}\bigcup\Big\{(x,y)\mid H^-(x,y)=\frac{h}{2},\ x<0\Big\},\quad 0<h<1.$$

For the sake of convenience, here we use $\frac{h}{2}$ instead of h. The limit L_0 of L_h as $h\to 0$ is a compound heteroclinic loop with two saddles $S_1(-1,0)$ and $S_2(1,0)$. And if $h\to 1$, L_h approaches the origin which is an elementary center of parabolic-parabolic type (see [9–10] for the definition). See Figure 2.

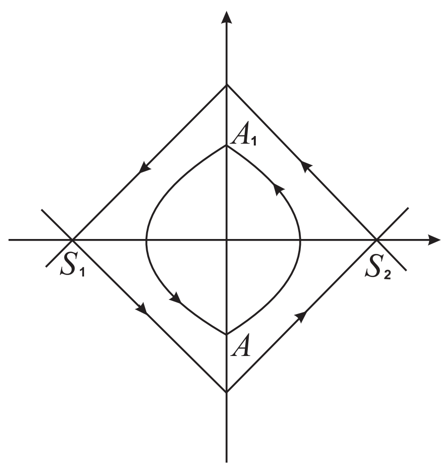

Figure 2 Phase portrait of system $(1.7)|_{\varepsilon=0}$.

Noticing $H_y^+(0,y)\equiv H_y^-(0,y)$ for $-1<y<1$, we have by Lemma 1.1 that the first-order Melnikov function of system (1.7) satisfies

$$M\Big(\frac{h}{2}\Big)=\int_{\widehat{AA_1}}q^+\mathrm{d}x-p^+\mathrm{d}y+\int_{\widehat{A_1A}}q^-\mathrm{d}x-p^-\mathrm{d}y\equiv\overline{M}(h), \tag{1.8}$$

where $0<h<1$, and

$$A=(0,-\sqrt{1-h}),\quad A_1=(0,\sqrt{1-h}),$$

$$\widehat{AA_1} = \Big\{(x,y) \mid H^+(x,y) = \frac{h}{2},\ x \geq 0\Big\},$$
$$\widehat{A_1A} = \Big\{(x,y) \mid H^-(x,y) = \frac{h}{2},\ x < 0\Big\}.$$

Let $Z(n)$ denote the maximal number of zeros of the non-zero function $\overline{M}(h)$ on the open interval $(0,1)$ for all possible p and q satisfying (1.6), which is the maximal number of limit cycles of (1.7) bifurcating from the periodic annulus $\bigcup\limits_{0<h<1} L_h$ for all possible p and q satisfying (1.6). Let $N_{\mathrm{Hopf}}(n)$ and $N_{\mathrm{heteroc}}(n)$ denote respectively the maximal number of limit cycles bifurcated in Hopf bifurcation near the origin and in heteroclinic bifurcation near L_0 for all possible p and q satisfying (1.6). Then our main results can be stated as follows.

Theorem 1.1 *For any $n \geq 1$ we have*
(1) $N_{\mathrm{Hopf}}(n) \geq n$.
(2) $N_{\mathrm{heteroc}}(n) \geq n$.

Theorem 1.2 *For $n = 1, 2, 3, 4$, we have $Z(n) = n$.*

Theorem 1.3 *For any $n \geq 5$ we have $n \leq Z(n) \leq n + [\frac{n+1}{2}]$.*

We conjecture that $Z(n) = n$ for all $n \geq 5$.

2 Preliminary Lemmas

In this section, we give an expression of the first-order Melnikov function $\overline{M}(h)$ in (1.8) for $0 < h < 1$, and provide two expansions of $\overline{M}(h)$ near the origin and the heteroclinic loop L_0 respectively. By (1.8) we set for $0 < h < 1$,

$$\begin{aligned} M^+\Big(\frac{h}{2}\Big) &= \int_{\widehat{AA_1}} q^+\mathrm{d}x - p^+\mathrm{d}y = \int_{H^+(x,y)=\frac{h}{2},\ x\geq 0} q^+\mathrm{d}x - p^+\mathrm{d}y, \\ M^-\Big(\frac{h}{2}\Big) &= \int_{\widehat{A_1A}} q^-\mathrm{d}x - p^-\mathrm{d}y = \int_{H^-(x,y)=\frac{h}{2},\ x<0} q^-\mathrm{d}x - p^-\mathrm{d}y. \end{aligned} \tag{2.1}$$

Since (1.7) in this paper and (1.9) in [14] have the same right subsystems, we directly have from (2.3)–(2.4) and (2.17) in [14] that

$$\begin{aligned} M^+\Big(\frac{h}{2}\Big) &= \sqrt{1-h}\,\mu_{[\frac{n}{2}]}(h) + \mu_{[\frac{n-1}{2}]}(h)I_{10}(h) \\ &= \phi_1(h)\ln h + \phi_2(h), \end{aligned} \tag{2.2}$$

where $\mu_j(h)$ denotes a polynomial of h with degree j, $\phi_2(h) \in C^\omega$ at $h = 0$, $\phi_1(h)$ is a polynomial in h with degree $[\frac{n+1}{2}]$, $\phi_1(0) = 0$, and

$$I_{10}(h) = h\varphi_0\Big(\sqrt{\frac{1-h}{h}}\Big), \quad \varphi_0(u) = \int_0^u \sqrt{1+x^2}\mathrm{d}x. \tag{2.3}$$

We know that the function $\varphi_0(u)$ is analytic on $\mathbb{R}$ and odd in u.

By making a change of $x = -u,\ y = v$, we get from (2.1)–(2.2) that

$$\begin{aligned} M^-\Big(\frac{h}{2}\Big) &= \int_{\widehat{A_1A}} q^-\mathrm{d}x - p^-\mathrm{d}y \\ &= \int_{\widehat{AA_1}} q^-(-u,v,\delta)\mathrm{d}u + p^-(-u,v,\delta)\mathrm{d}v \end{aligned}$$

$$
\begin{aligned}
&= \int_{\widehat{AA_1}} \Big(\sum_{i+j=0}^{n}(-1)^i b_{ij}^- u^i v^j\Big)\mathrm{d}u - \Big(\sum_{i+j=0}^{n}(-1)^{i+1} a_{ij}^- u^i v^j\Big)\mathrm{d}v\\
&= \sqrt{1-h}\,\widetilde{\mu}_{[\frac{n}{2}]}(h) + \widetilde{\mu}_{[\frac{n-1}{2}]}(h) I_{10}(h)\\
&= \widetilde{\phi}_1(h)\ln h + \widetilde{\phi}_2(h), \qquad (2.4)
\end{aligned}
$$

where $\widetilde{\mu}_j$ and $\widetilde{\phi}_i$ are similar to μ_j and ϕ_i in (2.2), $i = 1, 2$. Hence, from (2.2) and (2.4) we obtain the following lemma.

Lemma 2.1 *For the system* (1.7)*, the first-order Melnikov function has the following form:*

$$\overline{M}(h) = \sqrt{1-h}\, f_{[\frac{n}{2}]}(1-h) + g_{[\frac{n-1}{2}]}(1-h) I_{10}(h), \quad 0 < h < 1,$$

where $f_{[\frac{n}{2}]}(u)$ and $g_{[\frac{n-1}{2}]}(u)$ are polynomials in u of degrees $[\frac{n}{2}]$ and $[\frac{n-1}{2}]$ respectively.

In the following we study the expansions of $\overline{M}(h)$ near $h = 0, 1$. By (2.2) and (2.4), the lemma below holds immediately.

Lemma 2.2 *For the system* (1.7)*, the first-order of Melnikov function $\overline{M}(h)$ has the following expansion: for $0 < h \ll 1$*

$$\overline{M}(h) = \Big(\sum_{i=1}^{[\frac{n+1}{2}]} b_i^* h^i\Big)\ \ln h + \sum_{j\ge 0} b_j h^j,$$

where for $n = 2$,

$$b_1^* = -\tfrac{1}{2}(a_{10}^+ + b_{01}^+ + 2a_{20}^+ + b_{11}^+ + a_{10}^- + b_{01}^- - 2a_{20}^- - b_{11}^-),$$

$$b_0 = -a_{10}^+ - b_{01}^+ - \frac{2}{3}a_{20}^+ - \frac{1}{3}b_{11}^+ - 2a_{00}^+ - \frac{2}{3}a_{02}^+ - a_{10}^- - b_{01}^- + \frac{2}{3}a_{20}^- + \frac{1}{3}b_{11}^- + 2a_{00}^- + \frac{2}{3}a_{02}^-,$$

$$
\begin{aligned}
b_1 &= \Big(\frac{1}{2}+\ln 2\Big)a_{10}^+ + \Big(\frac{1}{2}+\ln 2\Big)b_{01}^+ + (2\ln 2 - 1)a_{20}^+ + \Big(\ln 2 - \frac{1}{2}\Big)b_{11}^+ + a_{00}^+ + a_{02}^+\\
&\quad + \Big(\frac{1}{2}+\ln 2\Big)a_{10}^- + \Big(\frac{1}{2}+\ln 2\Big)b_{01}^- + (1 - 2\ln 2)a_{20}^- + \Big(\frac{1}{2} - \ln 2\Big)b_{11}^- - a_{00}^- - a_{02}^-,
\end{aligned}
$$

$$
\begin{aligned}
b_2 &= \frac{1}{4}a_{20}^+ + \frac{1}{8}b_{11}^+ - \frac{1}{4}a_{02}^+ - \frac{1}{8}a_{10}^+ - \frac{1}{8}b_{01}^+ + \frac{1}{4}a_{00}^+ - \frac{1}{4}a_{20}^- - \frac{1}{8}b_{11}^- + \frac{1}{4}a_{02}^-\\
&\quad - \frac{1}{8}a_{10}^- - \frac{1}{8}b_{01}^- - \frac{1}{4}a_{00}^-,
\end{aligned}
$$

$\cdots\cdots$

Next, we give the expansion of $\overline{M}(h)$ near $h = 1$. Noticing that in (2.3) $\varphi_0(u) \in C^w$ on $\mathbb{R}$ and is odd in u, we can write for $|u|$ small

$$\varphi_0(u) = \sum_{i=0}^{\infty} \varsigma_i u^{2i+1},$$

where ς_i is a constant, $i \ge 0$. Then, it follows from (2.3), for $1 - h > 0$ small, that

$$I_{10}(h) = h\varphi_0\Big(\sqrt{\frac{1-h}{h}}\Big) = \sqrt{h(1-h)}\sum_{i=0}^{\infty}\varsigma_i\Big(\frac{1-h}{h}\Big)^i \equiv \sqrt{1-h}\,\phi_3(1-h), \qquad (2.5)$$

where $\phi_3(u) \in C^w$ at $u = 0$. Then by (2.5) and Lemma 2.1, we have the following lemma.

Lemma 2.3 *For the system* (1.7)*, the first order of Melnikov function* $\overline{M}(h)$ *has the following expansion: for* $0 < 1 - h \ll 1$,

$$\overline{M}(h) = \sqrt{1-h}\sum_{i\geq 0} c_i(1-h)^i,$$

where for $n = 2$,

$$\begin{aligned}
&c_0 = 2(a_{00}^- - a_{00}^+), \qquad c_1 = \frac{2}{3}(a_{02}^- - a_{10}^- - b_{01}^- - a_{02}^+ - a_{10}^+ - b_{01}^+),\\
&c_2 = \frac{2}{15}(2a_{20}^- + b_{11}^- - a_{10}^- - b_{01}^- - 2a_{20}^+ - b_{11}^+ - a_{10}^+ - b_{01}^+),\\
&\cdots\cdots
\end{aligned}$$

The following definition and lemma will be used in the proof of Theorem 1.2 in Section 3.

Definition 2.1 (see [15]) *Let* $f_0, f_1, \cdots, f_{m-1}$ *be analytic functions on an open interval* L *of* $\mathbb{R}$.

(a) $(f_0, f_1, \cdots, f_{m-1})$ *is said to be a Chebyshev system, provided that any nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{m-1} f_{m-1}(x)$$

has at most $m - 1$ *isolated zeros on* L.

(b) $(f_0, f_1, \cdots, f_{m-1})$ *is said to be a complete Chebyshev system, provided that* $(f_0, f_1, \cdots, f_{k-1})$ *is a Chebyshev system on* L *for all* $k = 1, 2, \cdots, m$.

(c) $(f_0, f_1, \cdots, f_{m-1})$ *is said to be an extended complete Chebyshev system, provided that any nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k - 1$ *isolated zeros on* L *counting multiplicity of zeros for all* $k = 1, 2, \cdots, m$.

(d) *The continuous Wronskian of* $(f_0, f_1, \cdots, f_{k-1})$ *at* $x \in L$ *is defined to be*

$$W[f_0, f_1, \cdots, f_{k-1}](x) = \det\big(f_j^{(i)}(x)\big)_{0\leq i,j\leq k-1} = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & f_1'(x) & \cdots & f_{k-1}'(x) \\ \vdots & \vdots & & \vdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

Lemma 2.4 (see [15]) $(f_0, f_1, \cdots, f_{m-1})$ *is an extended complete Chebyshev system on* L*, if and only if for each* $k = 1, 2, \cdots, m$,

$$W[f_0, f_1, \cdots, f_{k-1}](x) \neq 0 \quad for\ all\ x \in L.$$

3 Proof of the Main Results

Proof of Theorem 1.1 (1) For simplicity, we let $p^{\pm}$ and $q^{\pm}$ in (1.6) satisfy

$$p^+(x, y) = \sum_{i=0}^{n} a_i^+ x^i, \quad q^+(x, y) \equiv p^-(x, y) \equiv q^-(x, y) \equiv 0, \tag{3.1}$$

where $p^+(x,y)$ is independent of y. By Lemma 2.3, the first-order Melnikov function $\overline{M}(h)$ has the following expansion:

$$\overline{M}(h) = \sqrt{1-h}\sum_{j\geq 0} c_j(1-h)^j, \quad 0 < 1-h \ll 1. \tag{3.2}$$

Next, we give the formulas of coefficients c_i for $i \geq 0$. Comparing conditions (3.1) above and (3.1) in [14], we have from the proof of Theorem 1.1(1) in [14] that

$$M^+\Big(\frac{h}{2}\Big) = \sqrt{1-h}\sum_{j\geq 0} \widetilde{c}_j(1-h)^j, \quad 0 < 1-h \ll 1,$$

with

$$\widetilde{c}_j = \begin{cases} -2a_0^+, & j = 0, \\ -p_jB_j, & j \geq 1, \end{cases}$$
$$p_1 = \frac{1}{2}a_1^+, \quad p_2 = \frac{1}{4}a_2^+ + \frac{1}{8}a_1^+, \quad \cdots, \quad p_n = \frac{a_n^+}{2^n} + L(a_1^+, \quad \cdots, a_{n-1}^+), \quad \cdots,$$

where $L(\cdot)$ denotes a linear combination, B_j is a positive constant for $j \geq 1$. Under (3.1) we know $M^-\big(\frac{h}{2}\big) \equiv 0$ for $0 < h < 1$. Hence, by (1.8) we have $c_i = \widetilde{c}_i$ for $i \geq 0$. Further, it is easy to get that

$$\frac{\partial(c_0, c_1, \cdots, c_n)}{\partial(a_0^+, a_1^+, \cdots, a_n^+)} = \begin{pmatrix} -2 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{B_1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{B_2}{8} & -\frac{B_2}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & -\frac{B_n}{2^n} \end{pmatrix}. \tag{3.3}$$

Since the rank of this matrix is $n+1$, we can choose $c_j,\ 0 \leq j \leq n$ as free parameters such that $0 < |c_0| \ll |c_1| \ll \cdots \ll |c_n| \ll 1$ and $c_ic_{i+1} < 0,\ 0 \leq i \leq n-1$. Then by (3.2), $\overline{M}(h)$ has n positive simple zeros satisfying $0 < 1-h \ll 1$. Consequently, system (1.7) can have n limit cycles near the origin, which means $N_{\mathrm{Hopf}}(n) \geq n$.

(2) Here, we suppose that $p^\pm$ and $q^\pm$ in (1.6) satisfy

$$p^+(x,y) = \sum_{i=0}^{n} a_i^+y^i, \quad q^+(x,y) = \sum_{i=0}^{n} b_i^+y^i, \quad p^-(x,y) \equiv q^-(x,y) \equiv 0, \tag{3.4}$$

where $p^+(x,y)$ and $q^+(x,y)$ are univariate polynomials of the variable y. Since $(1-h)^{-\frac{1}{2}} \in C^\omega$ at $h = 0$, we can write by Lemma 2.2 that

$$\frac{\overline{M}(h)}{-2\sqrt{1-h}} = \sum_{k=0}^{\infty} v_kh^k + \sum_{k=0}^{\infty} v_k^*h^{k+1}\ln h \quad \text{for } 0 < h \ll 1. \tag{3.5}$$

Under (3.4), we know $M^-\big(\frac{h}{2}\big) \equiv 0$. Then $\overline{M}(h) = M^+\big(\frac{h}{2}\big)$ for this case. Notice that $p^+(x,y)$ and $q^+(x,y)$ in (3.4) are the same as those in (3.7) of [14]. Hence, by (3.9) and (3.11) in [14] we have

$$v_0 = a_0^+ + L(a_2^+, a_4^+, \cdots, a_{2[\frac{n}{2}]}^+, b_1^+, b_3^+, \cdots, b_{2[\frac{n-1}{2}]+1}^+),$$
$$v_1 = -\frac{a_2^+}{3} + L(a_4^+, \cdots, a_{2[\frac{n}{2}]}^+, b_1^+, b_3^+, \cdots, b_{2[\frac{n-1}{2}]+1}^+),$$

$$
\begin{aligned}
&v_2 = \frac{a_4^+}{5} + L(a_6^+, \cdots, a_{2[\frac{n}{2}]}^+, b_1^+, b_3^+, \cdots, b_{2[\frac{n-1}{2}]+1}^+),\\
&\cdots\cdots\\
&v_{[\frac{n}{2}]} = \frac{a_{2[\frac{n}{2}]}^+}{2[\frac{n}{2}]+1}(-1)^{[\frac{n}{2}]} + L(b_1^+, b_3^+, \cdots, b_{2[\frac{n-1}{2}]+1}^+),\\
&v_0^* = \frac{1}{4}b_1^+,\\
&v_1^* = \frac{3}{4}b_3^+\beta_1^* + L(b_1^+),\\
&v_2^* = \frac{5}{4}b_5^+\beta_2^* + L(b_1^+, b_3^+),\\
&\cdots\cdots\\
&v_{[\frac{n-1}{2}]}^* = \frac{2[\frac{n-1}{2}]+1}{4}b_{2[\frac{n-1}{2}]+1}^+\beta_{[\frac{n-1}{2}]}^* + L(b_1^+, b_3^+, \cdots, b_{2[\frac{n-1}{2}]-1}^+),
\end{aligned}
$$

and β_i^* is a nonzero constant for $1 \le i \le [\frac{n-1}{2}]$.

It follows that

$$
\begin{aligned}
A &\equiv \frac{\partial(v_0, v_1, \cdots, v_{[\frac{n}{2}]}, v_0^*, v_1^*, \cdots, v_{[\frac{n-1}{2}]}^*)}{\partial(a_0^+, a_2^+, \cdots, a_{2[\frac{n}{2}]}^+, b_1^+, b_3^+, \cdots, b_{2[\frac{n-1}{2}]+1}^+)}\\
&= \begin{pmatrix}
1 & * & * & \cdots & * & * & * & \cdots & * \\
0 & -\frac{1}{3} & * & \cdots & * & * & * & \cdots & * \\
0 & 0 & \frac{1}{5} & \cdots & * & * & * & \cdots & * \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & \frac{(-1)^{[\frac{n}{2}]}}{2[\frac{n}{2}]+1} & * & * & \cdots & * \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{4} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & * & \frac{3}{4}\beta_1^* & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & * & * & \cdots & \frac{2[\frac{n-1}{2}]+1}{4}\beta_{[\frac{n-1}{2}]}^*
\end{pmatrix}.
\end{aligned}
$$

By (3.5) we write

$$
\frac{\overline{M}(h)}{-2\sqrt{1-h}} = \begin{cases}
v_0 + v_0^* h\ln h + v_1 h + v_1^* h^2 \ln h + \cdots + v_{[\frac{n-1}{2}]}^* h^{[\frac{n-1}{2}]+1}\ln h + v_{[\frac{n}{2}]}h^{[\frac{n}{2}]} \\
\quad + O(h^{[\frac{n}{2}]+1}\ln h), \quad n \text{ even},\\
\\
v_0 + v_0^* h\ln h + v_1 h + v_1^* h^2 \ln h + \cdots + v_{[\frac{n}{2}]}h^{[\frac{n}{2}]} + v_{[\frac{n-1}{2}]}^* h^{[\frac{n-1}{2}]+1}\ln h \\
\quad + O(h^{[\frac{n-1}{2}]+1}), \quad n \text{ odd}.
\end{cases} \tag{3.6}
$$

Note that $\beta_i^* \neq 0,\ 1 \le i \le [\frac{n-1}{2}]$, which means $\mathrm{rank}(A) = n+1$. Therefore, $v_0, \cdots, v_{[\frac{n}{2}]}, v_0^*, \cdots, v_{[\frac{n-1}{2}]}^*$ can be chosen as free parameters such that

$$
0 < v_0 \ll v_0^* \ll \cdots \ll v_{[\frac{n-1}{2}]}^* \ll v_{[\frac{n}{2}]} \ll 1 \quad \text{or} \quad 0 < -v_0 \ll -v_0^* \ll \cdots \ll -v_{[\frac{n-1}{2}]}^* \ll -v_{[\frac{n}{2}]} \ll 1,
$$

if n is even; or

$$
0 < v_0 \ll v_0^* \ll \cdots \ll v_{[\frac{n}{2}]} \ll v_{[\frac{n-1}{2}]}^* \ll 1 \quad \text{or} \quad 0 < -v_0 \ll -v_0^* \ll \cdots \ll -v_{[\frac{n}{2}]} \ll -v_{[\frac{n-1}{2}]}^* \ll 1,
$$

if n is odd. Thus by (3.6), $\overline{M}(h)$ has n positive simple zeros near $h=0$. This means that the system (1.7) can have n limit cycles near the heteroclinic loop L_0. That is, $N_{\text{heteroc}}(n) \geq n$. The proof ends.

Proof of Theorem 1.2 By the proof of Theorem 1.1, it suffices to prove that $Z(n) \leq n$ for $n = 1, 2, 3$ and 4.

First, for $n=1$, by Lemma 2.1 and (2.3),

$$\overline{M}(h) = A\sqrt{1-h} + Bh\varphi_0\Big(\sqrt{\frac{1-h}{h}}\Big),$$

where A and B are constants. Letting $\lambda = \sqrt{1-h} \in (0,1)$ yields

$$\overline{M}(h) = (1-\lambda^2)\Big[A\frac{\lambda}{1-\lambda^2} + B\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big)\Big] \equiv (1-\lambda^2)M_1(\lambda), \tag{3.7}$$

where $M_1(\lambda) = A\frac{\lambda}{1-\lambda^2} + B\varphi_0\big(\frac{\lambda}{\sqrt{1-\lambda^2}}\big)$. Denote

$$f_0 = \frac{\lambda}{1-\lambda^2}, \quad f_1 = \varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big).$$

For $\lambda \in (0,1)$, we know that f_0 and f_1 are analytic. Moreover, by Definition 2.1(d), $W[f_0] = \frac{\lambda}{1-\lambda^2} > 0$ and

$$\begin{aligned} W[f_0, f_1] &= \begin{vmatrix} \dfrac{\lambda}{1-\lambda^2} & \varphi_0\Big(\dfrac{\lambda}{\sqrt{1-\lambda^2}}\Big) \\ \dfrac{1+\lambda^2}{(1-\lambda^2)^2} & \dfrac{1}{(1-\lambda^2)^2} \end{vmatrix} \\ &= -\frac{1+\lambda^2}{(1-\lambda^2)^2}\Big[\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) - \frac{\lambda}{1-\lambda^4}\Big] \\ &\equiv -\frac{1+\lambda^2}{(1-\lambda^2)^2}M_2(\lambda). \end{aligned}$$

Note that $M_2(0) = 0$ and $M_2'(\lambda) = [\varphi_0\big(\frac{\lambda}{\sqrt{1-\lambda^2}}\big) - \frac{\lambda}{1-\lambda^4}]' = \frac{2\lambda^2(1-\lambda^2)}{(1-\lambda^4)^2} > 0$. Then it follows that $W[f_0, f_1] < 0$, and further by Lemma 2.4, (f_0, f_1) is an extended complete Chebyshev system on the interval $(0,1)$. Hence, $M_1(\lambda)$ has at most one isolated zero in $(0,1)$ counted with multiplicities, which together with (3.7) means $\overline{M}(h)$ has at most one zero in the interval $(0,1)$. That is, $Z(1) \leq 1$.

For the case of $n=2$, it follows from Lemma 2.1 and (2.3) that

$$\overline{M}(h) = \sqrt{1-h}[A_0 + A_1\lambda^2 + B_0\varphi_1(\lambda)], \tag{3.8}$$

where $\varphi_1(\lambda) = \frac{1-\lambda^2}{\lambda}\varphi_0\big(\frac{\lambda}{\sqrt{1-\lambda^2}}\big)$, $\lambda = \sqrt{1-h} \in (0,1)$, and A_0, A_1 and B_0 are constants. Set $f_0 = 1$, $f_1 = \lambda^2$ and $f_2 = \varphi_1(\lambda)$. We get by Definition 2.1(d)

$$W[f_0] = 1, \quad W[f_0, f_1] = 2\lambda > 0,$$

and

$$W[f_0, f_1, f_2] = \begin{vmatrix} 1 & \lambda^2 & \varphi_1(\lambda) \\ 0 & 2\lambda & \varphi_1'(\lambda) \\ 0 & 2 & \varphi_1''(\lambda) \end{vmatrix}$$

$$= \frac{2(\lambda^2+3)}{\lambda^2}\Big[\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) - \frac{3\lambda}{(1-\lambda^2)(3+\lambda^2)}\Big]$$
$$\equiv \frac{2(\lambda^2+3)}{\lambda^2} M_3(\lambda) < 0,$$

since

$$M_3(0)=0, \quad M_3'(\lambda) = \Big[\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) - \frac{3\lambda}{(1-\lambda^2)(3+\lambda^2)}\Big]' = \frac{-8\lambda^4}{(1-\lambda^2)^2(3+\lambda^2)^2} < 0.$$

Hence, by Lemma 2.4 (f_0, f_1, f_2) is an extended complete Chebyshev system on $(0,1)$. This means, by (3.8), $\overline{M}(h)$ has at most two zeros in the interval (0,1), i.e., $Z(2) \le 2$.

For $n=3$, we also have by Lemma 2.1 and (2.3),

$$\overline{M}(h) = \sqrt{1-h}[(A_0 + A_1\lambda^2) + (B_0 + B_1\lambda^2)\varphi_1(\lambda)]. \tag{3.9}$$

For $\lambda \in (0,1)$, let

$$f_0 = 1, \quad f_1 = \lambda^2, \quad f_2 = \varphi_1(\lambda), \quad f_3 = \lambda^2\varphi_1(\lambda).$$

In this case, $W[f_0]$, $W[f_0, f_1]$ and $W[f_0, f_1, f_2]$ are equal to those in the case $n=2$. In order to prove that (f_0, f_1, f_2, f_3) is an extended complete Chebyshev system on $(0,1)$, we only need to check

$$W[f_0, f_1, f_2, f_3] \neq 0, \quad \lambda \in (0,1).$$

In fact,

$$W[f_0, f_1, f_2, f_3] = \begin{vmatrix} 1 & \lambda^2 & \varphi_1(\lambda) & \lambda^2\varphi_1(\lambda) \\ 0 & 2\lambda & \varphi_1'(\lambda) & \lambda^2\varphi_1'(\lambda) + 2\lambda\varphi_1(\lambda) \\ 0 & 2 & \varphi_1''(\lambda) & \lambda^2\varphi_1''(\lambda) + 4\lambda\varphi_1'(\lambda) + 2\varphi_1(\lambda) \\ 0 & 0 & \varphi_1'''(\lambda) & \lambda^2\varphi_1'''(\lambda) + 6\lambda\varphi_1''(\lambda) + 6\varphi_1'(\lambda) \end{vmatrix}$$
$$\equiv -\frac{12(\lambda^4+6\lambda^2+1)}{\lambda^4} M_4(\lambda),$$

where

$$M_4(\lambda) = \varphi_0^2\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) + \frac{22\lambda^7 - 30\lambda^5 + 2\lambda^3 + 6\lambda}{3(\lambda^4+6\lambda^2+1)(\lambda^2-1)^3}\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big)$$
$$+ \frac{7\lambda^4 - 3\lambda^2}{3(\lambda^4+6\lambda^2+1)(\lambda^2-1)^3}.$$

Notice that

$$M_4'(\lambda) = \frac{4\lambda^2(15\lambda^8 + 12\lambda^6 + 10\lambda^4 + 12\lambda^2 + 15)}{3(\lambda^4+6\lambda^2+1)^2(1-\lambda^2)^3}\Big[\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big)$$
$$+ \frac{\lambda(5\lambda^6 + 5\lambda^4 + 7\lambda^2 + 15)}{(\lambda^2-1)(15\lambda^8 + 12\lambda^6 + 10\lambda^4 + 12\lambda^2 + 15)}\Big]$$
$$\equiv \frac{4\lambda^2(15\lambda^8 + 12\lambda^6 + 10\lambda^4 + 12\lambda^2 + 15)}{3(\lambda^4+6\lambda^2+1)^2(1-\lambda^2)^3} M_5(\lambda)$$

and

$$M_5'(\lambda) = \frac{256(\lambda^4+6\lambda^2+1)\lambda^6}{(15\lambda^8 + 12\lambda^6 + 10\lambda^4 + 12\lambda^2 + 15)^2(1-\lambda^2)} > 0.$$

It is direct from $M_4(0)=0$ and $M_5(0)=0$ and the above that

$$W[f_0, f_1, f_2, f_3] < 0.$$

Hence, by Lemma 2.4 and (3.9), $\overline{M}(h)$ has at most 3 zeros in the interval $(0,1)$ for this case. It follows that $Z(3) \le 3$.

For $n = 4$, by using the same method as above, we have

$$\overline{M}(h) = \sqrt{1-h}[(A_0 + A_1\lambda^2 + A_2\lambda^4) + (B_0 + B_1\lambda^2)\varphi_1(\lambda)]. \tag{3.10}$$

Let

$$f_0 = 1, \quad f_1 = \lambda^2, \quad f_2 = \lambda^4, \quad f_3 = \varphi_1(\lambda), \quad f_4 = \lambda^2\varphi_1(\lambda).$$

Then

$$W[f_0] = 1, \quad W[f_0, f_1] = 2\lambda > 0, \quad W[f_0, f_1, f_2] = 16\lambda^3 > 0$$

and

$$\begin{aligned} W[f_0, f_1, f_2, f_3] &= \begin{vmatrix} 1 & \lambda^2 & \lambda^4 & \varphi_1(\lambda) \\ 0 & 2\lambda & 4\lambda^3 & \varphi_1'(\lambda) \\ 0 & 2 & 12\lambda^2 & \varphi_1''(\lambda) \\ 0 & 0 & 24\lambda & \varphi_1'''(\lambda) \end{vmatrix} \\ &= -\frac{48(\lambda^2+5)}{\lambda}\Big[\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) + \frac{\lambda(17\lambda^2-15)}{3(1-\lambda^2)^2(5+\lambda^2)}\Big] \\ &\equiv -\frac{48(\lambda^2+5)}{\lambda} M_6(\lambda). \end{aligned}$$

In view of $M_6(0) = 0$ and $M_6'(\lambda) = \frac{16\lambda^6}{(5+\lambda^2)^2(1-\lambda^2)^3} > 0$, we have $W[f_0, f_1, f_2, f_3] < 0$.

Next, we consider $W[f_0, f_1, f_2, f_3, f_4]$. By some computations, we get

$$\begin{aligned} W[f_0, f_1, f_2, f_3, f_4] &= \begin{vmatrix} 1 & \lambda^2 & \lambda^4 & \varphi_1(\lambda) & \lambda^2\varphi_1(\lambda) \\ 0 & 2\lambda & 4\lambda^3 & \varphi_1'(\lambda) & \lambda^2\varphi_1'(\lambda) + 2\lambda\varphi_1(\lambda) \\ 0 & 2 & 12\lambda^2 & \varphi_1''(\lambda) & \lambda^2\varphi_1''(\lambda) + 4\lambda\varphi_1'(\lambda) + 2\varphi_1(\lambda) \\ 0 & 0 & 24\lambda & \varphi_1'''(\lambda) & \lambda^2\varphi_1'''(\lambda) + 6\lambda\varphi_1''(\lambda) + 6\varphi_1'(\lambda) \\ 0 & 0 & 24 & \varphi_1^{(4)}(\lambda) & \lambda^2\varphi_1^{(4)}(\lambda) + 8\lambda\varphi_1'''(\lambda) + 12\varphi_1''(\lambda) \end{vmatrix} \\ &= -\frac{288(\lambda^4+10\lambda^2+5)}{\lambda^4}\Big[\varphi_0^2\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) \\ &\quad - \frac{2\lambda(21\lambda^6 - 15\lambda^4 - 5\lambda^2 + 15)}{3(1-\lambda^2)^3(\lambda^4+10\lambda^2+5)}\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) \\ &\quad + \frac{\lambda^2(41\lambda^4 - 30\lambda^2 + 45)}{9(1-\lambda^2)^4(\lambda^4+10\lambda^2+5)}\Big] \\ &\equiv -\frac{288(\lambda^4+10\lambda^2+5)}{\lambda^4} M_7(\lambda), \\ M_7'(\lambda) &= -\frac{8\lambda^4(15\lambda^8 + 36\lambda^6 + 58\lambda^4 + 100\lambda^2 + 175)}{3(1-\lambda^2)^4(\lambda^4+10\lambda^2+5)^2}\Big[\varphi_0\Big(\frac{\lambda}{\sqrt{1-\lambda^2}}\Big) \\ &\quad - \frac{\lambda(15\lambda^6 + 39\lambda^4 + 125\lambda^2 + 525)}{3(1-\lambda^2)(15\lambda^8 + 36\lambda^6 + 58\lambda^4 + 100\lambda^2 + 175)}\Big] \\ &\equiv -\frac{8\lambda^4(15\lambda^8 + 36\lambda^6 + 58\lambda^4 + 100\lambda^2 + 175)}{3(1-\lambda^2)^4(\lambda^4+10\lambda^2+5)^2} M_8(\lambda) \end{aligned}$$

and

$$M_8'(\lambda) = -\frac{2048\lambda^8(\lambda^4+10\lambda^2+5)}{(1-\lambda^2)^2(15\lambda^8 + 36\lambda^6 + 58\lambda^4 + 100\lambda^2 + 175)^2} < 0.$$

Then, by $M_7(0) = 0$ and $M_8(0) = 0$, it is clear that $W[f_0, f_1, f_2, f_3, f_4] < 0$. Thus, $(f_0, f_1, f_2, f_3, f_4)$ is also an extended complete Chebyshev system on $(0, 1)$. Therefore, by Definition 2.1(c) and (3.10), $\overline{M}(h)$ has at most 4 zeros in the interval $(0, 1)$, which yields $Z(4) \le 4$. The proof ends.

Proof of Theorem 1.3 Following Theorem 1.1, we only need to prove $Z(n) \le n + [\frac{n+1}{2}]$, $n \ge 5$. Similar to the method used in [19] for instance, we give the proof bellow. By (2.3) and Lemma 2.1, it follows that

$$\begin{aligned}\overline{M}(h) &= \sqrt{1-h} f_{[\frac{n}{2}]}(1-h) + g_{[\frac{n-1}{2}]}(1-h) I_{10}(h)\\ &= \sqrt{1-h}\widetilde{f}_{[\frac{n}{2}]}(h) + \widetilde{g}_{[\frac{n-1}{2}]}(h) h \varphi_0\Big(\sqrt{\frac{1-h}{h}}\Big)\\ &\equiv u_0\sqrt{1-h} + u_1 \varphi_0\Big(\sqrt{\frac{1-h}{h}}\Big),\end{aligned}$$

where $\widetilde{f}_{[\frac{n}{2}]}$ and $\widetilde{g}_{[\frac{n-1}{2}]}$ are polynomials in h with degrees $[\frac{n}{2}]$ and $[\frac{n-1}{2}]$ respectively, $h \in (0, 1)$, $u_0 = \widetilde{f}_{[\frac{n}{2}]}(h)$ and $u_1 = h\widetilde{g}_{[\frac{n-1}{2}]}(h)$. Thus,

$$\begin{aligned}F &\equiv \frac{\mathrm{d}}{\mathrm{d}h}\Big(\frac{\overline{M}(h)}{u_1}\Big)\\ &= \Big(\frac{u_0}{u_1}\sqrt{1-h} + \varphi_0\Big(\sqrt{\frac{1-h}{h}}\Big)\Big)'\\ &= \frac{2h^2(1-h)(u_0'u_1 - u_0u_1') - h^2 u_0 u_1 - u_1^2}{2h^2\sqrt{1-h}u_1^2}\\ &= \frac{2(1-h)(u_0'u_1 - u_0u_1') - u_0 u_1 - \widetilde{g}^2_{[\frac{n-1}{2}]}(h)}{2\sqrt{1-h}u_1^2}\\ &\equiv \frac{\vartheta(h)}{2\sqrt{1-h}u_1^2},\end{aligned}$$

where

$$\vartheta(h) = 2(1-h)(u_0'u_1 - u_0u_1') - u_0u_1 - \widetilde{g}^2_{[\frac{n-1}{2}]}(h).$$

Set $I = (0, 1)$. Introducing a notation $\sharp\{h \in I \mid f(h) = 0\}$ to indicate the number of zeros of the function f in the interval I and taking into account their multiplicities, we have by degree$(\vartheta(h)) = n$

$$\sharp\{h \in I \mid u_1(h) = 0\} \le \Big[\frac{n-1}{2}\Big], \quad \sharp\{h \in I \mid \vartheta(h) = 0\} \le n.$$

Therefore,

$$\begin{aligned}\sharp\{h \in I \mid \overline{M}(h) = 0\} &\le \sharp\{h \in I \mid u_1 = 0\} + \sharp\{h \in I \mid F = 0,\ u_1 \ne 0\} + 1\\ &\le \sharp\{h \in I \mid u_1 = 0\} + \sharp\{h \in I \mid \vartheta(h) = 0\} + 1\\ &\le n + \Big[\frac{n+1}{2}\Big].\end{aligned}$$

The proof is completed.

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