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Approximate Controllability of Neutral Functional Differential Systems with State-Dependent Delay*

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Abstract This paper deals with the approximate controllability of semilinear neutral functional differential systems with state-dependent delay. The fractional power theory and α-norm are used to discuss the problem so that the obtained results can apply to the systems involving derivatives of spatial variables. By methods of functional analysis and semigroup theory, sufficient conditions of approximate controllability are formulated and proved. Finally, an example is provided to illustrate the applications of the obtained results.

Keywords Approximate controllability, Neutral functional differential system, State-dependent delay, Analytic semigroup, Fractional power operator
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1 Introduction

In this paper, we consider the approximate controllability of systems represented in the following semilinear neutral functional differential systems with state-dependent delay:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t)+F(t,x_t)] = -Ax(t)+Bu(t)+G(t,x_{\rho(t,x_t)}), \quad t\in[0,T], \\ x_0=\phi\in\mathscr{B}_\alpha, \end{cases} \tag{1.1}$$

where the state variable $x(\cdot)$ takes values in a Hilbert space X and the control function $u(\cdot)$ is given in the Banach space $L^2([0,T];U)$, where U is also a Hilbert space. B is a bounded linear operator from U into X. The (unbounded) linear operator $-A: D(-A)\to X$ generates an analytic semigroup $(S(t))_{t\geq 0}$, and $F,G:[0,T]\times\mathscr{B}_\alpha\to X$ are appropriate functions to be specified later. $\mathscr{B}_\alpha\subset\mathscr{B}$, and $\mathscr{B}$ is a phase space given in the next section. The notation x_t represents the history function defined by $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, and belongs to some abstract phase space $\mathscr{B}_\alpha$ described axiomatically and $\rho:[0,T]\times\mathscr{B}_\alpha\to(-\infty,T]$ is a continuous function.

The controllability theory for abstract linear control systems in an infinite-dimensional space is well-developed, and the details can be found in various papers and monographs (see [4, 16] and references therein). Several authors have extended the controllability concepts to infinite-dimensional systems represented by nonlinear evolution equations (see [20–21, 35]). Most of

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the controllability results for nonlinear infinite-dimensional control systems concern the so-called semilinear control system that consists of a linear part and a nonlinear part. Zhou [35] studied approximate controllability of an abstract semilinear control system by assuming certain inequality conditions that are dependent on the properties of the system components. Naito [20–21] studied the approximate controllability of the same system. He showed that under a range condition on the control action operator, the semilinear control system is approximately controllable. Jeong et al. [14] and Wang [30] have extended the result to retarded systems with finite delays. Yamamoto and Park [32] discussed the same problem for parabolic equations with a uniformly bounded nonlinear part. Do [6], Joshi and Sukavanam [15] discussed approximate controllability for a class of semilinear abstract equations, while Muthukumar and Rajivganthi [19] investigated the controllability problem for a stochastic nonlinear third-order dispersion equation.

Bashirov and Mahmudov [2] showed that under an appropriate condition on resolvent operators, the approximate controllability of semilinear systems is implied by the approximate controllability of its linear part. This resolvent condition is convenient for application and it has been used in many papers to study the approximate controllability for nonlinear (functional) differential equations (see, for instance, [5, 8, 24–25]). In [5], by using the Schauder fixed point theorem and the resolvent condition, Dauer and Mahmudov studied the approximate controllability and complete controllability for the following semilinear abstract control system with finite delay:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t) + F(t, x_t, u), \quad t \in [0, T], \\ x_0 = \phi. \end{cases} \tag{1.2}$$

In [9, 18, 24–27, 33–34], the authors investigated the approximate controllability for semilinear impulsive systems and fractional order (stochastic) differential systems with (state-department) delay also by using the resolvent condition.

On the other hand, neutral partial functional differential systems appear in a great many practical mathematical models, such as some structured population models and systems of lossless transmission line networks (see [11, 31]). In recent years, existence results, asymptotic properties and controllability on this type of systems have been investigated by many authors (see [3, 7, 12]). We are going to discuss the approximate controllability for neutral partial functional differential systems with state-dependent delay. State-dependent delay differential equations can be met in various practical models. Some recent applications can be found in [1, 17]. In particular, the approximate controllability of fractional functionals and integro-differential equations with state-dependent delay has been studied in [24, 26, 33–34].

A motivation of the present paper is the approximate controllability problem of the following neutral partial differential control systems:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[z(t,x) + f\Big(t, z(\cdot,x), \dfrac{\partial z}{\partial x}(\cdot,x)\Big)\Big] = \dfrac{\partial^2 z}{\partial x^2}(t,x) + Bu(t) + g\Big(t, z(\cdot,x), \dfrac{\partial z}{\partial x}(\cdot,x)\Big), \\ z(t,0) = z(t,\pi) = 0, \quad t \geq 0, \\ z(\theta,x) = \phi(\theta,x), \quad 0 \leq x \leq \pi,\ \theta \leq 0. \end{cases} \tag{1.3}$$

System (1.3) arises as a model for nonlinear heat flow in materials with fading memory. Here $z(t,x)$ represents the temperature of a conduct of the point x and time t. Evidently, this system can be treated as the abstract equation (1.1), however, the results established in [5, 24–25] become invalid for this situation, since the functions $f,\ g$ in (1.3) involve spatial derivatives. In fact, as one will see in Section 4, if we take $X = L^2([0,\pi])$, then the third variables of f and g are defined on $\mathscr{C}_{g\frac{1}{2}}$ (induced by $X_{\frac{1}{2}}$) and so the solutions can not be discussed on X like in [24–25].

In this paper, inspired by the work in [7, 28–29], we shall discuss this problem by using the fractional power operators theory and α-norm techniques, that is, we shall restrict this equation in a Banach space X_α ($\subset X$) induced by fractional power operators. We first present the induced phase space $\mathscr{B}_\alpha$ for infinite delay, through which we investigate the existence of mild solutions and then we obtain the approximate controllability for (1.1) in space X. In this manner we overcome the above mentioned difficulty successfully and the achieved controllability results can be applied to the systems involving spatial derivatives (see the system (4.1) in Section 4). Hence our obtained results are more general in applications than those of [5, 24–25]. In addition, it can be seen that our techniques can also be adopted to study the approximate controllability of other kinds of control systems (such as fractional order and stochastic systems with infinite delay) to improve the existing results in, for instance, [9, 26, 33–34]. We would also like to point out here that the resolvent condition (H_0) employed in this paper is verified readily as shown in the example in Section 4, which is more advantageous than the range condition used in [14, 30], since it seems difficult to be verified for infinitely delayed control systems.

The whole article is organized as follows: We initially present some preliminaries about analytic semigroups and phase spaces for infinite delay in Section 2. Particularly, to make them still valid in our situation, we introduce the axioms of phase spaces on the space X_α. In Section 3, we first discuss the existence of mild solutions for System (1.1) by applying the fixed point theorem, and then we study the approximate controllability of (1.1) using limit arguments. Finally, in Section 4, an example is provided to show the applications of the obtained results.

2 Preliminaries

Throughout this paper, X is a Hilbert space with norm $\|\cdot\|$. And $-A : D(-A) \to X$ is the infinitesimal generator of a compact analytic semigroup $(S(t))_{t\geq 0}$ of uniformly bounded linear operators. Let $0 \in \rho(A)$. Then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$. Hereafter we denote by X_α the Banach space $D(A^\alpha)$ normed with $\|x\|_\alpha$. Then for each $\alpha > 0$, X_α is a Banach space, $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha$ and the imbedding is compact whenever the resolvent operator of A is compact.

For the analytic semigroup $(S(t))_{t\geq 0}$, the following properties will be used (see [22]): There exist constants $M \geq 1,\ M_\alpha > 0$ such that, for $t \in [0, T]$,

$$\|S(t)\| \leq M, \tag{2.1}$$

$$\|A^\alpha S(t)\| \leq \frac{M_\alpha}{t^\alpha}. \tag{2.2}$$

To study the system (1.1), we assume that the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belong to some abstract phase space $\mathscr{B}$, which is defined axiomatically. In this article, we employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato [10] and follow the terminology used in [13]. Thus, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$. We assume that $\mathscr{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + a)$ the followings hold:

(i) x_t is in $\mathscr{B}$;

(ii) $\|x(t)\| \le H\|x_t\|_{\mathscr{B}}$;

(iii) $\|x_t\|_{\mathscr{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$.

Here $H \ge 0$ is a constant, $K, M : [0, +\infty) \to [0, +\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and H, $K(\cdot)$, $M(\cdot)$ are independent of $x(t)$.

(A_1) For the function $x(\cdot)$ in (A), x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma + a]$.

(B) The space $\mathscr{B}$ is complete.

We denote by $\mathscr{B}_\alpha$ the set of all the elements in $\mathscr{B}$ that takes values in space X_α, that is,

$$\mathscr{B}_\alpha := \{\phi \in \mathscr{B} : \ \phi(\theta) \in X_\alpha \text{ for all } \theta \le 0\}.$$

Then $\mathscr{B}_\alpha$ becomes a subspace of $\mathscr{B}$ endowed with the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ which is induced by $\|\cdot\|_{\mathscr{B}}$ through $\|\cdot\|_\alpha$. More precisely, for any $\phi \in \mathscr{B}_\alpha$, the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ is defined by $\|A^\alpha\phi(\theta)\|$, instead of $\|\phi(\theta)\|$. For example, let the phase space $\mathscr{B} = C_r \times L^p(g : X)$, $r \ge 0,\ 1 \le p < \infty$ (see [13]), which consists of all classes of functions $\phi \in (-\infty, 0] \to X$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\phi(\cdot)\|^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\mathscr{B}$ is defined by

$$\|\phi\|_{\mathscr{B}} = \sup\{\phi(\theta) : -r \le \theta \le 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p \mathrm{d}\theta\Big)^{\frac{1}{p}}.$$

Then the seminorm in $\mathscr{B}_\alpha$ is defined by

$$\|\phi\|_{\mathscr{B}_\alpha} = \sup\{\|A^\alpha\phi(\theta)\| : -r \le \theta \le 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|A^\alpha\phi(\theta)\|^p \mathrm{d}\theta\Big)^{\frac{1}{p}}.$$

See also the space $\mathscr{C}_{g,\frac{1}{2}}$ presented in Section 4. Hence, since X_α is still a Hilbert space, we will assume that the subspace $\mathscr{B}_\alpha$ also satisfies the following conditions:

(A′) If $x : (-\infty, \sigma + a) \to X_\alpha$, $a > 0$ is continuous on $[\sigma, \sigma + a)$ (in α-norm) and $x_\sigma \in \mathscr{B}_\alpha$, then for every $t \in [\sigma, \sigma + a)$ the followings hold:

(i) x_t is in $\mathscr{B}_\alpha$;

(ii) $\|x(t)\|_\alpha \le H\|x_t\|_{\mathscr{B}_\alpha}$;

(iii) $\|x_t\|_{\mathscr{B}_\alpha} \le K(t-\sigma)\sup\{\|x(s)\|_\alpha : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}_\alpha}$.

Here H, $K(\cdot)$ and $M(\cdot)$ are as in (A)(iii) above.

(A'_1) For the function $x(\cdot)$ in (A′), x_t is a $\mathscr{B}_\alpha$-valued continuous function on $[\sigma, \sigma + a]$.

(B′) The space $\mathscr{B}_\alpha$ is complete.

For any $\phi \in \mathscr{B}_\alpha$, the notation $\phi_t,\ t \le 0$, represents the function $\phi_t(\theta) = \phi(t+\theta)$, $\theta \in (-\infty, 0]$. Then, if the function $x(\cdot)$ in axiom (A') with $x_0 = \phi$, we may extend the mapping $t \to x_t$ to the whole interval $(-\infty, T]$ by setting $x_t = \phi_t$ as $t \leq 0$. On the other hand, for the function $\rho : [0, T] \times \mathscr{B}_\alpha \to (-\infty, T]$, we introduce the set

$$Z(\rho^-) = \{\rho(s,\psi) : \rho(s,\psi) \leq 0, (s,\psi) \in [0,T] \times \mathscr{B}_\alpha\}$$

and give the following hypothesis on ϕ_t: The function $t \to \phi_t$ is continuous from $Z(\rho^-)$ into $\mathscr{B}_\alpha$ and there exists a continuous and bounded function $H^\phi : Z(\rho^-) \to (0, +\infty)$ such that, for each $t \in Z(\rho^-)$,

$$\|\phi_t\|_{\mathscr{B}_\alpha} \leq H^\phi(t)\|\phi\|_{\mathscr{B}_\alpha}.$$

Then we have the following lemma, which plays an important role in our proofs in the next section.

Lemma 2.1 (see [12]) *Let $x : (-\infty, T] \to X_\alpha$ be a function such that $x_0 = \phi$ and the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous. Then*

$$\|x_s\|_{\mathscr{B}_\alpha} \leq (H_3 + H)\|\phi\|_{\mathscr{B}_\alpha} + H_2 \sup\{\|x(\theta)\|_\alpha; \theta \in [0, \max\{0, s\}]\}, \quad s \in Z(\rho^-) \cup [0, T],$$

where

$$H_1 = \sup_{t \in Z(\rho^-)} H^\phi(t), \quad H_2 = \sup_{t \in [0,T]} K(t), \quad H_3 = \sup_{t \in [0,T]} M(t).$$

The mild solution to (1.1) expressed by the semigroup is defined as the following definition.

Definition 2.1 *The function $x(\cdot\ ; \phi, u) : (-\infty, T] \to D(A^\alpha)$, $T > 0$ is said to be a mild solution to* (1.1) *with initial value $\phi \in \mathscr{B}_\alpha$ (under control $u(t)$), if it is continuous in X_α-norm on $[0, T]$ and satisfies on $(-\infty, T]$ that*

$$x(t) = \begin{cases} S(t)[\phi(0) + F(0,\phi)] - F(t, x_t) + \displaystyle\int_0^t AS(t-s)F(s, x_s)\mathrm{d}s \\ \quad + \displaystyle\int_0^t S(t-s)[Bu(s) + G(s, x_{\rho(s,x_s)})]\mathrm{d}s, \qquad t \in [0,T], \\ \phi(t), \quad -\infty < t \leq 0. \end{cases} \tag{2.3}$$

Definition 2.2 *The system* (1.1) *is said to be approximately controllable on the interval $[0, T]$, if $\mathcal{R}(T, \phi)$ is dense in X, i.e.,*

$$\overline{\mathcal{R}(T,\phi)} = X,$$

where $\mathcal{R}(T, \phi) = \{x_T(\phi, u)(0),\ u(\cdot) \in L^2([0, T], U)\}$.

We shall study the approximate controllability for (1.1) by applying the results established in [2]. For this purpose, we need to introduce the following relevant operator:

$$\Gamma_T = \int_0^T S(T-s)BB^*S^*(T-s)\mathrm{d}s,$$
$$R(\lambda, \Gamma_T) = (\lambda I + \Gamma_T)^{-1},$$

where B^* denotes the adjoint of the operator B and $S^*(t)$ denotes the adjoint semigroup of $S(t)$. Because the operator Γ_T is positive, $R(\lambda, \Gamma_T)$ is well defined. Assume that

(H_0) $\lambda R(\lambda, \Gamma_T) \to 0$ as $\lambda \to 0$ in the strong operator topology.

From Theorem 2 of [2], the hypothesis (H_0) is equivalent to the fact that the following linear control system

$$\begin{cases} x'(t) = -Ax(t) + Bu(t), & t \in [0,T], \\ x(0) = \phi(0) \end{cases} \tag{2.4}$$

is approximately controllable on $[0,T]$.

We now end this section by stating some well-known theorems which will be used in the next section.

Theorem 2.1 (Lebesgue′s Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence in space $L^1(\Omega, X)$. Suppose that the sequence converges almost everywhere to a function f and is dominated by some function $g \in L^1(\Omega, X)$ in the sense that $\|f_n(x)\| \le g(x)$, for all $n \in \mathbb{N}$ and almost all points $x \in \Omega$. Then $f \in L^1(\Omega, X)$ and*

$$\lim_{n\to\infty} \int_D \|f_n - f\| \,\mathrm{d}\mu = 0 \quad \textit{for any} \ \ D \subset \Omega.$$

Theorem 2.2 (Infinite-Dimensional Version of Ascoli-Arzela Theorem) *Let*

$$\mathscr{F} \subset C([a,b];X)$$

satisfy that

(i) *the family $\{f(t) : f \in \mathscr{F}\}$ is uniformly bounded on X, that is, there is an $M > 0$ such that $\|f(t)\| \le M$ for all $f \in \mathscr{F}$ and $t \in [a,b]$;*

(ii) *$\mathscr{F}$ is equicontinuous on the interval $[a,b]$, that is, for any $\epsilon > 0$ and any $t \in [a,b]$, there exists $\delta > 0$ such that*

$$\|f(t) - f(s)\| < \epsilon \quad \textit{for any } s \in [a,b] \textit{ satisfying } |t-s| < \delta, \textit{ and all } f \in \mathscr{F};$$

(iii) *for any $t \in [a,b]$, the set $\{f(t) : f \in \mathscr{F}\}$ is relatively compact in X.*
Then $\mathscr{F}$ is relatively compact in space $C([a,b];X)$.

Theorem 2.3 (see [23]) *Let P be a condensing operator on a Banach space X, i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \le \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(H) \subset H$ for a convex, closed and bounded set H of X, then P has a fixed point in H (where $\alpha(\cdot)$ denotes Kuratowski's measure of non-compactness).*

3 Approximate Controllability

In this section we discuss the approximate controllability for (1.1). We firstly show that, for any $x^T \in X$, by choosing proper control u^λ (for any given $\lambda \in (0,1)$), there is a mild solution $x^\lambda(\cdot;\phi,u) : (-\infty,T] \to D(A^\alpha)$ to (1.1), and then we prove that $x^\lambda(T) \to x^T$ in X.

To guarantee the existence of mild solutions, we impose the following restrictions on (1.1). Assume $\alpha \in (0,1)$.

(H_1) $B \in \mathscr{L}(U,\ X)$, i.e., B is a bounded linear operator from U to X. Let $\|B\| = N$.

(H_2) The function $F : [0,T] \times \mathscr{B}_\alpha \to D(A^{\alpha+\beta})$ is a continuous function for some $\beta \in (0,1)$ with $\alpha + \beta \le 1$, and there exists $L > 0$ such that the function $A^\beta F$ satisfies

$$\|A^\beta F(s_1,\phi_1) - A^\beta F(s_2,\phi_2)\|_\alpha \le L\ (|s_1 - s_2| + \|\phi_1 - \phi_2\|_{\mathscr{B}_\alpha}) \tag{3.1}$$

for any $0 \le s_1, s_2 \le T$, $\phi_1, \phi_2 \in \mathscr{B}_\alpha$. Moreover, there exist $L_1 > 0$ and $\gamma_1 \in (0,1)$ such that the inequality

$$\|F(t,\phi)\|_{\alpha+\beta} \le L_1(\|\phi\|_{\mathscr{B}_\alpha}^{\gamma_1} + 1) \tag{3.2}$$

holds for any $t \in [0,T]$ and $\phi \in \mathscr{B}_\alpha$.

(H$_3$) The function $G : [0,T] \times \mathscr{B}_\alpha \to X$ satisfies the following conditions:

(a) Let $x : (-\infty, T] \to X_\alpha$ be such that $x_0 = \phi$ and the restriction of $x(\cdot)$ to the interval $[0,T]$ is continuous. The function $t \to G(s, x_{\rho(t,x_t)})$ is strongly measurable on $[0,T]$ and $t \to G(s, x_t)$ is continuous on $Z(\rho^-) \cup [0,T]$ for every $s \in [0,T]$.

(b) For each $r > 0$, there exists a function $g_r \in C([0,T], \mathbb{R}^+)$ such that

$$\sup_{\|\phi\|_{\mathscr{B}_\alpha} \le r} \|G(t,\phi)\| \le g_r(t), \quad t \in [0,T],\ \phi \in \mathscr{B}_\alpha.$$

And there exist $L_2 > 0$ and $\gamma_2 \in (0,1)$ such that

$$\|g_r(\cdot)\| \le L_2(r^{\gamma_2} + 1). \tag{3.3}$$

For any given $x^T \in X$ and $\lambda \in (0,1)$, we take the control function $u^\lambda(t)$, simply denoted by $u(t)$, as

$$\begin{aligned} u(t) := {}& B^* S^*(T-t) R(\lambda, \Gamma_T)\Big\{ x^T - S(T)[\phi(0) + F(0,\phi)] + F(T, x_T) \\ & - \int_0^T AS(T-\tau)F(\tau, x_\tau)\mathrm{d}\tau - \int_0^T S(T-\tau)G(\tau, x_{\rho(\tau,x_\tau)})\mathrm{d}\tau \Big\}. \end{aligned} \tag{3.4}$$

Using this control, we define the operator $Q^\lambda : (-\infty, T] \to X_\alpha$ as follows:

$$\begin{aligned} (Q^\lambda x)(t) &= \begin{cases} S(t)[\phi(0) + F(0,\phi)] - F(t, x_t) + \displaystyle\int_0^t AS(t-s)F(s,x_s)\mathrm{d}s \\ \quad + \displaystyle\int_0^t S(t-s)[Bu(s) + G(s, x_{\rho(s,x_s)})]\mathrm{d}s, \quad t \in [0,T], \\ \phi(t), \quad t \in (-\infty, 0], \end{cases} \\ &= \begin{cases} S(t)[\phi(0) + F(0,\phi)] - F(t, x_t) + \displaystyle\int_0^t AS(t-s)F(s,x_s)\mathrm{d}s \\ \quad + \displaystyle\int_0^t S(t-s)\Big\{ BB^*S^*(T-s)R(\lambda,\Gamma_T)\Big[x^T - S(T)[\phi(0) + F(0,\phi)] \\ \quad + F(T, x_T) - \displaystyle\int_0^T AS(T-\tau)F(\tau,x_\tau)\mathrm{d}\tau \\ \quad - \displaystyle\int_0^T S(T-\tau)G(\tau, x_{\rho(\tau,x_\tau)})\mathrm{d}\tau \Big] + G(s, x_{\rho(s,x_s)})\Big\}\mathrm{d}s, \quad t \in [0,T], \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases} \end{aligned} \tag{3.5}$$

At first we prove the following theorem.

Theorem 3.1 *Let $\phi \in \mathscr{B}_\alpha$. Suppose that assumptions* (H$_0$)–(H$_3$) *are satisfied. Then for each $0 < \lambda < 1$, the equation* (1.1) *admits one mild solution on* $(-\infty, T]$ *provided that*

$$L_0 := LH_2\Big(\|A^{-\beta}\| + \frac{1}{\beta} M_{1-\beta} T^\beta\Big) < 1. \tag{3.6}$$

Proof Let $y(\cdot):(-\infty,T]\to X_\alpha$ be the function defined by

$$y(t):=\begin{cases}S(t)\phi(0), & t\geq 0,\\ \phi(t), & -\infty<t<0,\end{cases}$$

so then $y_0=\phi$, $y_t\in\mathscr{B}_\alpha$ for any $t\in[0,T]$. It is easy to see that the map $t\to y(t)$ is continuous in α-norm on $[0,T]$, and hence $t\to y_t$ is also continuous in seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$.

We define the set

$$B(r):=\{z\in C([0,T];X_\alpha):\ z(0)=0,\ \|z(t)\|_\alpha\leq r,\ 0\leq t\leq T\}.$$

Then $B(r)$ is clearly a non-empty bounded, closed and convex subset of $C([0,T];X_\alpha)$. For each $z\in B(r)$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t):=\begin{cases}z(t), & 0\leq t\leq T,\\ 0, & -\infty<t<0.\end{cases}$$

Clearly, if $x(\cdot)$ satisfies the equation (1.1), we can decompose it as $x(t)=z(t)+y(t)$, $0\leq t\leq T$, which implies $x_t=\overline{z}_t+y_t$ for every $0\leq t\leq T$ and the function $z(\cdot)$ satisfies

$$\begin{aligned}z(t)&=S(t)F(0,\phi)-F(t,\overline{z}_t+y_t)+\int_0^t AS(t-s)F(s,\overline{z}_s+y_s)\mathrm{d}s\\&\quad+\int_0^t S(t-s)[Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\mathrm{d}s,\quad 0\leq t\leq T.\end{aligned}$$

Let P^λ, P_1^λ, P_2^λ be the operators on $B(r)$ defined, respectively, by

$$\begin{aligned}(P^\lambda z)(t)&:=S(t)F(0,\phi)-F(t,\overline{z}_t+y_t)+\int_0^t AS(t-s)F(s,\overline{z}_s+y_s)\mathrm{d}s\\&\quad+\int_0^t S(t-s)[Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\mathrm{d}s,\\(P_1^\lambda z)(t)&:=S(t)F(0,\phi)-F(t,\overline{z}_t+y_t)+\int_0^t AS(t-s)F(s,\overline{z}_s+y_s)\mathrm{d}s,\\(P_2^\lambda z)(t)&:=\int_0^t S(t-s)[Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\mathrm{d}s.\end{aligned}$$

Then, the assertion that (1.1) admits a mild solution is equivalent to the fact that the operator Q^λ has a fixed point. Obviously, the fact that the operator Q^λ has a fixed point is equivalent to that $P^\lambda=P_1^\lambda+P_2^\lambda$ has a fixed point. Next we prove that P^λ has a fixed point by using Theorem 2.1. For this purpose, we will show that P^λ maps $B(r)$ into itself, and P_1^λ verifies a contraction condition while P_2^λ is a completely continuous operator.

Step 1 For $0<\lambda<1$, there exists an $r(\lambda)>0$, such that $P^\lambda(B(r))\subset B(r)$. If this is not true, then, for every $r>0$, there exist $z\in B(r)$ and $t\in[0,T]$ such that $\|(P^\lambda z)(t)\|_\alpha>r$. Then, noting that (by (2.1)–(2.2) and H_2),

$$\begin{aligned}\|u(t)\|=\Big\|&B^*S^*(T-t)R(\lambda,\Gamma_T)\Big\{x^T-S(T)[\phi(0)+F(0,\phi)]+F(T,x_T)\\&-\int_0^T AS(T-\tau)F(\tau,x_\tau)\mathrm{d}\tau-\int_0^T S(T-\tau)G(\tau,x_{\rho(\tau,x_\tau)})\mathrm{d}\tau\Big\}\Big\|\end{aligned}$$

$$
\begin{aligned}
&= \Big\| B^* S^*(T-t) R(\lambda, \Gamma_T) \Big\{ x^T - S(T)[\phi(0) + F(0,\phi)] + A^{-(\alpha+\beta)} A^{\alpha+\beta} F(T, x_T) \\
&\quad - \int_0^T A^{1-(\alpha+\beta)} S(T-\tau) A^{\alpha+\beta} F(\tau, x_\tau) \mathrm{d}\tau - \int_0^T S(T-\tau) G(\tau, x_{\rho(\tau, x_\tau)}) \mathrm{d}\tau \Big\} \Big\| \\
&\le \frac{1}{\lambda} MN \Big[\|x^T\| + M\|\phi(0) + F(0,\phi)\| + \Big(\|A^{-(\alpha+\beta)}\| + \frac{M_{1-(\alpha+\beta)} T^{\alpha+\beta}}{\alpha+\beta} \Big) L_1({r_1}^{\gamma_1} + 1) \\
&\quad + M \int_0^T \|G(\tau, x_{\rho(\tau, x_\tau)})\| \mathrm{d}\tau \Big],
\end{aligned}
$$

where $r_1 := (H_2 r + H_2 M)\|\phi(0)\|_\alpha + H_3 \|\phi\|_{\mathscr{B}_\alpha}$, we have

$$
\begin{aligned}
r &< \|(P^\lambda z)(t)\|_\alpha = \Big\| S(t)F(0,\phi) - F(t, \overline{z}_t + y_t) + \int_0^t AS(t-s) F(s, \overline{z}_s + y_s) \mathrm{d}s \\
&\quad + \int_0^t S(t-s)[Bu_{\overline{z}+y}(s) + G(s, \overline{z}_{\rho(s, \overline{z}_s + y_s)} + y_{\rho(s, \overline{z}_s+y_s)})] \mathrm{d}s \Big\|_\alpha \\
&\le M\|F(0,\phi)\|_\alpha + \|A^{-\beta}\| \ \|F(t, \overline{z}_t + y_t)\|_{\alpha+\beta} + \int_0^t \|A^{1-\beta} S(t-s)\| \ \|F(s, \overline{z}_s + y_s)\|_{\alpha+\beta} \mathrm{d}s \\
&\quad + \int_0^t \|A^\alpha S(t-s)\| [N\|u\| + \|G(s, \overline{z}_{\rho(s, \overline{z}_s+y_s)} + y_{\rho(s, \overline{z}_s+y_s)})\|] \mathrm{d}s.
\end{aligned}
$$

For any $z \in B(r)$, it follows from Lemma 2.1 that,

$$
\|z_{\rho(s, x_s)}\|_{\mathscr{B}_\alpha} \le H_2 r,
$$

and then,

$$
\begin{aligned}
\|x_{\rho(s,x_s)}\|_{\mathscr{B}_\alpha} &= \|\overline{z}_{\rho(s,\overline{z}_s+y_s)} + y_{\rho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_\alpha} \le \|\overline{z}_{\rho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_\alpha} + \|y_{\rho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_\alpha} \\
&\le H_2 r + (H_3 + H_1)\ \|\phi\|_{\mathscr{B}_\alpha} + H_2 M \|\phi(0)\|_\alpha := \ r_2.
\end{aligned}
$$

Hence we obtain by axiom (A$'$), (3.2)–(3.3) that

$$
\begin{aligned}
r &< M\|A^{-\beta}\| L(\|\phi\|_{\mathscr{B}_\alpha} + 1) + \Big(\|A^{-\beta}\| + \frac{M_{1-\beta} T^\beta}{\beta} \Big) L_1({r_1}^{\gamma_1} + 1) \\
&\quad + \frac{M_\alpha T^{1-\alpha}}{1-\alpha} \Big\{ \frac{1}{\lambda} MN^2 \Big[\|x^T\| + M\|\phi(0) + F(0,\phi)\| \\
&\quad + \Big(\|A^{-(\alpha+\beta)}\| + \frac{M_{1-(\alpha+\beta)} T^{\alpha+\beta}}{\alpha+\beta} \Big) L_1({r_1}^{\gamma_1} + 1) + MT\|g_{r_2}(\cdot)\| \Big] + \|g_{r_2}(\cdot)\| \Big\} \\
&= K_1 {r_1}^{\gamma_1} + K_2 {r_2}^{\gamma_2} + K_3,
\end{aligned}
$$

where $K_1, K_2, K_3 > 0$ are constants independent of r. Thus,

$$
r - K_1 {r_1}^{\gamma_1} - K_2 {r_2}^{\gamma_2} < K_3. \tag{3.7}
$$

However, the left side of (3.7) may go to $+\infty$ as long as $r \to +\infty$ since $\gamma_1, \gamma_2 < 1$ by our assumption. This is an contradiction. Therefore, there is an $r(\lambda) > 0$ such that P^λ maps $B(r)$ into itself.

Step 2 To prove that P_1^λ satisfies a contraction condition, we take $z_1, z_2 \in B(r)$, and then, for each $t \in [0,T]$, by axiom A$'$(iii) and (3.1),

$$
\|(P_1^\lambda z_1)(t) - (P_1^\lambda z_2)(t)\|_\alpha
$$

$$\begin{aligned}
&\leq \|F(t,\overline{z}_{1,t}+y_t)-F(t,\overline{z}_{2,t}+y_t)\|_\alpha+\Big\|\int_0^t AS(t-s)[F(s,\overline{z}_{1,s}+y_s)-F(s,\overline{z}_{2,s}+y_s)]\mathrm{d}s\Big\|_\alpha\\
&\leq \|A^{-\beta}\|\ L\ \|\overline{z}_{1,t}-\overline{z}_{2,t}\|_{\mathscr{B}_\alpha}+\int_0^t \frac{M_{1-\beta}}{(t-s)^{1-\beta}}L\ \|\overline{z}_{1,s}-\overline{z}_{2,s}\|_{\mathscr{B}_\alpha}\mathrm{d}s\\
&\leq LH_2\Big(\|A^{-\beta}\|+\frac{1}{\beta}M_{1-\beta}T^\beta\Big)\sup_{0\leq s\leq T}\|z_1(s)-z_2(s)\|_\alpha\\
&= L_0\sup_{0\leq s\leq T}\|z_1(s)-z_2(s)\|_\alpha.
\end{aligned}$$

Thus

$$\|(P_1^\lambda z_1)(\cdot)-(P_1^\lambda z_2)(\cdot)\|_{C([0,T];\ X_\alpha)}\leq L_0\|z_1(\cdot)-z_2(\cdot)\|_{C([0,T];\ X_\alpha)},$$

and so from (3.6) P_1^λ satisfies the contraction condition.

Step 3 In order to prove that the operator P_2^λ is completely continuous, we firstly show that it is continuous on $B(r)$.

Let $\{z^n\}_{n\in\mathbb{N}^+}$ be a sequence in $B(r)$ such that $z^n\to z$ $(n\to+\infty)$, and then, we have that $z^n_{\rho(s,z^n_s)}\to z_{\rho(s,z_s)}$ as $n\to+\infty$ for every $s\in Z(\rho^-)\cup[0,T]$.

Then for all $s\in Z(\rho^-)\cup[0,T]$, by (A′),

$$\|z^n_s-z_s\|_{\mathscr{B}_\alpha}\leq H_2\sup_{0\leq s+\theta\leq T}\|(z^n(s+\theta)-z(s+\theta))\|_\alpha\leq H_2\|z^n-z\|\to 0,\quad n\to\infty.$$

Hence,

$$\begin{aligned}
&\|G(s,\overline{z}^n_{\rho(s,\overline{z}^n_s+y_s)}+y_{\rho(s,\overline{z}^n_s+y_s)})-G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})\|\\
\leq{}&\|G(s,\overline{z}^n_{\rho(s,\overline{z}^n_s+y_s)}+y_{\rho(s,\overline{z}^n_s+y_s)})-G(s,\overline{z}_{\rho(s,\overline{z}^n_s+y_s)}+y_{\rho(s,\overline{z}^n_s+y_s)})\|\\
&+\|G(s,\overline{z}_{\rho(s,\overline{z}^n_s+y_s)}+y_{\rho(s,\overline{z}^n_s+y_s)})-G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})\|\\
\to{}&0,\quad n\to\infty.
\end{aligned}$$

As above it is easy to calculate that

$$\begin{aligned}
\|(P_2^\lambda z^n)(t)-(P_2^\lambda z)(t)\|_\alpha&\leq\Big\|\int_0^t A^\alpha S(t-s)B[u^n(s)-u(s)]\mathrm{d}s\Big\|\\
&\quad+\Big\|\int_0^t A^\alpha S(t-s)\big[G(s,\overline{z}^n_{\rho(s,\overline{z}^n_s+y_s)}+y_{\rho(s,\overline{z}^n_s+y_s)})\\
&\quad-G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})\big]\mathrm{d}s\Big\|\\
&\leq 2\frac{M_\alpha T^{1-\alpha}}{1-\alpha}\Big\{\frac{1}{\lambda}MN^2\Big[\|x^T\|+M\|\phi(0)+F(0,\phi)\|\\
&\quad+\Big(\|A^{-(\alpha+\beta)}\|+\frac{M_{1-(\alpha+\beta)}T^{\alpha+\beta}}{\alpha+\beta}\Big)L_1({r_1}^{\gamma_1}+1)\\
&\quad+MT\|g_{r_2}(\cdot)\|\Big]+\|g_{r_2}(\cdot)\|\Big\},
\end{aligned}$$

where u_n,u are the corresponding controls to z_n,z, respectively (determined by (3.4)). Hence from Theorem 2.1 it follows that

$$\|(P_2^\lambda z^n)(t)-(P_2^\lambda z)(t)\|_\alpha\to 0\quad\text{as } n\to+\infty,$$

i.e., P_2^λ is continuous.

Step 4 We show that the operator P_2^λ maps $B(r)$ into a relatively compact subset of $C([0,T];X_\alpha)$. Firstly, we prove that the set $V(t)=\{(P_2^\lambda z)(t),\ z\in B(r)\}$ is relatively compact in X_α for every $t\in[0,T]$. Indeed, the case when $t=0$ is trivial. Now let $t\in(0,T]$ be fixed, and then

$$\begin{aligned}(P_2^\lambda z)(t)=\int_0^t S(t-s)\Big\{&BB^*S^*(T-s)R(\lambda,\Gamma_T)\Big[x^T-S(T)[\phi(0)+F(0,\phi)]\\&-\int_0^T AS(T-\tau)F(\tau,\overline{z}_\tau+y_\tau)\mathrm{d}\tau-\int_0^T S(T-\tau)G(\tau,\overline{z}_{\rho(\tau,\overline{z}_\tau+y_\tau)}+y_{\rho(\tau,\overline{z}_\tau+y_\tau)})\mathrm{d}\tau\Big]\\&+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})\Big\}\mathrm{d}s,\quad z\in B(r).\end{aligned}$$

Observe that, for $0<\alpha<\alpha'<1$,

$$\begin{aligned}\|A^{\alpha'}(P_2^\lambda z)(t)\|&\le\int_0^t\|A^{\alpha'}S(t-s)[Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\|\mathrm{d}s\\&\le\int_0^t M_{\alpha'}(t-s)^{-\alpha'}\mathrm{d}s\cdot[N\|u\|+\|g_{r_2}(\cdot)\|]\\&\le\frac{M_{\alpha'}T^{1-\alpha'}}{1-\alpha'}\Big(\frac{1}{\lambda}MN^2\Big[\|x^T\|+M\|\phi(0)+F(0,\phi)\|\\&\quad+\Big(\|A^{-(\alpha+\beta)}\|+\frac{M_{1-(\alpha+\beta)}T^{\alpha+\beta}}{\alpha+\beta}\Big)L_1({r_1}^{\gamma_1}+1)\Big]\\&\quad+\frac{1}{\lambda}M^2N^2T\|g_{r_2}(\cdot)\|+\|g_{r_2}(\cdot)\|\Big),\end{aligned}$$

which implies that $\{A^{\alpha'}V(t)\}$ is bounded in X. Hence we infer that $V(t)$ is relatively compact in X_α by the compactness of operator $A^{-\alpha'}:X\to X_\alpha$ (the imbedding $X_{\alpha'}\hookrightarrow X_\alpha$ is compact). Hence for each $t\in[0,T]$, $V(t)$ is relatively compact in X_α.

Next we prove that the family of functions $V=\{P_2^\lambda(z)(\cdot):z\in B(r)\}$ is equi-continuous on interval $(0,T]$. Let $0<t_1<t_2\le T$, and then

$$\begin{aligned}&\|(P_2^\lambda z)(t_2)-(P_2^\lambda z)(t_1)\|_\alpha\\=&\Big\|\int_0^{t_1}[S(t_2-s)-S(t_1-s)][Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\mathrm{d}s\\&+\int_{t_1}^{t_2}S(t_2-s)[Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\mathrm{d}s\Big\|_\alpha\\\le&\int_0^{t_1-\epsilon}\|[S(t_2-s)-S(t_1-s)][Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\|_\alpha\mathrm{d}s\\&+\int_{t_1-\epsilon}^{t_1}\|[S(t_2-s)-S(t_1-s)][Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\|_\alpha\mathrm{d}s\\&+\int_{t_1}^{t_2}\|A^\alpha S(t_2-s)[Bu(s)+G(s,\overline{z}_{\rho(s,\overline{z}_s+y_s)}+y_{\rho(s,\overline{z}_s+y_s)})]\|\mathrm{d}s\end{aligned}$$

or

$$\begin{aligned}&\|(P_2^\lambda z)(t_2)-(P_2^\lambda z)(t_1)\|_\alpha\\\le&\|S(t_2-t_1+\epsilon)-S(\epsilon)\|\int_0^{t_1-\epsilon}\|A^\alpha S(t_1-s-\epsilon)\|\end{aligned}$$

$$
\begin{aligned}
&\cdot \|Bu(s) + G(s, \overline{z}_{\rho(s,\overline{z}_s+y_s)} + y_{\rho(s,\overline{z}_s+y_s)})\|\mathrm{d}s \\
&+ \int_{t_1-\epsilon}^{t_1} \|A^\alpha[S(t_2-s) - S(t_1-s)][Bu(s) + G(s, \overline{z}_{\rho(s,\overline{z}_s+y_s)} + y_{\rho(s,\overline{z}_s+y_s)})]\|\mathrm{d}s \\
&+ \frac{M_\alpha}{1-\alpha}[N\|u\| + \|g_{r_2}(\cdot)\|](t_2-t_1)^{1-\alpha} \\
\leq\ & \frac{M_\alpha}{1-\alpha}[N\|u\| + \|g_{r_2}(\cdot)\|](t_1-\epsilon)^{1-\alpha}\|S(t_2-t_1+\epsilon) - S(\epsilon)\| \\
&+ \frac{M_\alpha}{1-\alpha}[N\|u\| + \|g_{r_2}(\cdot)\|][(t_2-t_1-\epsilon)^{1-\alpha} - (t_2-t_1)^{1-\alpha} + \epsilon^{1-\alpha}] \\
&+ \frac{M_\alpha}{1-\alpha}[N\|u\| + \|g_{r_2}(\cdot)\|](t_2-t_1)^{1-\alpha},
\end{aligned}
$$

where $\varepsilon > 0$ is sufficiently small. Since $\{S(t)\}_{t\geq 0}$ is strongly continuous, and the compactness of $S(t),\ t > 0$ implies the continuity in the uniform operator topology, it follows that $\|(P_2^\lambda z)(t_2) - (P_2^\lambda z)(t_1)\|_\alpha$ tends to zero as $t_2 - t_1 \to 0$, and hence $V = \{(P_2^\lambda z)(\cdot), z \in B(r)\}$ is equicontinuous. Accordingly, from Theorem 2.2, P_2^λ is a completely continuous operator on $\mathscr{B}_\alpha$.

These arguments enable us to infer that $P^\lambda = P_1^\lambda + P_2^\lambda$ is a condense mapping on $B(r)$, and by Theorem 2.3, we conclude that there exists a fixed point z^λ for P^λ on $B(r)$. Let $x^\lambda(t) = \overline{z}^\lambda(t) + y(t),\ t \in (-\infty, T]$, and then $x^\lambda(\cdot)$ is a fixed point of the operator Q^λ, which implies that equation (1.1) admits a mild solutions $x^\lambda(\cdot)$ on $(-\infty, T]$. The proof is completed.

Theorem 3.2 *Assume that the assumptions of Theorem* 3.1 *are satisfied with functions* $F(\cdot,\cdot)$ *and* $G(\cdot,\cdot)$ *uniformly bounded, and additionally suppose that the hypothesis* (H_0) *holds. Then* (1.1) *is approximately controllable on* $[0, T]$.

Proof Let $x^\lambda(\cdot)$ be a fixed point of Q^λ on $B(r)$, and then, as one can see above, x^λ is a mild solution to (1.1) on $(-\infty, T]$ under the control

$$
\begin{aligned}
u^\lambda(t) =& B^*S^*(T-t)R(\lambda, \Gamma_T)\Big[x^T - S(T)[\phi(0) + F(0,\phi)] + F(T, x_T^\lambda) \\
&- \int_0^T AS(T-\tau)F(\tau, x_\tau^\lambda)\mathrm{d}\tau - \int_0^T S(T-\tau)G(\tau, x^\lambda_{\rho(\tau, x_\tau^\lambda)})\mathrm{d}\tau\Big]
\end{aligned}
$$

and satisfies

$$
\begin{aligned}
x^\lambda(T) &= S(T)[\phi(0) + F(0,\phi)] - F(T, x_T^\lambda) + \int_0^T AS(T-s)F(s, x_s^\lambda)\mathrm{d}s \\
&\quad + \int_0^T S(T-s)[Bu^\lambda(s) + G(s, x^\lambda_{\rho(s,x_s^\lambda)})]\mathrm{d}s \\
&= S(T)[\phi(0) + F(0,\phi)] - F(T, x_T^\lambda) + \int_0^T AS(T-s)F(s, x_s^\lambda)\mathrm{d}s \\
&\quad + \int_0^T S(T-s)\Big\{BB^*S^*(T-s)R(\lambda,\Gamma_T)\Big[x^T - S(T)[\phi(0) + F(0,\phi)] + F(T, x_T^\lambda) \\
&\quad - \int_0^T AS(T-\tau)F(\tau, x_\tau^\lambda)\mathrm{d}\tau - \int_0^T S(T-\tau)G(\tau, x^\lambda_{\rho(\tau,x_\tau^\lambda)})\mathrm{d}\tau\Big] + G(s, x^\lambda_{\rho(s,x_s^\lambda)})\Big\}\mathrm{d}s \\
&= x^T + [\Gamma_T R(\lambda, \Gamma_T) - I]\Big\{x^T - S(T)[\phi(0) + F(0,\phi)] + F(T, x_T^\lambda) \\
&\quad - \int_0^T AS(T-s)F(s, x_s^\lambda)\mathrm{d}s - \int_0^T S(T-s)G(s, x^\lambda_{\rho(s,x_s^\lambda)})\mathrm{d}s\Big\}
\end{aligned}
$$

$$= x^T - \lambda R(\lambda, \Gamma_T)\Big\{x^T - S(T)[\phi(0) + F(0,\phi)] + F(T, x_T^\lambda) - \int_0^T AS(T-s)F(s, x_s^\lambda)\mathrm{d}s$$
$$- \int_0^T S(T-s)G(s, x^\lambda_{\rho(s,x_s^\lambda)})\mathrm{d}s\Big\}. \tag{3.8}$$

From the assumption we see that the sequences $\{F(s, x_s^\lambda) : \lambda \in (0,1)\}$ and $\big\{G(s, x^\lambda_{\rho(s,x_s^\lambda)}) : \lambda \in (0,1)\big\}$ are bounded (uniformly in λ) in X. Hence there are subsequences, still denoted by $F(s, x_s^\lambda)$ and $G(s, x^\lambda_{\rho(s,x_s^\lambda)})$, that weakly converge to, say, $f(s)$ and $g(s)$ in X for each $s \in [0,T]$, respectively.

Then, by the compactness of the semigroup again, it follows immediately that

$$\|S(T-s)[G(s, x^\lambda_{\rho(s,x_s^\lambda)}) - g(s)]\| \to 0 \quad \text{for all } s \in [0,T),$$

which implies

$$\Big\|\int_0^T S(T-s)[G(s, x^\lambda_{\rho(s,x_s^\lambda)}) - g(s)]\mathrm{d}s\Big\| \to 0$$

as $\lambda \to 0^+$. On the other hand, it is not difficult to show that the map $f(t) \to \int_0^t AS(t-s)f(s)\mathrm{d}s : L^2([0,T], X_{\alpha+\beta}) \to C([0,T], X)$ is compact, and we hence obtain

$$\Big\|\int_0^T AS(T-s)[F(s, x_s^\lambda) - f(s)]\mathrm{d}s\Big\|$$
$$= \Big\|\int_0^T A^{1-(\alpha+\beta)}S(T-s)[A^{\alpha+\beta}F(s, x_s^\lambda) - A^{\alpha+\beta}f(s)]\mathrm{d}s\Big\| \to 0$$

as $\lambda \to 0^+$. In addition, because the map $t \to F(t, x_t) : [0,T] \to C([0,T], X)$ is also compact, we may assume that there is an $F_T \in X$ such that

$$F(T, x_T^\lambda) \to F_T, \quad \lambda \to 0^+.$$

Thus by (3.8) we have that

$$\|x^\lambda(T) - x^T\|$$
$$= \Big\|\lambda R(\lambda, \Gamma_T)\Big\{x^T - S(T)[\phi(0) + F(0,\phi)] + F(T, x_T^\lambda)$$
$$- \int_0^T AS(T-s)F(s, x_s^\lambda)\mathrm{d}s - \int_0^T S(T-s)G(s, x^\lambda_{\rho(s,x_s^\lambda)})\mathrm{d}s\Big\}\Big\|$$
$$\le \Big\|\lambda R(\lambda, \Gamma_T)\Big\{x^T - S(T)[\phi(0) + F(0,\phi)] + F_T - \int_0^T AS(T-s)f(s)\mathrm{d}s$$
$$- \int_0^T S(T-s)g(s)\mathrm{d}s\Big\}\Big\| + \|\lambda R(\lambda, \Gamma_T)[F(T, x_T^\lambda) - F_T]\|$$
$$+ \Big\|\lambda R(\lambda, \Gamma_T)\int_0^T AS(T-s)[F(s, x_s^\lambda) - f(s)]\mathrm{d}s\Big\|$$
$$+ \Big\|\lambda R(\lambda, \Gamma_T)\int_0^T S(T-s)[G(s, x^\lambda_{\rho(s,x_s^\lambda)}) - g(s)]\mathrm{d}s\Big\| \to 0 \quad \text{as } \lambda \to 0^+. \tag{3.9}$$

So there holds $x^\lambda(T) \to x^T$ in X, and consequently we obtain the approximate controllability of (1.1). The proof is completed.

4 An Example

In order to apply Theorem 3.1 and Theorem 3.2, we consider the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[z(t,x)+\displaystyle\int_{-\infty}^{t}\int_0^{\pi} a\Big(s-t,x,z(s,y)+\dfrac{\partial}{\partial y}z(s,y)\Big)\mathrm{d}y\mathrm{d}s\Big]=\dfrac{\partial^2}{\partial x^2}z(t,x)+Bu(t) \\ \quad +b\Big(s-t,x,z(t-\sigma(\|z(t,x)\|),x),\dfrac{\partial z}{\partial x}(t-\sigma(\|z(t,x)\|),x)\Big), \\ \qquad\qquad\qquad\qquad 0\le t\le T,\ \ 0\le x\le\pi, \\ z(t,0)=z(t,\pi)=0, \quad 0\le t\le T, \\ z(\theta,x)=\phi(\theta,x), \quad \theta\le 0,\ \ 0\le x\le \pi, \end{cases} \tag{4.1}$$

where the functions a and b will be described below.

Let $X=L^2([0,\pi])$ and operator A be defined by

$$Af=-f''$$

with the domain

$$D(A)=H_0^2([0,\pi])=\{f(\cdot)\in X:\ \ f',f''\in X,\ f(0)=f(\pi)=0\}.$$

Then $-A$ generates a strongly continuous semigroup $(S(t))_{t\ge 0}$ which is analytic, compact and self-adjoint. Furthermore, $-A$ has a discrete spectrum, and the eigenvalues are $-n^2$, $n\in\mathbb{N}^+$, with the corresponding normalized eigenvectors $z_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$. Then the following properties hold:

(a) If $f\in D(A)$, then

$$Af=\sum_{n=1}^{\infty}n^2\langle f,z_n\rangle z_n.$$

(b) For every $f\in X$,

$$S(t)f=\sum_{n=1}^{\infty}\mathrm{e}^{-n^2t}\langle f,z_n\rangle z_n,$$
$$A^{-\frac12}f=\sum_{n=1}^{\infty}\frac{1}{n}\langle f,z_n\rangle z_n.$$

In particular, $\|S(t)\|\le \mathrm{e}^{-t}$, $\|A^{-1}\|=\|A^{-\frac12}\|=1$.

(c) The operator $A^{\frac12}$ is given by

$$A^{\frac12}f=\sum_{n=1}^{\infty}n\langle f,z_n\rangle z_n$$

on the space $D(A^{\frac12})=\big\{f(\cdot)\in X,\ \sum\limits_{n=1}^{\infty}n\langle f,z_n\rangle z_n\in X\big\}$.

Define an infinite-dimensional space U by

$$U=\Big\{u=\sum_{n=2}^{+\infty}u_nz_n(x)\Big|\sum_{n=2}^{+\infty}u_n<\infty\Big\},$$

and the norm in U is defined by $\|u\| = \big(\sum\limits_{n=2}^{+\infty} u_n^2\big)^{\frac{1}{2}}$. Now define the linear continuous mapping B from U to X as

$$Bu = 2u_2 z_1(x) + \sum_{n=2}^{+\infty} u_n z_n(x) \quad \text{for } u = \sum_{n=2}^{+\infty} u_n z_n(x) \in U.$$

From [4], the linear system corresponding to (4.1) is approximately controllable.

Here we take $\alpha = \beta = \frac{1}{2}$ and the phase space $\mathscr{B} = \mathscr{C}_g$, where the space $\mathscr{C}_g$ is defined as: Let g be a continuous function on $(-\infty, 0]$ with $g(0) = 1$, $\lim\limits_{\theta \to -\infty} g(\theta) = \infty$, and g is decreasing on $(-\infty, 0]$, so then

$$\mathscr{C}_g = \Big\{\phi \in C((-\infty, 0]; X) : \sup_{s \le 0} \frac{\|\phi(s)\|}{g(s)} < \infty\Big\},$$

and the norm is defined by, for $\phi \in \mathscr{C}_g$,

$$|\phi|_g = \sup_{s \le 0} \frac{\|\phi(s)\|}{g(s)}.$$

It is known that $\mathscr{C}_g$ satisfies the axioms (A), (A_1), and (B) (see [13]). Further, the subspace $\mathscr{C}_{g,\frac{1}{2}}$ is defined by

$$\mathscr{C}_{g,\frac{1}{2}} = \Big\{\phi \in C((-\infty, 0]; X_{\frac{1}{2}}) : \sup_{s \le 0} \frac{\|A^{\frac{1}{2}}\phi(s)\|}{g(s)} < \infty\Big\},$$

endowed with the norm $|\phi|_{g,\frac{1}{2}} = \sup\limits_{s \le 0} \frac{\|A^{\frac{1}{2}}\phi(s)\|}{g(s)}$. Clearly, $\mathscr{C}_{g\frac{1}{2}}$ satisfies correspondingly the axioms (A'), (A_1'), and (B'), and we may choose a proper g such that $H, K(\cdot), M(\cdot) \le 1$ (see [13]). Thus we can obtain $H_2 \le 1,\ H_3 \le 1$.

We assume that the following conditions hold:

(i) The function $a(\cdot,\cdot,\cdot) \in C^2$ with $a(\cdot, 0, \cdot) = a(\cdot, \pi, \cdot) \equiv 0$, and there is a function $a_1(\cdot,\cdot) \in L^1((-\infty, 0] \times \mathbb{R},\ \mathbb{R}^+)$ and a constant $\gamma_1 \in (0, 1)$ such that, for $\theta \in (-\infty, 0],\ x, y \in \mathbb{R}$,

$$\Big|\frac{\partial^2}{\partial x^2} a(\theta, x, y_2) - \frac{\partial^2}{\partial x^2} a(\theta, x, y_1)\Big| < a_1(\theta,\ x)|y_2 - y_1|,$$
$$\Big|\frac{\partial^2}{\partial x^2} a(\theta, x, y)\Big| < a_1(\theta,\ x)|y|^{\gamma_1}$$

and

$$\int_0^\pi \int_{-\infty}^0 (g(\theta))^{\gamma_1} a_1(\theta,\ x) \mathrm{d}\theta \mathrm{d}x < \infty.$$

(ii) The functions $b : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, \infty) \to [0, \infty)$ are continuous and there is a constant $\gamma_2 \in (0, 1)$ such that, for any $\phi_1, \phi_2 \in \mathscr{C}_{g\frac{1}{2}}$,

$$\Big(\int_0^\pi |b(\theta, \phi_1(\theta)(x), \phi_1(\theta)(x))|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \le l_2(\|\phi_1\|_{\mathscr{C}_{g\frac{1}{2}}}^{\gamma_2} + \|\phi_1\|_{\mathscr{C}_{g\frac{1}{2}}}^{\gamma_2} + 1).$$

(iii) The function ϕ defined by $\phi(\theta)(x) = \phi(\theta, x)$ belongs to $\mathscr{C}_{g,\frac{1}{2}}$.

Now define the abstract functions F on $\mathscr{C}_{g,\frac{1}{2}}$, G on $[0, T] \times \mathscr{C}_{g,\frac{1}{2}}$ and the state-dependent function $\rho(\cdot,\cdot) : [0, T] \times \mathscr{C}_{g,\frac{1}{2}} \to (-\infty, T]$ by

$$F(\phi)(x) = \int_{-\infty}^0 \int_0^\pi a\Big(\theta, x, \phi(\theta)(y) + \phi(\theta)'(y)\Big) \mathrm{d}y \mathrm{d}\theta,$$

$$G(t,\phi)(x) = b(t,\phi(x),\phi'(x)),$$
$$\rho(t,\phi) = t - \sigma(\|\phi(0)\|.$$

Then the system (4.1) is rewritten in the abstract form (1.1), and condition (i) implies that $R(F) \subset D(A)$, since

$$\begin{aligned}\langle F(\phi), z_n\rangle &= -\frac{1}{n}\Big\langle \int_{-\infty}^{0}\int_0^{\pi} \frac{\partial}{\partial x} a(\theta,x,\phi(\theta)(y)+\phi(\theta)'(y))\mathrm{d}y\mathrm{d}\theta, \widetilde{z_n}(x)\Big\rangle \\ &= \frac{1}{n^2}\Big\langle \int_{-\infty}^{0}\int_0^{\pi} \frac{\partial^2}{\partial x^2} a(\theta,x,\phi(\theta)(y)+\phi(\theta)'(y))\mathrm{d}y\mathrm{d}\theta, z_n(x)\Big\rangle,\end{aligned}$$

where $\widetilde{z_n}(x) = \sqrt{\frac{2}{\pi}}\cos(nx)$, $n = 1, 2, \cdots$. Noting that, for any $\theta \in (-\infty, 0]$,

$$\begin{aligned}\|\phi_2(\theta)(x) - \phi_1(\theta)(x)\|^2 &= \sum_{n=1}^{\infty} \langle \phi_2 - \phi_1, z_n\rangle^2 \\ &\le \sum_{n=1}^{\infty} n^2 \langle \phi_2 - \phi_1, z_n\rangle^2 \\ &\le \|\phi_2(\theta)(x) - \phi_1(\theta)(x)\|_{\frac{1}{2}}{}^2\end{aligned}$$

and

$$\begin{aligned}\|\phi_2(\theta)'(x) - \phi_1(\theta)'(x)\|^2 &= \sum_{n=1}^{\infty} \langle \phi_2' - \phi_1', z_n\rangle^2 \\ &= \sum_{n=1}^{\infty} \langle \phi_2 - \phi_1, z_n'\rangle^2 \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^2 \langle \phi_2 - \phi_1, z_n\rangle\langle \phi_2 - \phi_1, z_m\rangle\langle -z_n'', z_m'\rangle \\ &\le \|\phi_2(\theta)(x) - \phi_1(\theta)(x)\|_{\frac{1}{2}}{}^2,\end{aligned}$$

we see

$$|\phi_2(\cdot) - \phi_1(\cdot)|_g \le |\phi_2(\cdot) - \phi_1(\cdot)|_{g,\frac{1}{2}},$$
$$|\phi_2(\cdot)' - \phi_1(\cdot)'|_g \le |\phi_2(\cdot) - \phi_1(\cdot)|_{g,\frac{1}{2}}.$$

Thus, the condition (i) ensures that $AF(\cdot)$ satisfies the Lipschitz continuous on $\mathscr{C}_{g,\frac{1}{2}}$. In fact, one has

$$\begin{aligned}&\|AF(\phi_2) - AF(\phi_1)\|^2 \\ &= \int_0^{\pi} \Big| \int_{-\infty}^{0}\int_0^{\pi} \Big[\frac{\partial^2 a}{\partial x^2}(\theta,x,\phi_2(\theta)(y)+\phi_2(\theta)'(y)) - \frac{\partial^2 a}{\partial x^2}(\theta,x,\phi_1(\theta)(y)+\phi_1(\theta)'(y))\Big]\mathrm{d}y\mathrm{d}\theta\Big|^2\mathrm{d}x \\ &\le \int_0^{\pi}\Big[\int_{-\infty}^{0}\int_0^{\pi} a(\theta,x)(|\phi_2(\theta)(y) - \phi(\theta)(y)| + |\phi_2(\theta)'(y)) - \phi_1(\theta)'(y))|)\mathrm{d}y\mathrm{d}\theta\Big]^2\mathrm{d}x \\ &\le \int_0^{\pi}\Big[\int_{-\infty}^{0} g(\theta)a(\theta,x)\Big(\frac{\|\phi_2(\theta) - \phi_1(\theta)\|}{g(\theta)} + \frac{\|\phi_2(\theta)') - \phi_1(\theta)')\|}{g(\theta)}\Big)\mathrm{d}\theta\Big]^2\mathrm{d}x\end{aligned}$$

$$\leq \pi \int_0^\pi \Big[\int_{-\infty}^0 g(\theta)a(\theta,x)\mathrm{d}\theta\Big]^2 \mathrm{d}x(|\phi_2-\phi_1|_g + |\phi_2'-\phi_1'|_g)^2$$

$$\leq 2\pi \int_0^\pi \Big[\int_{-\infty}^0 g(\theta)a(\theta,x)\mathrm{d}\theta\Big]^2 \mathrm{d}x\,|\phi_2-\phi_1|^2_{g,\frac{1}{2}}\,,$$

which shows the claim. Observing that F and G also verify (3.2) and (3.3) due to the assumptions (i) and (ii), we see that hypotheses (H_2) and (H_3) are satisfied respectively. Consequently, Theorem 3.2 is now well applied and the system (4.1) is approximate controllable on $[0,T]$ provided that (3.6) is satisfied.

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