Chin. Ann. Math.
37B(2), 2016, 309–322
DOI: 10.1007/s11401-016-0945-9

Chinese Annals of Mathematics, Series B
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On a Quasilinear Elliptic Equation with Superlinear Nonlinearities*

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Abstract This work is devoted to studying a quasilinear elliptic boundary value problem with superlinear nonlinearities in a weighted Sobolev space in a domain of $\mathbb{R}^N$. Based on the Galerkin method, Brouwer's theorem and the weighted compact Sobolev-type embedding theorem, a new result about the existence of solutions is revealed to the problem.

Keywords Weighted Sobolev space, Superlinear, Quasilinear elliptic equation
2000 MR Subject Classification 35H30, 35J50, 65L60

1 Introduction

Let $\Omega \in \mathbb{R}^N (N \geq 1)$ be open (possibly unbounded) and consider a weak solution in $H^1_{p,q,\rho}(\Omega, \Gamma)$ (see Section 2) to the following quasilinear elliptic problem:

$$\begin{cases} \mathcal{Q}u = [\lambda_{j_0} u + f(x,u)]\rho - G, \quad x \in \Omega, \\ u \in H^1_{p,q,\rho}(\Omega, \Gamma), \end{cases} \tag{1.1}$$

where λ_{j_0} is the j_0th eigenvalue of $\mathcal{L}$ ((2.3) below) of multiplicity J_0, and $\mathcal{Q}$ is a singular quasilinear elliptic operator defined by

$$\mathcal{Q}u = -\sum_{i=1}^{N} D_i[p_i^{\frac{1}{2}}(x) A_i(x,u,Du)] + qB_0(x)u. \tag{1.2}$$

The nonlinear part $f(x,u)$ in (1.1) satisfies certain superlinear conditions.

In fact, there have been many results about quasilinear elliptic equations, under the conditions of which the nonlinearities satisfy sublinear or linear growth in weighted Sobolev spaces. One can refer to [1–8]. For example, Shapiro [7] investigated the problem (1.1) aiming at the first eigenvalue λ_1 whose corresponding eigenfunction space has better characters with superlinear nonlinearities. In [8], Jia and Huang proved the existence of solutions to elliptic equations similar to (1.1) for the semilinear operator with superlinear nonlinearities. Compared with the

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*This work was supported by the National Natural Science Foundation of China (No. 11171220), the Shanghai Leading Academic Discipline Project (No. XTKX2012) and the Hujiang Foundation of China (No. B14005).

previous work, this paper is supposed to gain an existence result of (1.1) for any eigenvalue $\lambda_{j_0}(j_0 \geq 1)$, where the nonlinearties of quasilinear elliptic equations satisfy certain superlinear growth conditions. Therefore we have to overcome the difficulties that the eigenvalue λ_{j_0} may be of multiplicity J_0 and that the corresponding eigenfunctions may be sign-changing.

(1.1) is one of the most useful sets of p-Laplacian equations (see [9]). In fact, there are some serious ones. It appears that certain nonlinear mathematical models lead to differential equations with the p-Laplacian. One of them describing the behavior of compressible fluid in a homogeneous isotropic rigid porous medium is presented below. But some purely mathematical properties of the p-Laplacian also seem to be a challenge for nonlinear analysis and their study leads to the development of new methods and approaches.

Our main ideas prove the existence of $\{u_n\}$ in the finite dimensional space S_n spanned by $\{\phi_1, \phi_2, \cdots, \phi_n\}$ via Brouwer's fixed point theorem at first. Then we obtain the uniform boundedness of $\{u_n\}$ under the norm of $\|\cdot\|_{p,q,\rho}$ by virtue of a new compactly embedding theorem established by Shapiro in [7]. Finally, by the projective technique, the conclusion for existence of solutions to (1.1) in S_n could be extended to $H^1_{p,q,\rho}(\Omega,\Gamma)$. To overcome the difficulties brought by λ_{j_0}, $\circledast$-relationship is put forward in Definition 2.3 between the operators Q and L, which is different from [7] and [8].

This paper is organized as follows. In Section 2, we introduce some necessary assumptions and the main results. In Section 3, four fundamental lemmas are established. In Section 4, the proofs of main results are given. Section 5 illustrates an example to cover Definition 2.3.

2 Assumptions and Main Results

In this section, we introduce some assumptions and give the main results in this paper.

Let $\Gamma \subset \partial\Omega$ be a fixed closed set (it may be an empty set) and $\rho(x), p_i(x)$ $(i = 1, \cdots, N)$, $q(x) \in C^0(\Omega)$ be weight functions. $q(x)$ is nonnegative (maybe identically zero). Denote by $p(x)$ the vector function $(p_1(x), p_2(x), \cdots, p_N(x))$.

Consider the following pre-Hilbert spaces

$$C^0_\rho(\Omega) = \Big\{ u \in C^0(\Omega) \Big| \int_\Omega |u|^2 \rho < \infty \Big\}$$

with inner product $\langle u, v\rangle_\rho = \int_\Omega uv\rho$, $\forall u, v \in C^0_\rho(\Omega)$, and

$$C^1_{p,q,\rho}(\Omega,\Gamma) = \Big\{ u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \Big| u(x) = 0, \forall x \in \Gamma; \int_\Omega \Big[\sum_{i=1}^N |D_i u|^2 p_i + u^2(q+\rho) \Big] < \infty \Big\}$$

with inner product

$$\langle u, v\rangle_{p,q,\rho} = \int_\Omega \Big(\sum_{i=1}^N p_i D_i u D_i v + (q+\rho)uv \Big) \tag{2.1}$$

$\forall u, v \in C^1_{p,q,\rho}(\Omega)$ and $D_i u = \dfrac{\partial u}{\partial x_i}$, $i = 1, \cdots, N$. Let $L^2_\rho(\Omega)$ be the Hilbert space obtained through the completion of $C^0_\rho(\Omega)$ by using the method of Cauchy sequences with respect to the norm $\|u\|_\rho = \langle u, u\rangle_\rho^{\frac{1}{2}}$, and $H^1_{p,q,\rho}(\Omega,\Gamma)$ be the completion of the space $C^1_{p,q,\rho}(\Omega,\Gamma)$ with the norm $\|u\|_{p,q,\rho} = \langle u, u\rangle_{p,q,\rho}^{\frac{1}{2}}$. Similarly, we may have $L^2_{p_i}(\Omega)$ $(i = 1, \cdots, N)$ and $L^2_q(\Omega)$. Consequently, (2.1) may lead to

$$\langle u, v\rangle_{p,q,\rho} = \sum_{i=1}^{N} \langle D_i u, D_i v\rangle_{p_i} + \langle u, v\rangle_\rho + \langle u, v\rangle_q. \tag{2.2}$$

Definition 2.1 *For the quasilinear differential operator $\mathcal{Q}$, the two-form is*

$$\mathcal{Q}(u, v) = \int_\Omega \sum_{i=1}^{N} p_i^{\frac{1}{2}} A_i(x, u, Du) D_i v + \langle B_0 u, v\rangle_q, \quad \forall u, v \in H^1_{p,q,\rho}(\Omega,\Gamma). \tag{2.3}$$

For the linear differential operator

$$\mathcal{L}u = -\sum_{i=1}^{N} D_i(p_i D_i u) + qu, \tag{2.4}$$

the two-form is

$$\mathcal{L}(u, v) = \int_\Omega \sum_{i=1}^{N} p_i D_i u D_i v + \langle u, v\rangle_q, \quad \forall u, v \in H^1_{p,q,\rho}(\Omega,\Gamma). \tag{2.5}$$

Remark 2.1 Observing (2.2) and the two-form of $\mathcal{L}$, we get

$$\mathcal{L}(u, v) + \langle u, v\rangle_\rho = \langle u, v\rangle_{p,q,\rho}. \tag{2.6}$$

Definition 2.2 (Ω,Γ) *is a new-*$\mathrm{V_L}$ *region if the following conditions* ($\mathrm{V_L}$-1)–($\mathrm{V_L}$-5) *hold:*

($\mathrm{V_L}$-1) *There exists a complete orthonormal system $\{\varphi_n\}_{n=1}^\infty$ in L^2_ρ. Also $\varphi_n \in H^1_{p,q,\rho}(\Omega,\Gamma)$ $\cap C^2(\Omega)$, $\forall n$.*

($\mathrm{V_L}$-2) *There exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$, corresponding to the orthonormal sequence $\{\varphi_n\}_{n=1}^\infty$, and satisfying $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \to \infty$ as $n \to \infty$, such that $\mathcal{L}(\varphi_n, v) = \lambda_n \langle \varphi_n, v\rangle_\rho$, $\forall v \in H^1_{p,q,\rho}(\Omega,\Gamma)$.*

($\mathrm{V_L}$-3) $\Omega = \Omega_1 \times \cdots \times \Omega_N$, *where $\Omega_i \subset \mathbb{R}$ is an open set for $i = 1, \cdots, N$.*

($\mathrm{V_L}$-4) *For each $p_i(x)$ and $\rho(x)$ in* ($\mathrm{V_L}$-1)–($\mathrm{V_L}$-2), *associated with each Ω_i there are positive functions $p_i^*(s), \rho_i^*(s) \in C^0(\Omega_i)$ satisfying $\int_{\Omega_i} [p_i^*(s) + \rho_i^*(s)] \mathrm{d}s < \infty$, and $\rho(x) = \rho_1^*(x_1) \cdots \rho_N^*(x_N)$, $p_i(x) = \rho_1^*(x_1) \cdots \rho_{i-1}^*(x_{i-1}) p_i^*(x_i) \rho_{i+1}^*(x_{i+1}) \cdots \rho_N^*(x_N)$ for $i = 1, \cdots, N$.*

($\mathrm{V_L}$-5) *For each Ω_i, p_i^*, ρ_i^* $(i = 1, \cdots, N)$, there exists $h_i \in C^0(\Omega_i) \cap L^\theta_{\rho_i^*}(\Omega_i)$ for $2 < \theta < \infty$ with the property*

$$|\Phi(t)| \le h_i(t) \|\Phi\|_{p_i^*, \rho_i^*}, \quad \forall \Phi \in C^1(\Omega_i),\ \forall t \in \Omega_i,$$

where $\|\Phi\|^2_{p_i^*, \rho_i^*} = \int_{\Omega_i} \Big[p_i^*(t) \Big|\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}\Big|^2 + \rho_i^*(t) \Phi^2(t)\Big] \mathrm{d}t$.

There are many examples to illustrate new-$\mathrm{V_L}$ region. One can refer to [7] and [10].

Remark 2.2 From ($\mathrm{V_L}$-3) and ($\mathrm{V_L}$-4), it is easy to see that $\rho(x), p_i(x)$ are positive and

$$\int_\Omega \rho(x) < \infty, \quad \int_\Omega p_i(x) < \infty, \quad i = 1, \cdots, N. \tag{2.7}$$

Definition 2.3 *$\mathcal{Q}$ is $\circledast$-related to $\mathcal{L}$ if the following condition is satisfied:*

$$\lim_{\|u\|_{p,q,\rho}\to\infty} \frac{\mathcal{Q}(u,v)-\Lambda_{j_0}\mathcal{L}(u,v)}{\|u\|_{p,q,\rho}} = 0, \quad \text{uniformly for } \|v\|_{p,q,\rho} \le 1,$$

where $\Lambda_{j_0} = \frac{\lambda_{j_0}}{\lambda_\dagger}$. Here $\lambda_\dagger$ is a positive constant not greater than the first eigenvalue of $\mathcal{L}$.

We make the following assumptions concerning A_i $(i = 1, \cdots, N)$ and B_0:

($\mathcal{Q}$-1) $A_i(x,s,\xi)\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfies the Caratheodory conditions (i.e., $A_i(x,s,\xi)$ is measurable about x in Ω for every fixed $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and is continuous in (s,ξ) for a.e. $x\in\Omega$).

($\mathcal{Q}$-2) There exist $h_i^*\in L^2_{p_i}$, $i = 1,\cdots,N$, and a positive constant c_1 so that a.e. $x\in\Omega$, $|A_i(x,s,\xi)|\le c_1\sum\limits_{j=1}^N p_j^{\frac12}(|\xi_j|+|h_j^*|)$.

($\mathcal{Q}$-3) There exists a positive constant c_2 such that $\sum\limits_{i=1}^N p_i^{\frac12}(x)A_i(x,s,\xi)\xi_i \ge c_2\sum\limits_{i=1}^N p_i(x)\xi_i^2$ for a.e. $x\in\Omega$ and $\forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

($\mathcal{Q}$-4) $\sum\limits_{i=1}^N p_i^{\frac12}(x)[A_i(x,s,\xi)-A_i(x,s,\xi')](\xi_i-\xi_i')>0$ for a.e. $x\in\Omega$, $\forall s\in\mathbb{R}$, and $\forall\xi,\xi'\in\mathbb{R}^N$ with $\xi\ne\xi'$.

($\mathcal{Q}$-5) $B_0(x)\in C^0(\Omega)\cap L^\infty(\Omega)$ with $B_0(x)\ge\sigma_0$ (a positive constant).

It is assumed throughout this paper that $f(x,s)$ meets:

(f-1) $f(x,s)$ satisfies the Caratheodory conditions.

(f-2) (Superlinear growth condition) There exists θ with $2<\theta<\frac{2N}{N-1}$ such that

$$|f(x,s)|\le h_0(x)+K|s|^{\theta-1}, \quad \forall s\in\mathbb{R},\ \text{a.e. } x\in\Omega,$$

where $h_0(x)\in L^{\theta^*}_\rho(\Omega)$, K is a nonnegative constant and $\theta^*=\frac{\theta}{\theta-1}$.

(f-3) There exists a nonnegative function $h_1(x)\in L^{\theta^*}_\rho(\Omega)$ and a constant $\beta>0$ such that

$$sf(x,s)\le-\beta|s|^2+h_1(x)|s|, \quad \forall s\in\mathbb{R},\ \text{a.e. } x\in\Omega.$$

Remark 2.3 Observing that for $N=2$, $f(x,s)=-g(x)s|s|^{\frac53}-\beta s$, where $g(x)\in C^0(\Omega)\cap L^\infty(\Omega)$ is a positive function, meets both (f-2) and (f-3).

Now we state our main results in this paper.

Theorem 2.1 *Assume that (Ω,Γ) is a new-$\mathrm{V_L}$ region, the operator $\mathcal{Q}$ satisfies ($\mathcal{Q}$-1)–($\mathcal{Q}$-5) and is $\circledast$-related to the operator $\mathcal{L}$, f meets (f-1)–(f-3), and $G\in[H^1_{p,q,\rho}(\Omega,\Gamma)]'$ (the dual of $H^1_{p,q,\rho}(\Omega,\Gamma)$). λ_{j_0} is the j_0th eigenvalue of $\mathcal{L}$. Then the problem (1.1) has at least one nontrivial weak solution, that is, there exists a $u^*\in H^1_{p,q,\rho}(\Omega,\Gamma)$ such that*

$$\mathcal{Q}(u^*,v)=\lambda_{j_0}\langle u^*,v\rangle_\rho+\int_\Omega f(x,u^*)v\rho-G(v) \quad \forall v\in H^1_{p,q,\rho}(\Omega,\Gamma).$$

To derive out Theorem 2.1, we first discuss the problem in S_n which is the subspace of $H^1_{p,q,\rho}(\Omega,\Gamma)$ spanned by $\varphi_1,\cdots,\varphi_n$. Then by virtue of Galerkin method, the results will be extended to $H^1_{p,q,\rho}(\Omega,\Gamma)$.

3 Fundamental Lemmas

In this section, we introduce and establish four fundamental lemmas. Lemmas 3.1–3.2 give two useful embedding theorems. Lemma 3.3 constructs some approximation solutions in S_n. Lemma 3.4 studies the properties of the approximation solutions.

Lemma 3.1 (see [7]) *Assume that $\mathcal{L}$ is given by* (2.4) *and* (Ω,Γ) *is a new-*$\mathrm{V_L}$ *region. For $N\ge 2$, then $H^1_{p,q,\rho}(\Omega,\Gamma)$ is compactly embedded in $L^\theta_\rho(\Omega)$ $\forall\theta$ $(2<\theta<\frac{2N}{N-1})$; for $N=1$, then $H^1_{p,q,\rho}(\Omega,\Gamma)$ is compactly embedded in $L^\theta_\rho(\Omega)$ $\forall\theta$ $(2<\theta<\infty)$.*

Lemma 3.2 (see [7]) *Assume that $\mathcal{L}$ is given by* (2.4) *and* (Ω,Γ) *is a new-*$\mathrm{V_L}$ *region. Then $H^1_{p,q,\rho}(\Omega,\Gamma)$ is compactly embedded in $L^2_\rho(\Omega)$.*

Lemma 3.3 *Let all the assumptions in Theorem* 2.1 *hold. Then for $n\ge j_0+J_0$, there exists a $u_n\in S_n$ such that*

$$\mathcal{Q}(u_n,v)=\Big(\lambda_{j_0}-\frac{1}{n}\Big)\langle u_n,v\rangle_\rho+\int_\Omega f(x,u_n)v\rho-G(v),\quad \forall v\in S_n. \tag{3.1}$$

Proof For fixed n $(n\ge j_0+J_0)$ and $\forall\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{R}^n$, set $u=\sum\limits_{k=1}^{n}\alpha_k\varphi_k$. From new-$\mathrm{V_L}$ conditions, we obtain

$$\mathcal{L}(u,u)=\sum_{k=1}^{n}\lambda_k\alpha_k^2,\quad \|u\|_\rho^2=\sum_{k=1}^{n}\alpha_k^2=|\alpha|^2. \tag{3.2}$$

And from Remark 2.1,

$$\|u\|^2_{p,q,\rho}=\mathcal{L}(u,u)+\|u\|^2_\rho\le(\lambda_n+1)\|u\|^2_\rho. \tag{3.3}$$

For $m\ge 2$, a positive integer, we put

$$f_m(x,s)=\begin{cases} f(x,m), & m\le s;\\ f(x,s), & -m\le s\le m;\\ f(x,-m), & s\le -m.\end{cases} \tag{3.4}$$

Note from (f-2) that $|f_m(x,s)|\le h_0(x)+K|m|^{\theta-1}$ $\forall s\in\mathbb{R}$, a.e. $x\in\Omega$. Also from $h_0(x)\in L^{\theta^*}_\rho$, Hölder inequality, Minkowski inequality and (2.7), we get $\forall v\in S_n$,

$$\int_\Omega|f_m(x,u)v\rho|\le\|f_m(x,u)\|_{L^{\theta^*}_\rho}\|v\|_{L^\theta_\rho}\le T_m\|v\|_{L^\theta_\rho}, \tag{3.5}$$

where T_m is a positive constant depending on m.

The remaining proof is separated into two parts. The first part is to prove the claim (3.6) for $f_m(x,s)$. The second part is to get the conclusion by leaving $m\to\infty$ based on (3.6).

Part 1 Fix m ($m\geq 2$). We are to show that there exists $u^*_{n,m}$ such that

$$\mathcal{Q}(u^*_{n,m},v)=\Big(\lambda_{j_0}-\frac{1}{n}\Big)\langle u^*_{n,m},v\rangle_\rho+\int_\Omega f_m(x,u^*_{n,m})v\rho-G(v),\quad \forall v\in S_n. \tag{3.6}$$

For $u=\sum\limits_{k=1}^{n}\alpha_k\varphi_k$, we set

$$F_k(\alpha)=\mathcal{Q}(u,\varphi_k)-\Big(\lambda_{j_0}-\frac{1}{n}\Big)\langle u,\varphi_k\rangle_\rho-\int_\Omega f_m(x,u)\varphi_k\rho+G(\varphi_k),\quad k=1,\cdots,n.$$

It is clear that $\sum\limits_{k=1}^{n}F_k(\alpha)\alpha_k=\mathrm{I}(\alpha)+\mathrm{II}(\alpha)$, where

$$\mathrm{I}(\alpha)=\mathcal{Q}(u,u)-\Lambda_{j_0}\mathcal{L}(u,u)-\int_\Omega f_m(x,u)u\rho+G(u), \tag{3.7}$$

$$\mathrm{II}(\alpha)=\Lambda_{j_0}\mathcal{L}(u,u)-\Big(\lambda_{j_0}-\frac{1}{n}\Big)\langle u,u\rangle_\rho. \tag{3.8}$$

For (3.7), observing the fact that the operator $\mathcal{Q}$ is $\divideontimes$-related to $\mathcal{L}$, $G\in[H^1_{p,q,\rho}(\Omega,\Gamma)]'$, by (3.2)–(3.3), (3.5) and Lemma 3.1 we conclude that

$$\lim_{|\alpha|\to\infty}\frac{\mathcal{Q}(u,u)-\Lambda_{j_0}\mathcal{L}(u,u)}{|\alpha|^2}=0,\quad \lim_{|\alpha|\to\infty}\frac{|\int_\Omega f_m(x,u)u\rho|}{|\alpha|^2}=0,\quad \lim_{|\alpha|\to\infty}\frac{|G(u)|}{|\alpha|^2}=0,$$

and $\lim\limits_{|\alpha|\to\infty}\dfrac{|\mathrm{I}(\alpha)|}{|\alpha|^2}=0$.

For (3.8), considering $\Lambda_{j_0}=\frac{\lambda_{j_0}}{\lambda_\dagger}$ and $\lambda_\dagger\leq\lambda_1$, it is clear that $\Lambda_{j_0}\lambda_k\geq\lambda_{j_0}$ ($k=1,\cdots,n$). By (3.2), we obtain

$$\mathrm{II}(\alpha)=\sum_{k=1}^{n}\Big(\Lambda_{j_0}\lambda_k-\lambda_{j_0}+\frac{1}{n}\Big)\alpha_k^2\geq\frac{1}{n}|\alpha|^2. \tag{3.9}$$

Consequently, $\sum\limits_{k=1}^{n}F_k(\alpha)\alpha_k\geq\dfrac{|\alpha|^2}{2n}$, where $|\alpha|\geq s_0$ (here s_0 is a large enough constant). By virtue of generalized Brouwer's theorem (see [11]), there exists $\gamma_{n,m}=(\gamma^{(1)}_{n,m},\cdots,\gamma^{(n)}_{n,m})$ such that $F_k(\gamma_{n,m})=0,\ k=1,\cdots,n$. Taking $u^*_{n,m}=\sum\limits_{k=1}^{n}\gamma^{(k)}_{n,m}\varphi_k$, then (3.6) holds.

Part 2 We claim that $\{\|u^*_{n,m}\|_\rho\}_{m=2}^\infty$ (n fixed) is uniformly bounded according to m. Arguing by contradiction, and without loss of generality, we suppose that

$$\lim_{m\to\infty}\|u^*_{n,m}\|_\rho=\infty. \tag{3.10}$$

Taking $v=u^*_{n,m}$ in (3.6),

$$\Lambda_{j_0}\mathcal{L}(u^*_{n,m},u^*_{n,m})-\lambda_{j_0}\langle u^*_{n,m},u^*_{n,m}\rangle_\rho+\frac{1}{n}\langle u^*_{n,m},u^*_{n,m}\rangle_\rho$$

$$= \int_\Omega f_m(x, u_{n,m}^*)u_{n,m}^*\rho - G(u_{n,m}^*) + \Lambda_{j_0}\mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \mathcal{Q}(u_{n,m}^*, u_{n,m}^*) \tag{3.11}$$

holds, that is,

$$\begin{aligned}&\sum_{k=1}^{n}(\Lambda_{j_0}\lambda_k - \lambda_{j_0})|\widehat{u}_{n,m}^*(k)|^2 + \frac{1}{n}\|u_{n,m}^*\|_\rho^2 \\ &= \int_\Omega f_m(x, u_{n,m}^*)u_{n,m}^*\rho - G(u_{n,m}^*) + \Lambda_{j_0}\mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \mathcal{Q}(u_{n,m}^*, u_{n,m}^*),\end{aligned} \tag{3.12}$$

where $\widehat{u}_{n,m}^*(k) = \langle \varphi_k, u_{n,m}^*\rangle_\rho$.

On the other hand, using (f-3), for $s \geq m$, we have

$$sf_m(x, s) = \frac{s}{m} \cdot mf(x, m) \leq h_1(x)|s|, \quad \text{a.e. } x \in \Omega. \tag{3.13}$$

Similarly we can also obtain the same conclusion, where $-m \leq s \leq m$ or $s \leq -m$. As a result,

$$sf_m(x, s) \leq h_1(x)|s|, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{3.14}$$

(3.12) and (3.14) imply that

$$\begin{aligned}\frac{1}{n}\|u_{n,m}^*\|_\rho^2 \leq \|h_1\|_{L_\rho^{\theta^*}}\|u_{n,m}^*\|_{L_\rho^\theta} + |G(u_{n,m}^*)| \\ + \Lambda_{j_0}\mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \mathcal{Q}(u_{n,m}^*, u_{n,m}^*).\end{aligned} \tag{3.15}$$

Dividing both sides of (3.15) by $\|u_{n,m}^*\|_\rho^2$ and leaving $m \to \infty$, we obtain from the fact that $\mathcal{Q}$ is $\divideontimes$-related to $\mathcal{L}$, $G \in [H_{p,q,\rho}^1(\Omega, \Gamma)]'$, $h_1(x) \in L_\rho^{\theta^*}(\Omega)$ together with Lemma 3.1 and (3.3) that $\frac{1}{n} \leq 0$. However, n is a positive integer. So we have arrived at a contradiction. (3.10) does not hold, i.e.,

$$\exists K_1 > 0, \quad \|u_{n,m}^*\|_\rho \leq K_1, \quad \forall m \geq 2. \tag{3.16}$$

(3.3) and (3.16) imply that there is a subsequence of $\{u_{n,m}^*\}_{m=2}^\infty$ (for easy notation, take the full sequence) and a $u_n \in S_n$ (see [12]) such that

$$\begin{cases}\lim\limits_{m\to\infty} \|u_{n,m}^* - u_n\|_{p,q,\rho} = 0, \\ \lim\limits_{m\to\infty} u_{n,m}^*(x) = u_n(x), & \text{a.e. } x \in \Omega, \\ \lim\limits_{m\to\infty} D_i u_{n,m}^*(x) = D_i u_n(x), & \text{a.e. } x \in \Omega,\ i = 1, \cdots, N.\end{cases} \tag{3.17}$$

Therefore from (3.17) with ($\mathcal{Q}$-1)–($\mathcal{Q}$-5), we obtain

$$\lim_{m\to\infty} \mathcal{Q}(u_{n,m}^*, v) = \mathcal{Q}(u_n, v), \quad \forall v \in S_n. \tag{3.18}$$

And recall from Lemma 3.1 that

$$\lim_{m\to\infty} \int_\Omega |u_{n,m}^* - u_n|^\theta \rho = 0. \tag{3.19}$$

Moreover, we can get that there exists $W(x) \in L^\theta_\rho$ and a subsequence $\{u^*_{n,m_j}\}_{j=1}^\infty \subset \{u^*_{n,m}\}_{m=2}^\infty$ such that $|u^*_{n,m_j}(x)| \le W(x)$, a.e. $x \in \Omega\ \forall j$.

By virtue of Hölder inequality, (f-1)–(f-2) and Lebesgue-dominated convergence theorem, we get

$$\lim_{j\to\infty}\int_\Omega f_{m_j}(x,u^*_{n,m_j})v\rho = \int_\Omega f(x,u_n)v\rho, \quad \forall v \in S_n.$$

Now replacing m by m_j in (3.6) and taking the limit as $j \to \infty$ on both sides of the equation, we consequently obtain that (3.1) holds and Lemma 3.3 is established.

Lemma 3.4 *Let all the assumptions in Theorem* 2.1 *hold. Then the sequence* $\{u_n\}$ *obtained in Lemma* 3.3 *is uniformly bounded in* $H^1_{p,q,\rho}(\Omega,\Gamma)$.

Based on Lemma 3.3 and ⁎-relationship, for the proofs of Lemma 3.4, one can refer to [7] or [8].

4 Proof of Theorem 2.1

Since $H^1_{p,q,\rho}(\Omega,\Gamma)$ is a separable Hilbert space, from Lemmas 3.1–3.2, we conclude that there exists a subsequence of $\{u_n\}_{n=j_0+J_0}^\infty$ (for easy notation, we take the full sequence) and a function $u^* \in H^1_{p,q,\rho}(\Omega,\Gamma)$ with the following properties (see [12]):

$$\lim_{n\to\infty}\Big[\|u_n-u^*\|_\rho+\int_\Omega |u_n-u^*|^\theta\rho\Big]=0, \tag{4.1}$$

$$\exists W'(x) \in L^2_\rho(\Omega)\cap L^\theta_\rho(\Omega), \quad \text{s.t. } |u_n(x)| \le W'(x),\ \text{a.e. } x\in\Omega, \tag{4.2}$$

$$\lim_{n\to\infty} u_n(x)=u^*(x), \quad \text{a.e. } x\in\Omega; \tag{4.3}$$

$$\lim_{n\to\infty}\langle D_iu_n,v\rangle_{p_i}=\langle D_iu^*,v\rangle_{p_i}, \quad \forall v\in L^2_{p_i},\ i=1,\cdots,N; \tag{4.4}$$

$$\lim_{n\to\infty}\langle u_n,v\rangle_q=\langle u^*,v\rangle_q, \quad \forall v\in L^2_q; \tag{4.5}$$

$$\lim_{n\to\infty} G(u_n)=G(u^*). \tag{4.6}$$

We prove the theorem through three steps.

Step 1 We intend to show the following that there exists a subsequence $\{u_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} Du_{n_j}(x)=Du^*(x), \quad \text{a.e. } x\in\Omega. \tag{4.7}$$

Before establishing (4.7), we prefer to show the following two facts first.

(1) There exists a subsequence $\{u_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty}\sum_{i=1}^N p_i^{\frac12}(x)[A_i(x,u_{n_j},Du_{n_j})-A_i(x,u_{n_j},Du^*)][D_iu_{n_j}(x)-D_iu^*(x)]=0 \tag{4.8}$$

for a.e. $x\in\Omega$.

(2) With $\{u_{n_j}\}_{j=1}^{\infty}$ designating the same subsequence as in (1),

$$\{|Du_{n_j}(x)|\}_{j=1}^{\infty} \text{ is pointwise bounded for a.e. } x \in \Omega. \tag{4.9}$$

Firstly, to show (4.8), we observe from ($\mathcal{Q}$-2), (4.3) and Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^{N} |A_i(x, u_n, Du^*) - A_i(x, u^*, Du^*)|^2 = 0. \tag{4.10}$$

Using (4.4), we obtain

$$\lim_{n\to\infty} \int_\Omega A_i(x, u^*, Du^*)(D_i u_n - D_i u^*) p_i^{\frac{1}{2}} = 0, \quad i = 1, \cdots, N. \tag{4.11}$$

It follows from Lemma 3.4 , (4.10)–(4.11) that

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^{N} A_i(x, u_n, Du^*)(D_i u_n - D_i u^*) p_i^{\frac{1}{2}} = 0, \quad i = 1, \cdots, N. \tag{4.12}$$

In addition, from (4.5) and ($\mathcal{Q}$-5), we have

$$\lim_{n\to\infty} \int_\Omega B_0(x) u^*(u_n - u^*) q = 0. \tag{4.13}$$

Now if we can show that

$$\lim_{n\to\infty} \mathcal{Q}(u_n, u_n - u^*) = 0, \tag{4.14}$$

then it will follow from (2.3), (4.12)–(4.13) that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \Big(\sum_{i=1}^{N} [A_i(x, u_n, Du_n) - A_i(x, u_n, Du^*)](D_i u_n - D_i u^*) p_i^{\frac{1}{2}} \\ &+ B_0(x)(u_n - u^*)(u_n - u^*) q \Big) = 0. \end{aligned} \tag{4.15}$$

Observing ($\mathcal{Q}$-4) and ($\mathcal{Q}$-5), we have that the integrand in (4.15) is nonnegative almost everywhere in Ω. Hence the integrand in (4.15) converges to zero in $L^1(\Omega)$. However, from [13], we have a subsequence of the integrand converging to zero almost everywhere in Ω. And from (4.3), $B_0(x)|u_n - u^*|^2$ converges to zero almost everywhere in Ω. We conclude that (4.8) is indeed true. So it remains to establish (4.14).

Observing $u^* \in H^1_{p,q,\rho}$, we define a projection $P_n : H^1_{p,q,\rho} \to S_n$, i.e.,

$$P_n u^* = \sum_{k=1}^{n} \widehat{u}^*(k) \varphi_k \in S_n. \tag{4.16}$$

From the definition, we get

$$\lim_{n\to\infty} \|P_n u^* - u^*\|_{p,q,\rho} = 0. \tag{4.17}$$

It is easy to know

$$\mathcal{Q}(u_n, P_n u^* - u^*) = \int_\Omega \sum_{i=1}^N A_i(x, u_n, Du_n) D_i(P_n u^* - u^*) p_i^{\frac{1}{2}} + \langle B_0 u_n, P_n u^* - u^* \rangle_q. \tag{4.18}$$

From ($\mathcal{Q}$-2), ($\mathcal{Q}$-5), Lemma 3.4, (4.17)–(4.18), we have

$$\lim_{n\to\infty} \mathcal{Q}(u_n, P_n u^* - u^*) = 0. \tag{4.19}$$

In view of (4.19), (4.14) will follow once we can show that

$$\lim_{n\to\infty} \mathcal{Q}(u_n, u_n - P_n u^*) = 0. \tag{4.20}$$

From Lemma 3.3 and (4.16), we obtain

$$\begin{aligned}\mathcal{Q}(u_n, u_n - P_n u^*) = &\Big(\lambda_{j_0} - \frac{1}{n}\Big)\langle u_n, u_n - P_n u^* \rangle_\rho \\ &+ \int_\Omega f(x, u_n)(u_n - P_n u^*)\rho - G(u_n - P_n u^*).\end{aligned} \tag{4.21}$$

Observing

$$\langle u_n, u_n - P_n u^* \rangle_\rho = \langle u_n, u_n - u^* \rangle_\rho + \langle u_n, u^* - P_n u^* \rangle_\rho, \tag{4.22}$$

from (4.1)–(4.2) and (4.17), we see $\big(\lambda_{j_0} - \frac{1}{n}\big)\langle u_n, u_n - P_n u^* \rangle_\rho \to 0$ as $n \to \infty$. Also from (f-2),

$$\int_\Omega |f(x, u_n)(u_n - P_n u^*)\rho| \le \|h_0(x)\|_{L_\rho^{\theta^*}} \|u_n - P_n u^*\|_{L_\rho^\theta} + \|u_n\|_{L_\rho^\theta}^{\frac{\theta}{\theta^*}} \|u_n - P_n u^*\|_{L_\rho^\theta}. \tag{4.23}$$

Recall Lemma 3.1, Lemma 3.4 and (4.17), $\int_\Omega f(x, u_n)(u_n - P_n u^*)\rho \to 0$ as $n \to \infty$. It is also clear from (4.17) that $G(u_n - P_n u^*) \to 0$ as $n \to \infty$. Therefore (4.20) holds and hence (4.8) is established.

Secondly, to establish (4.9), set Ω_1 to be the set meeting the following three conditions simultaneously:

(O_1) $u^*(x)$, $|Du^*(x)|$, $h_j^*(x)$, $u_{n_j}(x)$, $A_i(x, u_{n_j}(x), Du_{n_j}(x))$ and $A_i(x, u_{n_j}(x), Du^*(x))$ are finite-valued for $i = 1, \cdots, N$ and $j = 1, \cdots$;

(O_2) ($\mathcal{Q}$-2) and ($\mathcal{Q}$-3) hold;

(O_3) the limits in (4.3) and (4.8) exist.

Then

$$\text{meas}(\Omega - \Omega_1) = 0. \tag{4.24}$$

Suppose, to the contrary, that $\{|Du_{n_j}(x)|\}_{j=1}^\infty$ is not pointwise bounded in Ω_1. Then there exists $x_0 \in \Omega_1$ and a subsequence $\{|Du_{n_{j_k}}(x_0)|\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} |Du_{n_{j_k}}(x_0)| = \infty. \tag{4.25}$$

Set $c_5 = \min\{p_1(x_0), \cdots, p_N(x_0)\}$, and then $c_5 > 0$. From ($\mathcal{Q}$-3), we get

$$\sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) A_i(x_0, u_{n_{j_k}}, Du_{n_{j_k}}) D_i u_{n_{j_k}}(x_0) \geq c_2 c_5 |Du_{n_{j_k}}(x_0)|^2. \tag{4.26}$$

Considering the left side of (4.26), we have

$$\begin{aligned}
&\sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) A_i(x_0, u_{n_{j_k}}, Du_{n_{j_k}}) D_i u_{n_{j_k}}(x_0) \\
= &\sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) A_i(x_0, u_{n_{j_k}}, Du_{n_{j_k}}) D_i u^*(x_0) \\
&+ \sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) A_i(x_0, u_{n_{j_k}}, Du^*)[D_i u_{n_{j_k}}(x_0) - D_i u^*(x_0)] \\
&+ \sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) [A_i(x_0, u_{n_{j_k}}, Du_{n_{j_k}}) - A_i(x_0, u_{n_{j_k}}, Du^*)][D_i u_{n_{j_k}}(x_0) - D_i u^*(x_0)].
\end{aligned} \tag{4.27}$$

Divide both sides of (4.26) by $|Du_{n_{j_k}}(x_0)|^{\frac{3}{2}}$ and leave $k \to \infty$. From (4.27), ($\mathcal{Q}$-2), (4.3), (4.8) and the definition of Ω_1, we obtain

$$0 \geq c_2 c_5 |Du_{n_{j_k}}(x_0)|^{\frac{1}{2}}. \tag{4.28}$$

It is clear that $c_2 c_5 > 0$. Therefore $\lim\limits_{k\to\infty} |Du_{n_{j_k}}(x_0)| = 0$ which contradicts (4.28).

Consequently, $\{|Du_{n_j}(x)|\}_{j=1}^{\infty}$ is pointwise bounded in Ω_1. This fact in conjunction with (4.24) establishes (4.9).

Now we have that (4.8)–(4.9) hold which will imply (4.7). Let Ω_2 be the subset of Ω, where ($\mathcal{Q}$-1), ($\mathcal{Q}$-4), (4.8) and (4.9) hold simultaneously. Consequently,

$$\operatorname{meas}(\Omega - \Omega_2) = 0. \tag{4.29}$$

Suppose that there exists $x_0 \in \Omega_2$ such that (4.7) does not hold. Hence by (4.9) there exists a further sequence $\{Du_{n_{j_k}}(x_0)\}_{k=1}^{\infty}$ and a $\xi^{\natural} \in \mathbb{R}^N$ with

$$Du^*(x_0) \neq \xi^{\natural} \tag{4.30}$$

such that $\lim\limits_{k\to\infty} Du_{n_{j_k}}(x_0) = \xi^{\natural}$. From (4.3), we have

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) \{[A_i(x_0, u_{n_{j_k}}, Du_{n_{j_k}}) - A_i(x_0, u_{n_{j_k}}, Du^*)][D_i u_{n_{j_k}}(x_0) - D_i u^*(x_0)]\} \\
= &\sum_{i=1}^{N} p_i^{\frac{1}{2}}(x_0) \{[A_i(x_0, u^*(x_0), \xi^{\natural}) - A_i(x_0, u^*(x_0), Du^*(x_0))][\xi^{\natural} - D_i u^*(x_0)]\}.
\end{aligned}$$

Observing $x_0 \in \Omega_2$ and (4.8), the limit on the left side of (4.31) is zero. But from ($\mathcal{Q}$-4), the right side of (4.31) is strictly greater than zero, which has led to a contradiction. Hence (4.7) holds at every point in Ω_2, and consequently by (4.26), it holds almost everywhere in Ω. So the statement (4.7) is fully established.

Step 2 We proceed with the proof and let $v_J \in S_J$, where $J \geq j_0 + J_0$ is a fixed but arbitrary positive integer. From (4.3), (4.7) and ($\mathcal{Q}$-1) we can get

$$\lim_{j\to\infty} |A_i(x, u_{n_j}, Du_{n_j}) - A_i(x, u^*, Du^*)| = 0, \quad \text{a.e. } x \in \Omega. \tag{4.31}$$

Recalling Lemma 3.4, ($\mathcal{Q}$-2) and by virtue of the Lebesgue-dominated convergence theorem, we obtain

$$\lim_{j\to\infty} \int_\Omega |A_i(x, u_{n_j}, Du_{n_j}) - A_i(x, u^*, Du^*)|^2 = 0. \tag{4.32}$$

Also from $D_i v_J \in L^2_{p_i}$, and by applying Schwarz's inequality, we have

$$\lim_{j\to\infty} \int_\Omega [A_i(x, u_{n_j}, Du_{n_j}) - A_i(x, u^*, Du^*)] D_i v_J p_i^{\frac{1}{2}} = 0. \tag{4.33}$$

Observing (4.5) and (4.34), we get

$$\lim_{j\to\infty} \mathcal{Q}(u_{n_j}, v_J) = \mathcal{Q}(u^*, v_J). \tag{4.34}$$

From (f-2), (4.1) and (4.2), using the Lebesgue-dominated convergence theorem, we obtain

$$\lim_{j\to\infty} \int_\Omega f(x, u_{n_j}) v_J \rho = \int_\Omega f(x, u^*) v_J \rho. \tag{4.35}$$

Replacing v by v_J, n by n_j in (3.1) and leaving $j \to \infty$, from (4.1), (4.6), (4.35)–(4.36), we have

$$\mathcal{Q}(u^*, v_J) = \lambda_{j_0} \langle u^*, v_J \rangle_\rho + \int_\Omega f(x, u^*) v_J \rho - G(v_J). \tag{4.36}$$

Step 3 Given $v \in H^1_{p,q,\rho}(\Omega, \Gamma)$, from the definition of projection P_n, we see

$$P_J v = \sum_{k=1}^{J} \widehat{v}(k) \varphi_k \in S_J, \tag{4.37}$$

where $\widehat{v}(k) = \langle \varphi_k, v \rangle_\rho$. It is easy to get $\lim\limits_{J\to\infty} \|P_J v - v\|_{p,q,\rho} = 0$. As a result, there hold

$$\begin{cases} \lim\limits_{J\to\infty} \mathcal{Q}(u^*, P_J v) = \mathcal{Q}(u^*, v), \\ \lim\limits_{J\to\infty} \langle u^*, P_J v \rangle_\rho = \langle u^*, v \rangle_\rho, \\ \lim\limits_{J\to\infty} \int_\Omega f(x, u^*) P_J v \rho = \int_\Omega f(x, u^*) v \rho, \\ \lim\limits_{J\to\infty} G(P_J v) = G(v). \end{cases} \tag{4.38}$$

Replacing v_J by $P_J v$ in (4.36), passing to the limit as $J \to \infty$ on both sides, and using (4.38), we can obtain

$$\mathcal{Q}(u^*, v) = \lambda_{j_0} \langle u^*, v \rangle_\rho + \int_\Omega f(x, u^*) v \rho - G(v), \quad \forall v \in H^1_{p,q,\rho}(\Omega, \Gamma).$$

Hence the proof of Theorem 2.1 is complete.

5 An Example for $\divideontimes$-related

Now we give an example of $\mathcal{Q}$ which is $\divideontimes$-related to $\mathcal{L}$.

Take $N = 1$, $\Omega = (-1, 1)$, $\Gamma =$ the empty set, $p(s) = 1 - s^2$, $\rho(s) = 1$, $q(s) = (1 - s^2)^{-1}$. From the definition of $\mathcal{L}$, we get

$$\mathcal{L}u = -D_1(1 - s^2)D_1u + (1 - s^2)^{-1}u. \tag{5.1}$$

Then

$$\mathcal{L}\Phi_{n,1}(s) = n(n + 1)\Phi_{n,1}(s), \quad n = 1, 2, \cdots, \tag{5.2}$$

where $\Phi_{n,1}(s)$ is the first-order associated Legendre function of degree n (see [14]). And $\{\frac{\Phi_{n,1}(s)}{a_n}\}_{j=1}^{\infty}$ properly normalized forms a CONS on Ω with respect to the weight $\rho(s)$, where $a_n^2 = \frac{2n(n+1)}{2n+1}$. Since Ω is 1-dimensional, (Ω, Γ) is a new-$\mathrm{V_L}$ region.

Set $F(t) = \frac{t^2}{1+t^2}$. $F(t)$ is a real-valued function with the following properties:

(i) $F(t) \in C^0([0, \infty))$ is nondecreasing and positive; (ii) $\lim\limits_{t\to\infty} t[1 - F(t)] = 0$.

Given j_0, observing $0 < \lambda_\dagger \le \lambda_1$, we take $\lambda_\dagger = 1$ and $\Lambda_{j_0} = n(n + 1),\ n = 1, 2, \cdots$. Then the $A_i(x, s, \xi)$ of $\mathcal{Q}$ in (1.2) are defined to be

$$A_1(x, s, \xi) = \frac{\Lambda_{j_0}}{2}[1 + F(|\xi_1|)]p^{\frac{1}{2}}\xi_1, \quad B_0(x) = \Lambda_{j_0}.$$

With this definition, it is clear that $\mathcal{Q}$ meets ($\mathcal{Q}$-1)–($\mathcal{Q}$-5) and that

$$\mathcal{Q}u = -\frac{\Lambda_{j_0}}{2}D_1[1 + F(|D_1u|)]pD_1u + \Lambda_{j_0}qu. \tag{5.3}$$

As a consequence,

$$\mathcal{Q}(u, v) - \Lambda_{j_0}\mathcal{L}(u, v) = \frac{\Lambda_{j_0}}{2}\int_\Omega p[F(|D_1u|) - 1]D_1uD_1v.$$

Now it follows from (ii) that there is a constant K_3 such that $|1 - F(t)|t \le K_3,\ \forall t \in (0, \infty)$. Consequently, if $\|v\|_{p,q,\rho} \le 1$, we obtain that

$$|\mathcal{Q}(u, v) - \Lambda_{j_0}\mathcal{L}(u, v)| \le \frac{K_3\Lambda_{j_0}}{2}\Big(\int_\Omega p\Big)^{\frac{1}{2}} \le K_4, \quad \forall u \in H^1_{p,q,\rho},$$

where K_4 is a constant.

That $\mathcal{Q}$ is $\divideontimes$-related to $\mathcal{L}$ then follows immediately from this last inequality.

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