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Musical Isomorphisms and Problems of Lifts*

Rabia CAKAN[1] Kursat AKBULUT[1] Arif SALIMOV[2]

Abstract Using the complete lift on tangent bundles, the authors construct the complete lift on cotangent bundles of tensor fields with the aid of a musical isomorphism. In this new framework, the authors have a new intrepretation of the complete lift of tensor fields on cotangent bundles.

Keywords Tensor fields, Cotangent bundles, Complete lift, Anti-Hermitian metric, Riemannian extension
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1 Introduction

Let (M,g) be a smooth pseudo-Riemannian manifold of dimension n. We denote by $TM = \bigcup\limits_{x\in M_n} T_xM$ and $T^*M = \bigcup\limits_{x\in M_n} T_x^*M$ the tangent and cotangent bundles over M with local coordinates $(x^i, x^{\overline{i}}) = (x^i, y^i)$ and $(x^i, \widetilde{x}^{\overline{i}}) = (x^i, p_i)$, $i = 1,\cdots,n$; $\overline{i} = n+1,\cdots,2n$, respectively, where $y_x = y^i\frac{\partial}{\partial x^i} \in T_xM$ and $p_x = p_i\mathrm{d}x^i \in T_x^*M$, $\forall x\in M$.

A very important feature of any pseudo-Riemannian metric g is that it provides musical isomorphisms $g^\flat : TM \to T^*M$ and $g^\sharp : T^*M \to TM$ between the tangent and cotangent bundles. Some properties of geometric structures on cotangent bundles with respect to the musical isomorphisms are proved in [1–5].

The musical isomorphisms $g^\flat$ and $g^\sharp$ are expressed by

$$g^\flat : x^I = (x^i, x^{\overline{i}}) = (x^i, y^i) \to \widetilde{x}^K = (x^k, \widetilde{x}^{\overline{k}}) = (\delta_i^k x^i, p_k = g_{ki}y^i)$$

and

$$g^\sharp : \widetilde{x}^K = (x^k, \widetilde{x}^{\overline{k}}) = (x^k, p_k) \to x^I = (x^i, x^{\overline{i}}) = (\delta_k^i x^k, y^i = g^{ik}p_k)$$

with respect to the local coordinates, respectively. The Jacobian matrices of $g^\flat$ and $g^\sharp$ are given by

$$(g_*^\flat) = (\widetilde{A}_I^K) = \Big(\frac{\partial \widetilde{x}^K}{\partial x^I}\Big) = \begin{pmatrix} \delta_i^k & 0 \\ y^s\partial_i g_{ks} & g_{ki} \end{pmatrix} \tag{1.1}$$

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[1]Department of Mathematics, Ataturk University, Erzurum 25240, Turkey.
E-mail: rabia.cakan@atauni.edu.tr kakbulut@atauni.edu.tr
[2]Department of Mathematics, Ataturk University, Erzurum 25240, Turkey; Azerbaijan National Academy of Sciences, Institute of Mathematics and Mechanics, Az1001, 30, Istiglaliyyat str., Baku, Azerbaijan Republic. E-mail: asalimov@atauni.edu.tr
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and

$$(g_*^\sharp) = (A_K^I) = \left(\frac{\partial x^I}{\partial \widetilde{x}^K}\right) = \begin{pmatrix} \delta_k^i & 0 \\ p_s \partial_k g^{is} & g^{ik} \end{pmatrix}, \tag{1.2}$$

respectively, where δ is the Kronecker delta.

We denote by $\Im_q^p(M)$ the set of all differentiable tensor fields of type (p,q) on M. Let ${}^C X_T \in \Im_0^1(TM)$, ${}^C \varphi_T \in \Im_1^1(TM)$ and ${}^C S_T \in \Im_2^1(TM)$ be complete lifts of tensor fields $X \in \Im_0^1(M)$, $\varphi \in \Im_1^1(M)$ and $S \in \Im_2^1(M)$ to the tangent bundle TM.

The aim of this paper is to study the lift properties of cotangent bundles of Riemannian manifolds. The results are significant for a better understanding of the geometry of the cotangent bundle of a Riemannian manifold. In this paper, we transfer via the differential $g_*^\flat$ the complete lifts ${}^C X_T \in \Im_0^1(TM)$, ${}^C \varphi_T \in \Im_1^1(TM)$ and ${}^C S_T \in \Im_2^1(TM)$ from the tangent bundle TM to the cotangent bundle T^*M. The transferred lifts $g_*^\flat\ {}^C X_T$, $g_*^\flat\ {}^C \varphi_T$ and $g_*^\flat\ {}^C S_T$ are compared with the complete lifts ${}^C X_{T^*} \in \Im_0^1(T^*M)$, ${}^C \varphi_{T^*} \in \Im_1^1(T^*M)$ and ${}^C S_{T^*} \in \Im_2^1(T^*M)$ in the cotangent bundle and we show that (a) $g_*^\flat {}^C X_T = {}^C X_{T^*}$ if and only if the vector field X is a Killing vector field, (b) $g_*^\flat {}^C \varphi_T = {}^C \varphi_{T^*}$ if and only if the triple (M, g, φ), $\varphi^2 = -Id_M$ is an anti-Kähler manifold, (c) $g_*^\flat {}^C S_T = {}^C S_{T^*}$ if and only if the metric g satisfies the Yano-Ako equations. Also we give a new interpretation of the Riemannian extension ${}^\nabla g \in \Im_2^0(T^*M)$, i.e., ${}^\nabla g$ should be considered as the pullback: ${}^\nabla g = (g^\sharp)^*\, {}^C g$, where ${}^C g$ is the complete lift of g to the tangent bundle TM.

2 Transfer of Complete Lifts of Vector Fields

Let $X = X^i \partial_i$ be the local expression in $U \subset M$ of a vector field $X \in \Im_0^1(M)$. Then the complete lift ${}^C X_T$ of X to the tangent bundle TM is given by

$${}^C X_T = X^i \partial_i + y^s \partial_s X^i \partial_{\bar{i}} \tag{2.1}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$.

Using (1.1) and (2.1), we have

$$\begin{aligned} g_*^\flat {}^C X_T &= \begin{pmatrix} \delta_i^k & 0 \\ y^s \frac{\partial g_{ks}}{\partial x^i} & g_{ki} \end{pmatrix} \begin{pmatrix} X^i \\ y^s \partial_s X^i \end{pmatrix} \\ &= \begin{pmatrix} X^k \\ X^i y^s \partial_i g_{ks} + g_{ki} y^s \partial_s X^i \end{pmatrix} \\ &= \begin{pmatrix} X^k \\ y^s((L_X g)_{sk} - (\partial_k X^i) g_{is} - (\partial_s X^i) g_{ki}) + g_{ik} y^s \partial_s X^i \end{pmatrix} \\ &= \begin{pmatrix} X^k \\ y^s (L_X g)_{sk} - p_i(\partial_k X^i) \end{pmatrix}, \end{aligned} \tag{2.2}$$

where L_X is the Lie derivation of g with respect to the vector field X:

$$(L_X g)_{sk} = X^i \partial_i g_{sk} + (\partial_s X^i) g_{ik} + (\partial_k X^i) g_{si}.$$

In a manifold (M, g), a vector field X is called a Killing vector field if $L_X g = 0$. It is well known that the complete lift ${}^C X_{T^*}$ of X to the cotangent bundle T^*M is given by

$$ {}^C X_{T^*} = X^k \partial_k - p_s \partial_k X^s \partial_{\overline{k}} \,. $$

From (2.2) we find

$$ g_*^{\flat}\, {}^C X_T = {}^C X_{T^*} + \gamma(L_X g)\,, $$

where $\gamma(L_X g)$ is defined by

$$ \gamma(L_X g) = \begin{pmatrix} 0 \\ y^s (L_X g)_{sk} \end{pmatrix}. $$

Thus we have the following theorem.

Theorem 2.1 *Let (M, g) be a pseudo-Riemannian manifold, and let ${}^C X_T$ and ${}^C X_{T^*}$ be complete lifts of a vector field X to the tangent and cotangent bundles, respectively. Then the differential (pushforward) of ${}^C X_T$ by $g^{\flat}$ coincides with ${}^C X_{T^*}$, i.e.,*

$$ g_*^{\flat}\, {}^C X_T = {}^C X_{T^*} $$

if and only if X is a Killing vector field.

Let X and Y be Killing vector fields on M. Then we have

$$ L_{[X,Y]} g = [L_X, L_Y] g = L_X \circ L_Y g - L_Y \circ L_X g = 0, $$

i.e., $[X, Y]$ is a Killing vector field. Since ${}^C [X, Y]_T = [{}^C X_T, {}^C Y_T]$ and ${}^C [X, Y]_{T^*} = [{}^C X_{T^*}, {}^C Y_{T^*}]$, from Theorem 2.1 we have the following result.

Corollary 2.1 *If X and Y are Killing vector fields on M, then*

$$ g_*^{\flat} [{}^C X_T, {}^C Y_T] = [{}^C X_{T^*},\ {}^C Y_{T^*}]\,, $$

where $g_*^{\flat}$ is a differential (pushforward) of the musical isomorphism $g^{\flat}$.

3 Transfer of Complete Lifts of Almost Complex Structures

Let (M, φ) be a $2n$-dimensional, almost complex manifold, where φ $(\varphi^2 = -I)$ denotes its almost complex structure. A semi-Riemannian metric g of the neutral signature (n, n) is an anti-Hermitian (also known as a Norden) metric if

$$ g(\varphi X, Y) = g(X, \varphi Y) $$

for any $X, Y \in \Im_0^1(M)$. An almost complex manifold (M, φ) with an anti-Hermitian metric is referred to as an almost anti-Hermitian manifold. Structures of this kind have also been studied under the name: Almost complex structures with pure (or B-)metric. An anti-Kähler (Kähler-Norden) manifold can be defined as a triple (M, g, φ) which consists of a smooth manifold M endowed with an almost complex structure φ and an anti-Hermitian metric g such that

$\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g. It is well known that the condition $\nabla\varphi = 0$ is equivalent to $\mathbb{C}$-holomorphicity (analyticity) of the anti-Hermitian metric g (see [6]), i.e.,

$$(\Phi_\varphi g)(X,Y,Z) = (L_{\varphi X} g - L_X G)(Y,Z) = 0$$

for any $X,Y,Z \in \Im^1_0(M)$, where $\Phi_\varphi g \in \Im^0_3(M)$ and $G(Y,Z) = (g\circ\varphi)(Y,Z) = g(\varphi Y, Z)$ is the twin anti-Hermitian metric. It is a remarkable fact that (M,g,φ) is anti-Kähler if and only if the twin anti-Hermitian structure (M,G,φ) is anti-Kähler. This is of special significance for anti-Kähler metrics since in such case g and G share the same Levi-Civita connection.

Let $\varphi = \varphi^i_j\,\partial_i \otimes \mathrm{d}x^j$ be the local expression in $U \subset M$ of an almost complex strucure φ. Then the complete lift ${}^C\varphi_T$ of φ to the tangent bundle TM is given by (see [8, p. 21])

$${}^C\varphi_T = ({}^C\varphi^I_J) = \begin{pmatrix} \varphi^i_j & 0 \\ y^s\partial_s\varphi^i_j & \varphi^i_j \end{pmatrix} \tag{3.1}$$

with respect to the induced coordinates $(x^i, x^{\bar{i}}) = (x^i, y^i)$ in TM. It is well known that ${}^C\varphi_T$ defines an almost complex structure on TM, if and only if so does φ on M.

Using (1.1)–(1.2) and (3.1), we have

$$\begin{aligned} g^\flat_*\,{}^C\varphi_T &= (\widetilde{\varphi}^{\,J}_{\,L}) = (A^J_I \widetilde{A}^{\,K}_{\,L}\,{}^C\varphi^I_K) \\ &= \begin{pmatrix} \varphi^j_l & 0 \\ y^s(\partial_i g_{js})\varphi^i_l + g_{ji}y^s\partial_s\varphi^i_l + g_{ji}p_s(\partial_l g^{ks})\varphi^i_k & g_{ji}g^{kl}\varphi^i_k \end{pmatrix}. \end{aligned} \tag{3.2}$$

Since $g = (g_{ij})$ and $g^{-1} = (g^{ij})$ are pure tensor fields with respect to φ, we find

$$g_{ji}g^{kl}\varphi^i_k = g_{ji}g^{ik}\varphi^l_k = \delta^k_j\varphi^l_k \tag{3.3}$$

and

$$\begin{aligned} &y^s(\partial_i g_{js})\varphi^i_l + g_{ji}y^s\partial_s\varphi^i_l + g_{ji}p_s(\partial_l g^{ks})\varphi^i_k \\ &= y^s(\Phi_l g_{js} + \partial_l(g\circ\varphi)_{js} - g_{is}\partial_j\varphi^i_l) + g_{ji}p_s(\partial_l g^{ks})\varphi^i_k \\ &= y^s\Phi_l g_{sj} + y^s\partial_l(g\circ\varphi)_{js} - p_i\partial_j\varphi^i_l + g_{ji}p_s(\partial_l g^{ks})\varphi^i_k \\ &= y^s\Phi_l g_{sj} - p_i\partial_j\varphi^i_l + y^s\partial_l(g\circ\varphi)_{js} + g_{ji}p_s(\partial_l g^{ks})\varphi^i_k \\ &= y^s\Phi_l g_{sj} - p_i\partial_j\varphi^i_l + y^s\partial_l(g_{sm}\varphi^m_j) + g_{ji}p_s(\partial_l g^{ks})\varphi^i_k \\ &= y^s\Phi_l g_{sj} - p_i\partial_j\varphi^i_l + y^s(\partial_l g_{sm})\varphi^m_j + y^s(\partial_l\varphi^m_j)g_{sm} + g_{jm}p_s(\partial_l g^{ks})\varphi^m_k \\ &= y^s\Phi_l g_{sj} - p_i\partial_j\varphi^i_l + y^s(\partial_l g_{sm})\varphi^m_j + y^s(\partial_l\varphi^m_j)g_{sm} + g_{mk}p_s(\partial_l g^{ks})\varphi^m_j \\ &= y^s\Phi_l g_{sj} - p_i\partial_j\varphi^i_l + y^s(\partial_l g_{sm})\varphi^m_j + y^s(\partial_l\varphi^m_j)g_{sm} - g^{ks}p_s(\partial_l g_{mk})\varphi^m_j \\ &= y^s\Phi_l g_{sj} - p_i\partial_j\varphi^i_l + y^s(\partial_l g_{sm})\varphi^m_j + p_m(\partial_l\varphi^m_j) - y^k(\partial_l g_{mk})\varphi^m_j \\ &= y^s\Phi_l g_{sj} + p_s(\partial_l\varphi^s_j - \partial_j\varphi^s_l), \end{aligned} \tag{3.4}$$

where

$$\Phi_k g_{ij} = \varphi^m_k\partial_m g_{ij} - \partial_k(g\circ\varphi)_{ij} + g_{mj}\partial_i\varphi^m_k + g_{im}\partial_j\varphi^m_k\,.$$

Substituting (3.3)–(3.4) into (3.2), we obtain

$$g_*^{\flat\, C}\varphi_T = \begin{pmatrix} \varphi_l^j & 0 \\ y^s\Phi_l g_{sj} + p_s(\partial_l\varphi_j^s - \partial_j\varphi_l^s) & \varphi_j^l \end{pmatrix}.$$

It is well known that the complete lift ${}^C\varphi_{T^*}$ of $\varphi \in \Im_0^1(M)$ to the cotangent bundle is given by (see [8, p. 242])

$${}^C\varphi_{T^*} = \begin{pmatrix} \varphi_l^j & 0 \\ p_s(\partial_l\varphi_j^s - \partial_j\varphi_l^s) & \varphi_j^l \end{pmatrix}$$

with respect to the induced coordinates in T^*M. Thus we obtain

$$g_*^{\flat\, C}\varphi_T = {}^C\varphi_{T^*} + \gamma(\Phi_\varphi g),$$

where

$$\gamma(\Phi_\varphi g) = \begin{pmatrix} 0 & 0 \\ y^s\Phi_l g_{sj} & 0 \end{pmatrix}.$$

From here, we have the following theorem.

Theorem 3.1 *Let (M, g, φ) be an almost anti-Hermitian manifold, and let ${}^C\varphi_T$ and ${}^C\varphi_{T^*}$ be complete lifts of an almost complex structure φ to the tangent and cotangent bundles, respectively. Then the differential of ${}^C\varphi_T$ by $g^\flat$ coincides with ${}^C\varphi_{T^*}$, i.e., $g_*^{\flat\, C}\varphi_T = {}^C\varphi_{T^*}$ if and only if (M, g, φ) is an anti-Kähler ($\Phi_\varphi g = 0$) manifold.*

4 Transfer of Complete Lifts of the Vector-Valued 2-Form

Let S be a vector-valued 2-form on M. A semi-Riemannian metric g is called pure with respect to S if

$$g(S_Y X_1, X_2) = g(X_1, S_Y X_2)$$

for any $X_1, X_2, Y \in \Im_0^1(M)$, where S_Y denotes a tensor field of type $(1,1)$ such that

$$S_Y(Z) = S(Y,Z) = -S(Z,Y) = -S_Z(Y)$$

for any $Y, Z \in \Im_0^1(M)$. The condition of purity of g may be expressed in terms of the local components as follows:

$$g_{mi_2} S_{i_1 l}^m = g_{i_1 m} S_{i_2 l}^m.$$

We now define the Yano-Ako operator

$$\Phi_S : \Im_2^0(M) \to \Im_4^0(M)$$

associated with S and applied to a pure tensor field g by (see [6–7])

$$\begin{aligned}(\Phi_S g)(X_1, X_2, Y_1, Y_2) = {} & (L_{S(X_1,X_2)} g)(Y_1, Y_2) - (L_{X_1}(g \circ S))(Y_1, X_2, Y_2) \\ & - (L_{X_2}(g \circ S))(X_1, Y_1, Y_2) + (g \circ S)([X_1, X_2], Y_1, Y_2),\end{aligned}$$

where $(g \circ S)(X, Y_1, Y_2) = g(S(X, Y_1), Y_2)$. The Yano-Ako operator has the following components with respect to the natural coordinate system:

$$(\Phi_S g)_{jihs} = S_{ji}^m \partial_m g_{hs} - (\partial_j S_{hi}^m) g_{ms} - (\partial_j g_{ms}) S_{hi}^m - (\partial_i S_{jh}^m) g_{ms} \\ - (\partial_i g_{ms}) S_{jh}^m + (\partial_h S_{ji}^m) g_{ms} + (\partial_s S_{ji}^m) g_{hm}. \tag{4.1}$$

The non-zero components of the complete lift ${}^C S_T$ of S to the tangent bundle TM are given by (see [8, p. 22])

$${}^C S_{ji}^h = {}^C S_{\bar{j}\,i}^{\bar{h}} = {}^C S_{j\,\bar{i}}^{\bar{h}} = S_{ji}^h, \quad {}^C S_{ji}^{\bar{h}} = x^{\overline{m}} \partial_m S_{ji}^h\,.$$

Using (1.1) and (1.2), we can easily verify that

$$g_*^\flat {}^C S_T = (\widetilde{S}_{JI}^H) = (A_M^H \widetilde{A}_J^K \widetilde{A}_I^P {}^C S_{KP}^M),$$

and $I, J, \cdots = 1, \cdots, 2n$ has non-zero components of the form

$$\begin{aligned}
\widetilde{S}\,{}_{ji}^{h} &= \delta_m^h \delta_j^k \delta_i^t {}^C S_{kt}^m = S_{ji}^h, \\
\widetilde{S}\,{}_{\bar{j}\,i}^{\bar{h}} &= g_{hm} g^{kj} \delta_i^t {}^C S_{\bar{k}\,t}^{\overline{m}} = g_{mk} g^{kj} S_{hi}^m = \delta_m^j S_{hi}^m = S_{hi}^j, \\
\widetilde{S}\,{}_{j\,\bar{i}}^{\bar{h}} &= g_{hm} \delta_j^k g^{ti} {}^C S_{k\,\bar{t}}^{\overline{m}} = g_{mt} g^{ti} S_{jh}^m = \delta_m^i S_{jh}^m = S_{jh}^i, \\
\widetilde{S}_{ji}^{\bar{h}} &= y^s (\partial_m g_{hs}) \delta_j^k \delta_i^t {}^C S_{kt}^m + g_{hm} \delta_j^k \delta_i^t {}^C S_{kt}^{\overline{m}} + g_{hm} p_s (\partial_j g^{ks}) {}^C S_{\bar{k}t}^{\overline{m}} + g_{hm} \delta_j^k p_s (\partial_i g^{ts}) {}^C S_{k\bar{t}}^{\overline{m}} \\
&= y^s (\partial_m g_{hs}) \delta_j^k \delta_i^t S_{kt}^m + g_{hm} \delta_j^k \delta_i^t y^s \partial_s S_{kt}^m + g_{hm} p_s (\partial_j g^{ks}) S_{kt}^m + g_{hm} \delta_j^k p_s (\partial_i g^{ts}) S_{kt}^m \\
&= y^s (\partial_m g_{hs}) S_{ji}^m + g_{hm} y^s \partial_s S_{ji}^m + g_{hm} p_s (\partial_j g^{ks}) S_{ki}^m + g_{hm} p_s (\partial_i g^{ts}) S_{jt}^m \\
&= y^s (\Phi_S g)_{jihs} + y^s (\partial_j S_{hi}^m) g_{ms} + y^s (\partial_j g_{ms}) S_{hi}^m + y^s (\partial_i S_{jh}^m) g_{ms} \\
&\quad + y^s (\partial_i g_{ms}) S_{jh}^m - y^s (\partial_h S_{ji}^m) g_{ms} - y^k (\partial_j g_{km}) S_{hi}^m - y^t (\partial_i g_{tm}) S_{jh}^m \\
&= y^s (\Phi_S g)_{jihs} + y^s (\partial_j S_{hi}^m) g_{ms} + y^s (\partial_i S_{jh}^m) g_{ms} - y^s (\partial_h S_{ji}^m) g_{ms} \\
&= y^s (\Phi_S g)_{jihs} - p_m (\partial_j S_{ih}^m + \partial_i S_{hj}^m + \partial_h S_{ji}^m),
\end{aligned}$$

i.e., the transfer $g_*^\flat {}^C S_T$ coincides with the complete lift ${}^C S_{T^*}$ of the vector-valued 2-form $S \in \wedge_2(M)$ to the cotangent bundle if and only if

$$(\Phi_S g)_{jihs} = 0.$$

Thus we have the following theorem.

Theorem 4.1 *Let g be a pure pseudo-Riemanian metric with respect to the vector-valued 2-form $S \in \wedge_2(M)$, and let ${}^C S_T$ and ${}^C S_{T^*}$ be complete lifts of S to the tangent and cotangent bundles, respectively. Then*

$$g_*^\flat {}^C S_T = {}^C S_{T^*}$$

if and only if g satisfies the following Yano-Ako equation:

$$(\Phi_S g)_{jihs} = 0\,,$$

where $\Phi_S g$ is the operator defined by (4.1).

5 Transfer of Complete Lifts of Metrics

Let ${}^Cg$ be a complete lift of a pseudo-Riemannian metric g to TM with components

$$ {}^Cg = ({}^Cg_{IJ}) = \begin{pmatrix} y^s\partial_s g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}. \tag{5.1} $$

Using (1.2) and (5.1) we see that the pullback of ${}^Cg$ by $g^\sharp$ is the $(0,2)$-tensor field $(g^\sharp)^*\,{}^Cg$ on T^*M and has components

$$ \begin{aligned} (((g^\sharp)^*\,{}^Cg)_{KL}) &= (A^I_K A^J_L {}^Cg_{IJ}) \\ &= \begin{pmatrix} A^i_k A^j_l {}^Cg_{i\,j} + A^{\bar i}_k A^j_l {}^Cg_{\bar i\,j} + A^i_k A^{\bar j}_l {}^Cg_{i\,\bar j} & A^i_k A^{\bar j}_{\bar l} {}^Cg_{i\,\bar j} \\ A^{\bar i}_{\bar k} A^j_l {}^Cg_{\bar i\,j} & 0 \end{pmatrix} \\ &= \begin{pmatrix} y^s\partial_s g_{kl} + p_s((\partial_k g^{is})g_{i\,l} + (\partial_l g^{js})g_{k\,j}) & \delta^l_k \\ \delta^k_l & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_t g^{st}\partial_s g_{kl} - p_s(g^{is}\partial_k g_{i\,l} + g^{js}\partial_l g_{k\,j}) & \delta^l_k \\ \delta^k_l & 0 \end{pmatrix} \\ &= \begin{pmatrix} -p_s g^{ts}(\partial_l g_{tk} + \partial_k g_{lt} - \partial_t g_{kl}) & \delta^l_k \\ \delta^k_l & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2p_s\Gamma^s_{kl} & \delta^l_k \\ \delta^k_l & 0 \end{pmatrix}. \end{aligned} \tag{5.2} $$

On the other hand, a new pseudo-Riemannian metric ${}^\nabla g \in \Im^0_2(T^*M)$ on T^*M is defined by the equation (see [8, p. 268])

$$ {}^\nabla g({}^CX, {}^CY) = -\gamma(\nabla_X Y + \nabla_Y X) $$

for any $X, Y \in \Im^1_0(M)$, where $\gamma(\nabla_X Y + \nabla_Y X)$ is a function in $\pi^{-1}(U) \subset T^*M$ with a local expression

$$ \gamma(\nabla_X Y + \nabla_Y X) = p_h(X^i\nabla_i Y^h + Y^i\nabla_i X^h), $$

and is called a Riemannian extension of the Levi-Civita connection ${}^\nabla g$ to T^*M. The Riemannian extension ${}^\nabla g$ has components of the form

$$ {}^\nabla g = ({}^\nabla g_{IJ}) = \begin{pmatrix} -2p_m\Gamma^m_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix} \tag{5.3} $$

with respect to the natural frame $\{\partial_i, \partial_{\bar i}\}$. Thus, from (5.2) and (5.3) we obtain $(g^\sharp)^*\,{}^Cg = {}^\nabla g$, i.e., we have the following theorem.

Theorem 5.1 *The Riemannian extension* ${}^\nabla g \in \Im^0_2(T^*M)$ *is a pullback of the complete lift* ${}^Cg \in \Im^0_2(TM)$*.*

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