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Analytic Feynman Integrals of Functionals in a Banach Algebra Involving the First Variation*

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Abstract This paper deals with the analytic Feynman integral of functionals on a Wiener space. First the authors establish the existence of the analytic Feynman integrals of functionals in a Banach algebra $\mathcal{S}_\alpha$. The authors then obtain a formula for the first variation of integrals. Finally, various analytic Feynman integration formulas involving the first variation are established.

Keywords Analytic Feynman integral, Banach algebra, First variation, Cameron-Storvick theorem, Wiener space
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Let $C_0[0,T]$ denote a one-parameter Wiener space; that is the space of continuous real-valued functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by $\int_{C_0[0,T]} F(x)\mathrm{d}m(x)$.

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided that ρB is $\mathcal{M}$-measurable for all $\rho>0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided that $m(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (for short s-a.e.) (see [16]). Throughout this paper we will assume that each functional $F: C_0[0,T]\to\mathbb{C}$ that we consider is scale-invariant measurable and that

$$\int_{C_0[0,T]} |F(\rho x)|\mathrm{d}m(x)<\infty$$

for each $\rho>0$.

In 1948, Feynman assumed the existence of an integral over a space of paths, and he used his integral in a formal way in his approach to quantum mechanics (see [15]). Many mathematicians have attempted to give rigorously meaningful definitions of the Feynman integral with appropriate existence theorems and have expressed solutions of the Schrödinger equation in terms of their integrals. One of these approaches was based on the similarity between the Wiener and the Feynman integrals, and procedures were set up by many mathematicians to obtain Feynman integrals from Wiener integrals by analytic extension from the real axis to

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the imaginary axis. For the procedure of analytic continuation to define the analytic Feynman integral, see [15]. Since the analytic Feynman integral was introduced, many mathematicians studied the analytic Feynman integral of functionals in several classes of functionals (see [1–5, 7, 13]). Also, they obtained various analytic Feynman integration formulas involving the first variation. In particular, in [2], the authors introduced a Banach algebra $\mathcal{S}$ which contains many functionals used in quantum mechanics, and then established analytic Feynman integrals of functionals in $\mathcal{S}$ involving the first variation.

In this paper we first recall the Banach algebra $\mathcal{S}_\alpha$ which was introduced in [11] and modified in [9, 14]. We then obtain the existence of the analytic Feynman integral and the first variation of a functional F in $\mathcal{S}_\alpha$. Also, we obtain analytic Feynman integration formulas involving the first variation. The results in [2] are special case of this paper when $\alpha = i$. The formulas and results in this paper are more complicated and more generalized than the results and formulas in [2]. In fact, when $\alpha = i$, all conditions hold, so they could be omitted naturally.

1 Definitions and Preliminaries

In this section, we recall some definitions and properties from [1–5, 7, 13].

Definition 1.1 *Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let $F : C_0[0,T] \to \mathbb{C}$ be a measurable functional such that for each $\lambda > 0$, the Wiener integral*

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}} x) \mathrm{d}m(x)$$

exists. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_0[0,T]}^{anw_\lambda} F(x) \mathrm{d}m(x).$$

Let $q \neq 0$ be a real number and let F be a functional such that $J^*(\lambda)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with a parameter q and we write

$$\int_{C_0[0,T]}^{anf_q} F(x) \mathrm{d}m(x) = \lim_{\lambda \to -\mathrm{i}q} \int_{C_0[0,T]}^{anw_\lambda} F(x) \mathrm{d}m(x)$$

where $\lambda \to -\mathrm{i}q$ through values in $\mathbb{C}_+$.

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund (for short PWZ) stochastic integral. Then we have the following assertions:

(1) For each $v \in L_2[0,T]$, $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$.

(2) If $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)\mathrm{d}x(t)$ for s-a.e. $x \in C_0[0,T]$.

(3) The PWZ stochastic integral $\langle v, x\rangle$ has the expected linearity property.

(4) The PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$.

For a more detailed study of the PWZ stochastic integral, see [5–6, 8–12].

The following is a well-known integration formula which is used several times in this paper.

For each $\alpha \in \mathbb{C}$ and for $v \in L_2[0,T]$,

$$\int_{C_0[0,T]} \exp\{\alpha\langle v,x\rangle\}\mathrm{d}m(x) = \exp\Big\{\frac{\alpha^2}{2}\|v\|_2^2\Big\}. \tag{1.1}$$

Now we recall a modified Banach algebra $\mathcal{S}_\alpha$ of functionals which was introduced in [11] and used in [9, 14].

For each complex number α with $\mathrm{Re}(\alpha^2) \le 0$, let $\mathcal{S}_\alpha$ be the class of functionals of the form

$$F(x) = \int_{L_2[0,T]} \exp\{\alpha\langle v,x\rangle\}\mathrm{d}f(v) \tag{1.2}$$

for s-a.e. $x \in C_0[0,T]$, where f is in $M(L_2[0,T])$, the class of all complex valued countably additive Borel measures on $L_2[0,T]$.

Remark 1.1 Using the techniques similar to those used in [2], we can show that for each $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha^2) \le 0$, the class $\mathcal{S}_\alpha$ is a Banach algebra with the norm

$$\|F\| = \|f\| = \int_{L_2[0,T]} |\mathrm{d}f(v)|, \quad f \in M(L_2[0,T]).$$

One can show that the correspondence $f \to F$ is injective, and carries convolution into pointwise multiplication and that for each complex number α with $\mathrm{Re}(\alpha^2) \le 0$, the space $\mathcal{S}_\alpha$ is a Banach algebra. In particular, if $\alpha = i$, then $\mathcal{S}_i$ is the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [2]. For more details, see [9, 11, 14].

2 Analytic Feynman Integrals of Functionals in $\mathcal{S}_\alpha$

In this section, we establish the analytic Feynman integral of functionals in $\mathcal{S}_\alpha$.

Remark 2.1 (1) Let $\gamma_1 = a + \mathrm{i}b$ and $\gamma_2 = c + \mathrm{i}d$ be nonzero complex numbers with $a \le 0$ and $c \ge 0$. First, we note that

$$\mathrm{Re}\Big(\frac{\gamma_1}{\gamma_2}\Big) = \frac{ac+bd}{c^2+d^2} \le 0$$

implies that $ac + bd \le 0$. This tells us that there are many nonzero complex numbers γ_1 and γ_2 so that $\mathrm{Re}\big(\frac{\gamma_1}{\gamma_2}\big) \le 0$. For example, if we take $\gamma_1 = -1 + \mathrm{i}$ and $\gamma_2 = 1 + \mathrm{i}$, then $\mathrm{Re}\big(\frac{\gamma_1}{\gamma_2}\big) = 0$. Also, if we take $\gamma_1 = -3 + 2\mathrm{i}$ and $\gamma_2 = 4 + 3\mathrm{i}$, then $\mathrm{Re}\big(\frac{\gamma_1}{\gamma_2}\big) = -6 \le 0$.

(2) Let α be a complex number with $\mathrm{Re}(\alpha^2) \le 0$ and let λ be an element of $\mathbb{C}_+$. Throughout this paper, we will consider a subregion Γ_α of $\mathbb{C}_+$, where

$$\Gamma_\alpha = \Big\{\lambda \in \mathbb{C}_+ : \mathrm{Re}\Big(\frac{\alpha^2}{\lambda}\Big) \le 0\Big\}. \tag{2.1}$$

In view of (1), the region Γ_α has sufficiently many complex numbers λ.

(3) Now we describe the region Γ_α for each α. Let $\alpha^2 = a + \mathrm{i}b$ and $\lambda = c + \mathrm{i}d$ be complex numbers with $a \le 0$ and $c > 0$. Then for each α, we can describe the region Γ_α as follows:

(i) When $d = 0$ or $b = 0$, $\Gamma_\alpha = \mathbb{C}_+$.

(ii) When $d \ne 0$ and $b \ne 0$, for a given α, if $b > 0$, then the region Γ_α is given by $\{\lambda : d \le -\frac{a}{b}c\}$ and if $b < 0$, then the region Γ_α is given by $\{\lambda : d \ge -\frac{a}{b}c\}$.

(iii) The region Γ_α always contains all positive real numbers.

In particular, when α is pure imaginary, $\Gamma_\alpha = \mathbb{C}_+$.

In our first lemma, we establish the existence of the analytic Wiener integral of a functional F in $\mathcal{S}_\alpha$.

Lemma 2.1 *Let $F \in \mathcal{S}_\alpha$ be given by* (1.2). *Then the analytic Wiener integral*

$$\int_{C_0[0,T]}^{anw_\lambda} F(x)\mathrm{d}m(x)$$

of functional F exists for each $\lambda \in \Gamma_\alpha$ and is given by the formula

$$\int_{L_2[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\Big\}\mathrm{d}f(v). \tag{2.2}$$

Proof First, we note that for all $\lambda > 0$, using formula (1.1), it follows that

$$\begin{aligned} J(\lambda) &\equiv \int_{C_0[0,T]}\int_{L_2[0,T]} \exp\{\lambda^{-\frac{1}{2}}\alpha\langle v, x\rangle\}\mathrm{d}f(v)\mathrm{d}m(x) \\ &= \int_{L_2[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\Big\}\mathrm{d}f(v). \end{aligned}$$

Therefore, for all $\lambda > 0$,

$$|J(\lambda)| \le \int_{L_2[0,T]} \exp\Big|\Big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\Big\}\Big||\mathrm{d}f(v)| \le \|f\|$$

because the real part of $\frac{\alpha^2}{2\lambda}$ is nonpositive. Next let

$$J^*(\lambda) = \int_{L_2[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\Big\}\mathrm{d}f(v),$$

where $\lambda \in \Gamma_\alpha$ is given by (2.1). Moreover, the function $J^*(\lambda)$ is well-defined on the region Γ_α. In fact, $|J^*(\lambda)| \le \|f\|$ for all $\lambda \in \Gamma_\alpha$. Also, $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$. At last, we will show that $J^*(\lambda)$ is analytic on Γ_α. To do this, let Λ be any simple closed contour in Γ_α. Then using the Fubini theorem and the Cauchy theorem, we have

$$\begin{aligned} &\int_\Lambda J^*(\lambda)\mathrm{d}\lambda = \int_\Lambda\int_{L_2[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\Big\}\mathrm{d}f(v)\mathrm{d}\lambda \\ &= \int_{L_2[0,T]}\int_\Lambda \exp\Big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\Big\}\mathrm{d}\lambda\mathrm{d}f(v) \\ &= 0 \end{aligned}$$

because the function $\big\{\frac{\alpha^2}{2\lambda}\|v\|_2^2\big\}$ is analytic on Γ_α for each $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha^2) \le 0$. Hence using the Morera's theorem, $J^*(\lambda)$ is analytic on Γ_α, so we complete the proof of Lemma 2.1.

The following theorem is the first main theorem in this paper.

Theorem 2.1 *Let F and f be as in Lemma* 2.1 *and let q be a nonzero real number such that*

$$\begin{cases} \mathrm{sign}(q) = -\mathrm{sign}(\mathrm{Im}(\alpha^2)), & \textit{if } \mathrm{Im}(\alpha^2) \ne 0, \\ q \in \mathbb{R}, & \textit{if } \mathrm{Im}(\alpha^2) = 0, \end{cases} \tag{2.3}$$

where sign *denotes the signum function defined by* $\operatorname{sign}(a) = \begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$ *and* $\mathbb{R}$ *is the set of all real numbers. Assume that*

$$\int_{L_2[0,T]} \exp\Big\{-\frac{\operatorname{Im}(\alpha^2)}{2|q|}\|v\|_2^2\Big\}|\mathrm{d}f(v)| < \infty. \tag{2.4}$$

Then the analytic Feynman integral $\int_{C_0[0,T]}^{anf_q} F(x)\mathrm{d}m(x)$ *of* F *exists and is given by the formula*

$$\int_{L_2[0,T]} \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}f(v). \tag{2.5}$$

Proof From Lemma 2.1, the analytic Wiener integral $\int_{C_0[0,T]}^{anw_\lambda} F(x)\mathrm{d}m(x)$ of F exists for each $\lambda \in \Gamma_\alpha$. To complete the proof of Theorem 2.1, we have to show that

$$\lim_{\lambda\to -\mathrm{i}q} J^*(\lambda) = \int_{L_2[0,T]} \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}f(v). \tag{2.6}$$

To do this, we recall the region Γ_α as in Remark 2.1. Note that for a given nonzero real number q which satisfies the condition (2.3), there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ in Γ_α so that $\lambda_n \to -\mathrm{i}q$ as $n\to\infty$. In fact, if we let $\lambda_n = \frac{1}{n} - \mathrm{i}q,\ n = 1, 2, \cdots$, then $\{\lambda_n\}_{n=1}^\infty$ in Γ_α and $\lambda_n \to -\mathrm{i}q$ as $n\to\infty$. By Remark 2.1 and Lemma 2.1, $|J^*(\lambda_l)| \le \|f\|$ for all $l = 1, 2, \cdots$. Hence using the dominated convergence theorem, for every nonzero real number q which satisfies the condition (2.3) above,

$$\begin{aligned}\lim_{\lambda_l\to -\mathrm{i}q} J^*(\lambda) &= \lim_{\lambda_l\to -\mathrm{i}q}\int_{L_2[0,T]} \exp\Big\{\frac{\alpha^2}{2\lambda_l}\|v\|_2^2\Big\}\mathrm{d}f(v) \\ &= \int_{L_2[0,T]} \exp\Big\{\frac{\alpha^2}{2(-\mathrm{i}q)}\|v\|_2^2\Big\}\mathrm{d}f(v) \\ &= \int_{L_2[0,T]} \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}f(v),\end{aligned}$$

which establishes Equations (2.5) and (2.6) as desired. Furthermore,

$$\Big|\int_{C_0[0,T]}^{anf_q} f(x)\mathrm{d}m(x)\Big| \le \int_{L_2[0,T]} \exp\Big\{-\frac{\operatorname{Im}(\alpha^2)}{2|q|}\|v\|_2^2\Big\}|\mathrm{d}f(v)| < \infty.$$

Hence we complete the proof of Theorem 2.1.

The result in [2, Theorem 5.1] is obtained immediately from Theorem 2.1.

Corollary 2.1 *When* $\alpha = \mathrm{i}$, $\Gamma_\alpha = \mathbb{C}_+$ *and Equation* (2.4) *holds, for every nonzero real number* q*, the analytic Feynman integral* $\int_{C_0[0,T]}^{anf_q} F(x)\mathrm{d}m(x)$ *of* $F \in \mathcal{S}$ *exists and is given by the formula*

$$\int_{L_2[0,T]} \exp\Big\{-\frac{\mathrm{i}}{2q}\|v\|_2^2\Big\}\mathrm{d}f(v).$$

3 The First Variation and Analytic Feynman Integrals

In this section we establish the existence of first variation of F in $\mathcal{S}_\alpha$ and establish various analytic Feynman integrals involving first variations of F.

We start this section by giving the definition of the first variation of a functional F on $C_0[0,T]$.

Definition 3.1 *Let F be a functional defined on $C_0[0,T]$. Then the first variation of F is defined by the formula*

$$\delta F(x|u) = \frac{\partial}{\partial k} F(x+ku)\Big|_{k=0}, \quad x, u \in C_0[0,T], \tag{3.1}$$

if it exists.

Let

$$\mathcal{A} = \Big\{ u \in C_0[0,T] : u(t) = \int_0^t z(s)\mathrm{d}s \text{ for some } z \in L_2[0,T] \Big\}.$$

Note that for all $w, v \in L_2[0,T]$, we have

$$|(w,v)_2| \le \|w\|_2 \|v\|_2.$$

Furthermore, for $u \in \mathcal{A}$ and $v \in L_2[0,T]$, the PWZ integral $\langle v, u\rangle$ exists and is given by the formula

$$\langle v, u\rangle = \int_0^T v(s)z(s)\mathrm{d}s = (v,z)_2$$

and hence $|\langle v,u\rangle| \le \|v\|_2\|z\|_2$.

The following observation will be very useful in our study. For $F \in \mathcal{S}_\alpha$, we will assume that the associated measure f in $M(L_2[0,T])$ of F always satisfies the following inequality:

$$\int_{L_2[0,T]} \|v\|_2 |\mathrm{d}f(v)| < \infty. \tag{3.2}$$

We state an interesting observation involving the first variation.

Remark 3.1 First we could consider the following integral:

$$\int_{L_2[0,T]} \alpha\langle v,u\rangle \exp\{\alpha\langle v,x\rangle\}\mathrm{d}f(v). \tag{3.3}$$

Since $\mathrm{Re}(\alpha^2) \le 0$, by assumption (3.2),

$$\int_{L_2[0,T]} \alpha\langle v,u\rangle \mathrm{d}f(v) < \infty \tag{3.4}$$

and

$$\int_{L_2[0,T]} \exp\{\alpha\langle v,x\rangle\}\mathrm{d}f(v)$$

exists for s-a.e. $x \in C_0[0,T]$. However, the integral (3.3) might not exist because the product of L_1-functionals might not be in L_1. Hence we should give a condition for f as follows: If $\mathrm{Re}(\alpha^2) \le 0$ and Equation (3.2) holds, then the integral (3.3) always exists.

In our next theorem, we obtain the formula for the first variation of functionals from $\mathcal{S}_\alpha$ into $\mathcal{S}_\alpha$.

Theorem 3.1 *Let F and f be as in Lemma 2.1 and let $u \in \mathcal{A}$. Assume that*

$$\Big|\frac{\partial}{\partial k}\exp\{\alpha\langle v, x+ku\rangle\}\Big| \le L(x), \tag{3.5}$$

where $L(x)$ is integrable on $C_0[0,T]$. Then the first variation $\delta F(x|u)$ of F exists and is given by the formula

$$\delta F(x|u) = \int_{L_2[0,T]} \alpha\langle v, u\rangle \exp\{\alpha\langle v, x\rangle\}\mathrm{d}f(v) \tag{3.6}$$

for s-a.e. $x \in C_0[0,T]$. Furthermore, as a function of x, δF is an element of $\mathcal{S}_\alpha$. In fact,

$$\delta F(x|u) = \int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle\}\mathrm{d}\phi(v),$$

where ϕ is an element of $M(L_2[0,T])$.

Proof Using Equation (3.1), it follows that for s-a.e. $x \in C_0[0,T]$,

$$\begin{aligned}\delta F(x|u) &= \frac{\partial}{\partial k}F(x+ku)\Big|_{k=0}\\ &= \frac{\partial}{\partial k}\Big(\int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle + \alpha k\langle v, u\rangle\}\mathrm{d}f(v)\Big)\Big|_{k=0}\\ &= \int_{L_2[0,T]} \alpha\langle v, u\rangle \exp\{\alpha\langle v, x\rangle\}\mathrm{d}f(v).\end{aligned} \tag{3.7}$$

The second equality in (3.7) follows from the condition (3.5) and thus by using Remark 3.1, the last expression in Equation (3.7) exists. Thus we have established Equation (3.6). Now let ϕ be a set function defined by

$$\phi(E) = \int_E \alpha\langle v, u\rangle\mathrm{d}f(v)$$

for $E \in \mathcal{B}(L_2[0,T])$. Then we see that ϕ is an element of $M(L_2[0,T])$ by using Equation (3.2) and the last expression in Equation (3.7) becomes

$$\int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle\}\mathrm{d}\phi(v).$$

Hence δF is an element of $\mathcal{S}_\alpha$.

The following theorem is the second main result in this paper.

Theorem 3.2 *Let F, f and q be as in Theorem 2.1, and let u and ϕ be as in Theorem 3.1. Then the analytic Feynman integral $\int_{C_0[0,T]}^{anf_q} \delta F(x|u)\mathrm{d}m(x)$ of $\delta F(x|u)$ exists and is given by the formula*

$$\int_{L_2[0,T]} \alpha\langle v, u\rangle \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}f(v). \tag{3.8}$$

Furthermore, the expression (3.8) can be expressed by the formula

$$\int_{L_2[0,T]} \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}\phi(v).$$

Proof From Theorem 3.1, the first variation $\delta F(x|u)$ as a function of x is an element of $\mathcal{S}_\alpha$. Hence using Equation (2.5), we have

$$\int_{C_0[0,T]}^{anf_q} \delta F(x|u)\mathrm{d}m(x) = \int_{L_2[0,T]} \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}\phi(v).$$

From the definition of ϕ, we can rewrite the above expression in the following way:

$$\int_{L_2[0,T]} \alpha\langle v,u\rangle \exp\Big\{\frac{\mathrm{i}\alpha^2}{2q}\|v\|_2^2\Big\}\mathrm{d}f(v).$$

The existence of the last expression comes from Remark 3.1. Hence we have the desired results.

Let F and G be elements of $\mathcal{S}_\alpha$ whose associated measures f and g satisfy

$$\int_{L_2[0,T]} \|v\|_2[|\mathrm{d}f(v)| + |\mathrm{d}g(v)|] < \infty. \tag{3.9}$$

Then using Remark 3.1, for each $u \in A$,

$$L(x) \equiv F(x)\delta G(x|u) + \delta F(x|u)G(x)$$

is an element of $\mathcal{S}_\alpha$. In addition, the condition

$$\int_{L_2[0,T]} \exp\Big\{-\frac{\mathrm{Im}(\alpha^2)}{2|q|}\|v\|_2^2\Big\}[|\mathrm{d}f(v)| + |\mathrm{d}g(v)|] < \infty$$

implies the existence of the analytic Feynman integral

$$\int_{C_0[0,T]}^{anf_q} L(x)\mathrm{d}m(x) = \int_{C_0[0,T]}^{anf_q} [F(x)\delta G(x|u) + \delta F(x|u)G(x)]\mathrm{d}m(x)$$

for every nonzero real numbers q as in Theorem 3.2.

The following lemma was established in [1].

Lemma 3.1 *Let $u \in \mathcal{A}$ and let F be a Wiener integrable functional. Furthermore, assume that $F(x)$ has a first variation $\delta F(x|u)$ for all $x \in C_0[0,T]$ such that for some $\eta > 0$, $\sup_{|h|\le\eta} |\delta F(x+hu|u)|$ is Wiener integrable. Then*

$$\int_{C_0[0,T]} \delta F(x|u)\mathrm{d}m(x) = \int_{C_0[0,T]} F(x)\langle z,x\rangle \mathrm{d}m(x). \tag{3.10}$$

In addition, for any nonzero real number q,

$$\int_{C_0[0,T]}^{anf_q} \delta F(x|u)\mathrm{d}m(x) = -\mathrm{i}q\int_{C_0[0,T]}^{anf_q} F(x)\langle z,x\rangle \mathrm{d}m(x). \tag{3.11}$$

Equation (3.11) *is called the Cameron-Storvick formula on $C_0[0,T]$.*

In the last main result, we obtain the Cameron-Storvick formula for analytic Feynman integral of functionals F and G in $\mathcal{S}_\alpha$.

Theorem 3.3 *Let F and G be elements of $\mathcal{S}_\alpha$ whose associated measures f and g satisfy the condition* (3.9) *and let q be as in Theorem* 3.2*. Then*

$$\begin{aligned}&\int_{C_0[0,T]}^{anf_q}[F(x)\delta G(x|u)+\delta F(x|u)G(x)]\mathrm{d}m(x)\\ &=-\mathrm{i}q\int_{C_0[0,T]}^{anf_q}F(x)G(x)\langle z,x\rangle\mathrm{d}m(x).\end{aligned}\tag{3.12}$$

Proof For F and G in $\mathcal{S}_\alpha$, let $L(x)=F(x)G(x)$. Then using Equation (3.11) with F being replaced by L, we obtain Equation (3.12). In fact, the left-hand side of Equation (3.12) always exists because the associated measures f and g satisfy the condition (3.9).

The following corollaries immediately follow from Theorem 3.3.

Corollary 3.1 *Let F, f and q be as in Theorem* 3.3*. Then*

$$2\int_{C_0[0,T]}^{anf_q}F(x)\delta F(x|u)\mathrm{d}m(x)=-\mathrm{i}q\int_{C_0[0,T]}^{anf_q}F^2(x)\langle z,x\rangle\mathrm{d}m(x).\tag{3.13}$$

Proof We easily obtain Equation (3.13) by replacing G with F in Equation (3.12).

Corollary 3.2 *Let F, f and q be as in Theorem* 3.3*. Then*

$$\begin{aligned}&\int_{C_0[0,T]}^{anf_q}[F(x)\delta F^{n-1}(x|u)+\delta F(x|u)F^{n-1}(x)]\mathrm{d}m(x)\\ &=-\mathrm{i}q\int_{C_0[0,T]}^{anf_q}F^n(x)\langle z,x\rangle\mathrm{d}m(x).\end{aligned}\tag{3.14}$$

Proof We easily obtain Equation (3.14) by replacing G with F^n in Equation (3.12) above and using the mathematical induction for n.

Remark 3.2 (1) As mentioned in Section 2, if $\alpha=\mathrm{i}$, then $\mathcal{S}_\alpha$ is the Banach algebra $\mathcal{S}$ introduced in [2]. Furthermore, all conditions in the previous sections and this section hold, so all results and formulas in [2] are corollaries of our results and formulas in this paper.

(2) In [2], the authors obtained the existence of the analytic Wiener integral of F in $\mathcal{S}$ on $\mathbb{C}_+$. We obtained the existence of the analytic Wiener integral of F in $\mathcal{S}_\alpha$ on Γ_α which is a subset of $\mathbb{C}_+$ because α may not be a pure imaginary number. In fact, if α is a pure imaginary number, then we can obtain the existence of the analytic Wiener integral of F in $\mathcal{S}_\alpha$ on $\mathbb{C}_+$.

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