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Orientable Small Covers over a Product Space*

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Abstract A small cover is a closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope P^n. Let Δ^n denote an n-simplex and $P(m)$ an m-gon. This paper gives formulas for calculating the number of D-J equivalent classes and equivariant homeomorphism classes of orientable small covers over the product space $\Delta^{n_1} \times \Delta^{n_2} \times P(m)$, where n_1 is odd.

Keywords $(\mathbb{Z}_2)^n$-Action, Small cover, Equivariant homeomorphism, Polytope
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1 Introduction

The geometry of toric varieties is one of the fascinating topics in algebraic geometry and has found applications in many branches of mathematical sciences. From a combinatorial viewpoint, it is known that there is a one-to-one correspondence between fans in $\mathbb{R}^n$ and toric varieties of complex dimension n. Given an n-polytope P^n with vertices in the integer lattice $\mathbb{Z}^n$, a fan is generated by the set of normal vectors corresponding to faces of codimension 1 of P^n. According to the above correspondence between fans and toric varieties, there is a toric variety M_P associated to the fan. Let T^n denote a torus $(S^1)^n$. It turns out that as a topological space, $M_P = T^n \times P^n/\sim$ for some equivalent relation $\sim$. The torus T^n naturally acts on M_P, and P^n is the orbit space. Inspired by the above identification space description of a toric variety, Davis and Januszkiewicz introduced a topological counterpart, namely, the study of small covers and quasitoric manifolds in [5].

A small cover (see [5]) is a closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope P^n, where $\mathbb{Z}_2$ denotes the cyclic group of order 2. For instance, the real projective space $\mathbb{R}P^n$ with a natural $(\mathbb{Z}_2)^n$-action is a small cover over an n-simplex. This makes the research on the equivariant topology of small covers possible through the combinatorial structure of the orbit space.

In [7], Lü and Masuda showed that the equivariant homeomorphism class of a small cover over a simple convex polytope P^n agrees with the equivalent class of its corresponding char-

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acteristic functions under the action of the automorphism group of face poset (i.e., a partially ordered set by inclusion) of P^n. This finding also holds true for orientable small covers by the orientability condition (see Theorem 2.2). However, it is a hard task to obtain general formulas for calculating the number of equivariant homeomorphism classes of (orientable) small covers over an arbitrary simple convex polytope.

In recent years, several studies have attempted to calculate the number of equivalent classes of all small covers over a specific polytope. Garrison and Scott used a computer program to calculate the number of homeomorphism classes of all small covers over a dodecahedron (see [6]). Cai, Chen and Lü calculated the number of equivariant homeomorphism classes of small covers over prisms (see [2]). However, little is known about orientable small covers. Choi calculated the number of D-J equivalent classes of orientable small covers over cubes (see [4]). Chen and Wang calculated the number of equivariant homeomorphism classes of orientable small covers over a product of at most three simplices (see [3]). From [9], we know the existence of orientable small covers over the polytope $\Delta^{n_1} \times \Delta^{n_2} \times P(m)$, where Δ^{n_i} denotes a simplex of dimension n_i and $P(m)$ an m-gon. The objective of this paper is to determine the number of D-J equivalent classes and equivariant homeomorphism classes of all orientable small covers over $\Delta^{n_1} \times \Delta^{n_2} \times P(m)$, where n_1 is odd.

This paper is organized as follows. In Section 2, we review the fundamental knowledge on small covers and list several known theorems. In Section 3, we calculate the number of D-J equivalent classes of the orientable small covers over the product space and the number of orientable characteristic functions corresponding to orientable small covers. In Section 4 we obtain a formula for calculating the number of equivariant homeomorphism classes of all orientable small covers over the product space.

2 Preliminaries

The standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ is that

$$\begin{aligned}&(\mathbb{Z}_2)^n \times \mathbb{R}^n \to \mathbb{R}^n,\\&(g_1, \cdots, g_n) \times (x_1, \cdots, x_n) \mapsto ((-1)^{g_1} x_1, \cdots, (-1)^{g_n} x_n),\end{aligned}$$

and the orbit space is $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}$.

A convex polytope P^n of dimension n is said to be simple if every vertex of P^n is the intersection of exactly n facets (i.e., faces of dimension $(n-1)$) (see [10]). An n-dimensional closed manifold M^n is said to be a small cover if it admits a $(\mathbb{Z}_2)^n$-action such that the action is locally isomorphic to a standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ and the orbit space $M^n/(\mathbb{Z}_2)^n$ is a simple convex polytope of dimension n (see [5]).

Let P^n be a simple convex polytope of dimension n and $\mathcal{F}(P^n) = \{F_1, \cdots, F_l\}$ be the set of facets of P^n. Suppose that $\pi : M^n \to P^n$ is a small cover over P^n. Then there are l connected submanifolds $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_l)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a subgroup $\mathbb{Z}_2(F_i)$ of rank 1 in $(\mathbb{Z}_2)^n$, so that each facet F_i corresponds to a subgroup $\mathbb{Z}_2(F_i)$ of rank 1. Obviously, the subgroup $\mathbb{Z}_2(F_i)$ actually agrees with an element ν_i in $(\mathbb{Z}_2)^n$ as a vector space.

For each face F of codimension u of P^n, since P^n is simple, there are u facets $F_{i_1}, \cdots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}), \cdots, \pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbb{Z}_2(F)$ of $\pi^{-1}(F)$ is generated by $\mathbb{Z}_2(F_{i_1}), \cdots, \mathbb{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1}, \cdots, \nu_{i_u}$ in $(\mathbb{Z}_2)^n$), and has rank u. Thus, this gives a function (see [5]):

$$\lambda : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^n,$$

which is defined by $\lambda(F_i) = \nu_i$. This function satisfies the non-singularity condition: $\lambda(F_{i_1}), \cdots, \lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$ as a vector space whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_u}$ is non-empty. We call λ, which satisfies the non-singularity condition, a characteristic function on P^n.

In fact, Davis and Januszkiewicz [5] also gave a reconstruction process of a small cover using a characteristic function $\lambda : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^n$. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, we denote the minimal face containing p in its relative interior by $F(p)$. Assuming that $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbb{Z}_2(F(p)) = \bigoplus_{j=1}^{u} \mathbb{Z}_2(F_{i_j})$, then $\mathbb{Z}_2(F(p))$ is a subgroup of rank u in $(\mathbb{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbb{Z}_2)^n / \sim$, where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^n$ on $P^n \times (\mathbb{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient P^n. Thus, $M(\lambda)$ is a small cover over P^n.

Two small covers M_1 and M_2 over P^n are said to be weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are equivariantly homeomorphic. Following [5], two small covers M_1 and M_2 over P^n are said to be Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P^n.

By $\Lambda(P^n)$, we denote the set of characteristic functions on P^n. We have the following result.

Theorem 2.1 (see [5]) *All small covers over P^n are given by $\{M(\lambda) \mid \lambda \in \Lambda(P^n)\}$, i.e., for each small cover M^n over P^n, there is a characteristic function λ with an equivariant homeomorphism $M(\lambda) \to M^n$ covering the identity on P^n.*

Nakayama and Nishimura gave an orientability condition for a small cover in [9].

Theorem 2.2 (see [9]) *For a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \to \mathbb{Z}_2 = \{0, 1\}$ is defined by $\varepsilon(e_i) = 1$ $(i = 1, \cdots, n)$. A small cover $M(\lambda)$ over a simple convex polytope P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.*

From Theorem 2.2, we know that a small cover $M(\lambda)$ over P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that $\lambda(F_i) = e_{i_1} + e_{i_2} + \cdots + e_{i_{2h_i+1}}, 1 \leq i_1 < i_2 < \cdots < i_{2h_i+1} \leq n$ for any $F_i \in \mathcal{F}(P^n)$. A characteristic function which satisfies the orientability condition in Theorem 2.2 is called an orientable characteristic function. We immediately have that all orientable small covers over P^n are given by orientable characteristic functions on P^n. So, we know the existence of an orientable small cover over P^n by the existence of an orientable characteristic function. However, it is worth noting that different orientable characteristic functions may correspond to the same orientable small cover.

In order to classify orientable small covers over P^n up to D-J equivalence, by $O(P^n)$ we denote the set of orientable characteristic functions on P^n, and consider a free action of $GL(n,\mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\sigma \times \lambda \mapsto \sigma \circ \lambda$. We assume that $F_1, \cdots, F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1, \cdots, e_n$ be the standard basis of $(\mathbb{Z}_2)^n$ and $B(P^n) = \{\lambda \in O(P^n) \mid \lambda(F_i) = e_i, i = 1, \cdots, n\}$. Then $B(P^n)$ is the orbit space of $O(P^n)$ under the action of $GL(n,\mathbb{Z}_2)$. In fact, for $\lambda \in B(P^n)$ and $n+1 \le i \le l$, we have $\lambda(F_i) = e_{i_1} + e_{i_2} + \cdots + e_{i_{2h_i+1}}, 1 \le i_1 < i_2 < \cdots < i_{2h_i+1} \le n$. The relation between the number of elements of $O(P^n)$ and the number of elements of $B(P^n)$ is given by the following lemma.

Lemma 2.1 $|O(P^n)| = |B(P^n)| \times |GL(n,\mathbb{Z}_2)|$.

It is easy to check that two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma \in GL(n,\mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. Thus the number of D-J equivalent classes of orientable small covers over P^n is $|B(P^n)| = |O(P^n)|/|GL(n,\mathbb{Z}_2)|$. From [1], we know $|GL(n,\mathbb{Z}_2)| = \prod_{k=1}^{n} (2^n - 2^{k-1})$.

For calculating the number of equivariant homeomorphism classes of orientable small covers over a simple convex polytope P^n, we consider a poset consisting of faces of P^n (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^n)$ is a bijection from $\mathcal{F}(P^n)$ to itself that preserves the poset structure of all faces of P^n. By $\mathrm{Aut}(\mathcal{F}(P^n))$, we denote the group of automorphisms of $\mathcal{F}(P^n)$. We define a right action of $\mathrm{Aut}(\mathcal{F}(P^n))$ on $O(P^n)$ by $\lambda \times h \mapsto \lambda \circ h$, where $\lambda \in O(P^n)$ and $h \in \mathrm{Aut}(\mathcal{F}(P^n))$. By improving the classifying result on unoriented small covers in [7], we have the following theorem.

Theorem 2.3 *Two orientable small covers over an n-dimensional simple convex polytope P^n are equivariantly homeomorphic if and only if there is $h \in \mathrm{Aut}(\mathcal{F}(P^n))$ such that $\lambda_1 = \lambda_2 \circ h$, where λ_1 and λ_2 are their corresponding orientable characteristic functions on P^n.*

Proof It is proven true by combining Lemma 5.4 in [7] with Theorem 2.2.

According to Theorem 2.3, the number of equivariant homeomorphism classes of orientable small covers over P^n is equal to the number of orbits of $O(P^n)$ under the action of $\mathrm{Aut}(\mathcal{F}(P^n))$. The famous Burnside Lemma is very useful in enumerating the number of orbits.

Lemma 2.2 (Burnside Lemma) *Let G be a finite group acting on a set X. Then the number of orbits under the action of G equals $\frac{1}{|G|} \sum_{g \in G} |X_g|$, where $X_g = \{x \in X \mid gx = x\}$.*

To determine the number of the orbits of $O(P^n)$ under the action of $\mathrm{Aut}(\mathcal{F}(P^n))$, the Burnside Lemma suggests that the structure of $\mathrm{Aut}(\mathcal{F}(P^n))$ should be understood.

Here we shall particularly be concerned with the case $P^n = \Delta^{n_1} \times \Delta^{n_2} \times P(m)$, where Δ^{n_i} denotes a simplex of dimension n_i and $P(m)$ an m-gon. In this case, by $O(n_1, n_2, m)$, $B(n_1, n_2, m)$, $\mathcal{F}(n_1, n_2, m)$ and $\mathrm{Aut}(\mathcal{F}(n_1, n_2, m))$, we denote $O(\Delta^{n_1} \times \Delta^{n_2} \times P(m))$, $B(\Delta^{n_1} \times \Delta^{n_2} \times P(m))$,

$\mathcal{F}(\Delta^{n_1}\times\Delta^{n_2}\times P(m))$ and $\mathrm{Aut}(\mathcal{F}(\Delta^{n_1}\times\Delta^{n_2}\times P(m)))$ respectively. Then we have the following lemma.

Lemma 2.3 (see [8]) *The automorphism group* $\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$ *is isomorphic to*

$$\begin{cases} S_{n_1+1}\times\mathbb{Z}_2\times\mathcal{D}_m, & n_1>2,n_2=1,m=3;\ n_1\geq 2,n_2=1,m>4;\\ S_{n_1+1}\times(\mathbb{Z}_2)^3\times S_3, & n_1\geq 2,n_2=1,m=4;\\ (S_3)^2\times(\mathbb{Z}_2)^2, & n_1=2,n_2=1,m=3;\\ S_{n_1+1}\times S_{n_2+1}\times\mathcal{D}_m, & n_1>n_2=2,m>3;\ n_1>n_2>2,m\geq 3;\\ S_{n_1+1}\times S_{n_2+1}\times\mathcal{D}_m\times\mathbb{Z}_2, & n_1=n_2=2,m>3;\ n_1=n_2>2,m\geq 3;\ n_1>n_2=2,m=3;\\ (S_3)^4, & n_1=n_2=2,m=3,\end{cases}$$

where $\mathcal{D}_m$ *is the dihedral group of order* $2m$ *and* S_{n+1} *is the symmetric group on* $n+1$ *symbols.*

3 D-J Equivalent Classes and Orientable Characteristic Functions

Let $\{a_1,\cdots,a_{n_1},a_{n_1+1}\}$ be the set of facets of $\Delta^{n_1}\times\Delta^{n_2}\times P(m)$ corresponding to $\mathcal{F}(\Delta^{n_1})\times\Delta^{n_2}\times P(m)$, $\{b_1,\cdots,b_{n_2},b_{n_2+1}\}$ the set of facets corresponding to $\Delta^{n_1}\times\mathcal{F}(\Delta^{n_2})\times P(m)$, and $\{c_1,c_2,\cdots,c_m\}$ the set of facets corresponding to $\Delta^{n_1}\times\Delta^{n_2}\times\mathcal{F}(P(m))$ in their general order. Then $\mathcal{F}(n_1,n_2,m)=\{a_1,\cdots,a_{n_1},a_{n_1+1}\}\cup\{b_1,\cdots,b_{n_2},b_{n_2+1}\}\cup\{c_1,c_2,\cdots,c_m\}$. Without loss of generality, we assume that $a_1,\cdots,a_{n_1},b_1,\cdots,b_{n_2},c_1$ and c_2 are facets of $\Delta^{n_1}\times\Delta^{n_2}\times P(m)$ which meet at a vertex. Let $e_1,e_2,\cdots,e_{n_1+n_2+2}$ be the standard basis of $(\mathbb{Z}_2)^{n_1+n_2+2}$. By definition,

$$\begin{aligned}B(n_1,n_2,m)=\{\lambda\in O(n_1,n_2,m)\mid \lambda(a_i)=e_i,i=1,\cdots,n_1,\lambda(b_j)=e_{n_1+j},j=1,\cdots,n_2,\\ \lambda(c_1)=e_{n_1+n_2+1},\ \lambda(c_2)=e_{n_1+n_2+2}\}.\end{aligned}$$

Let $f_i(n_1,n_2,m)$ be the recursive functions which are listed in Appendix 1. Now we arrive at the first main result.

Theorem 3.1 *Let* n_1, n_2 *and* m *be positive integers, with* n_1 *odd,* $n_1\geq 2$, $n_2\geq 1$, $n_1\geq n_2$ *and* $m\geq 3$. *Then the number of D-J equivalent classes of orientable small covers over* $\Delta^{n_1}\times\Delta^{n_2}\times P(m)$ *is*

$$|B(n_1,n_2,m)|=\sum_{i\in I_1}f_i(n_1,n_2,m)+\sum_{i\in I_2}f_i(n_1,n_2,m),$$

where $I_1=\{1,2,\cdots,24\}$ *and* $I_2=\{8,12,17,19\}$.

Proof By the non-singularity condition of orientable characteristic functions, we have

$$\lambda(a_{n_1+1})=\sum_{i=1}^{n_1}e_i+\sum_{i=1}^{n_2}\varepsilon_ie_{n_1+i}+\varepsilon_{n_2+1}e_{n_1+n_2+1}+\varepsilon_{n_2+2}e_{n_1+n_2+2},$$

$$\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+\sum_{i=1}^{n_1}\delta_ie_i+\delta_{n_1+1}e_{n_1+n_2+1}+\delta_{n_1+2}e_{n_1+n_2+2},$$

where ε_i, $\delta_j=0$ or 1, $i=1,\cdots,n_2+2$, $j=1,\cdots,n_1+2$, and

$$\lambda(c_m)=\sum_{i=1}^{n_1+n_2+2}\gamma_ie_i,\quad \lambda(c_{m-1})=\sum_{i=1}^{n_1+n_2+2}\theta_ie_i,\quad \lambda(c_{m-2})=\sum_{i=1}^{n_1+n_2+2}\mu_ie_i$$

with $\gamma_i, \theta_j, \mu_p = 0$ or 1.

The calculation of $|B(n_1, n_2, m)|$ is divided into eight cases. Write

$$\begin{cases} B_1(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i\Big\}, \\ B_2(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + \sum\limits_{i=1}^{n_2} \varepsilon_i e_{n_1+i}, \exists \varepsilon_i \neq 0\Big\}, \\ B_3(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + e_{n_1+n_2+1}\Big\}, \\ B_4(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + e_{n_1+n_2+1} + \sum\limits_{i=1}^{n_2} \varepsilon_i e_{n_1+i}, \exists \varepsilon_i \neq 0\Big\}, \\ B_5(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + e_{n_1+n_2+2}\Big\}, \\ B_6(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + e_{n_1+n_2+2} + \sum\limits_{i=1}^{n_2} \varepsilon_i e_{n_1+i}, \exists \varepsilon_i \neq 0\Big\}, \\ B_7(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + e_{n_1+n_2+1} + e_{n_1+n_2+2}\Big\}, \\ B_8(n_1,n_2,m) = \Big\{\lambda \in O(n_1,n_2,m) \mid \lambda(a_{n_1+1}) = \sum\limits_{i=1}^{n_1} e_i + e_{n_1+n_2+1} + e_{n_1+n_2+2} \\ \qquad\qquad + \sum\limits_{i=1}^{n_2} \varepsilon_i e_{n_1+i}, \exists \varepsilon_i \neq 0\Big\}. \end{cases}$$

Then $|B(n_1,n_2,m)| = \sum\limits_{i=1}^{8} |B_i(n_1,n_2,m)|$. Now we calculate $|B_i(n_1,n_2,m)|$ $(1 \le i \le 8)$.

Case 1 Calculation of $|B_1(n_1,n_2,m)|$.

(1) $B_{11}(n_1,n_2,m) = \big\{\lambda \in B_1(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + \sum\limits_{i=1}^{n_1} \delta_i e_i\big\}$.

According to the non-singularity condition and the orientability condition of characteristic functions, the number of $\delta_i = 1$ is even for n_2 odd, the number of $\delta_i = 1$ is odd for n_2 even, and the coefficients in $\lambda(c_m)$ and $\lambda(c_{m-1})$ are listed in Table 1.

Table 1

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1, \cdots, \gamma_{n_1+n_2}, 0, 1$ $(\lvert\{j \mid \gamma_j \neq 0\}\rvert = 2k)$	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\lvert\{j \mid \theta_j \neq 0\}\rvert = 2k)$
	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 1$ $(\lvert\{j \mid \theta_j \neq 0\}\rvert = 2k+1)$
$\gamma_1, \cdots, \gamma_{n_1+n_2}, 1, 1$ $(\lvert\{j \mid \gamma_j \neq 0\}\rvert = 2k+1)$	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\lvert\{j \mid \theta_j \neq 0\}\rvert = 2k)$
	$\theta_1, \cdots, \theta_{n_1+n_2}, 0, 1$ $(\lvert\{j \mid \theta_j \neq 0\}\rvert = 2k)$

Set

$$B_{11}^1(n_1,n_2,m) = \Big\{\lambda \in B_{11}(n_1,n_2,m) \mid \lambda(c_{m-1}) = \sum_{i=1}^{n_1+n_2} \theta_i e_i + e_{n_1+n_2+1}\Big\},$$
$$B_{11}^2(n_1,n_2,m) = B_{11}(n_1,n_2,m) \setminus B_{11}^1(n_1,n_2,m).$$

If $\lambda \in B_{11}^1(n_1,n_2,m)$, then the values of λ restricted to c_{m-1} have $2^{n_1+n_2-1}$ possible choices and the values of λ restricted to c_m have $2^{n_1+n_2}$ possible choices. So, we obtain a recursive equation

$$|B_{11}^1(n_1,n_2,m)| = 2^{2n_1+2n_2-1}|B_{11}(n_1,n_2,m-2)|.$$

If $\lambda \in B_{11}^2(n_1,n_2,m)$, then $\lambda(c_{m-1})$ has $2^{n_1+n_2-1}$ possible values. We have $|B_{11}^2(n_1,n_2,m)| = 2^{n_1+n_2-1}|B_{11}(n_1,n_2,m-1)|$. So

$$|B_{11}(n_1,n_2,m)| = 2^{n_1+n_2-1}|B_{11}(n_1,n_2,m-1)| + 2^{2n_1+2n_2-1}|B_{11}(n_1,n_2,m-2)|.$$

Since $\lambda(b_{n_2+1})$ has 2^{n_1-1} possible values, a direct computation shows that $|B_{11}(n_1,n_2,3)| = (2^{n_1+n_2-1})\cdot 2^{n_1-1} = 2^{2n_1+n_2-2}$, $|B_{11}(n_1,n_2,4)| = 3\cdot 2^{2n_1+2n_2-2}\cdot 2^{n_1-1} = 3\cdot 2^{3n_1+2n_2-3}$. So

$$f_1(n_1,n_2,m) = |B_{11}(n_1,n_2,m)|.$$

(2) $B_{12}(n_1,n_2,m) = \left\{\lambda \in B_1(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+1} + \sum_{i=1}^{n_1}\delta_i e_i\right\}$.

A similar argument shows that the number of $\delta_i = 1$ is odd for n_2 odd, the number of $\delta_i = 1$ is even for n_2 even, and the coefficients in $\lambda(c_m)$ and $\lambda(c_{m-1})$ are listed in Table 2.

Table 2

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1,\cdots,\gamma_{n_1},0,\cdots,0,1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$
$\gamma_1,\cdots,\gamma_{n_1+n_2},0,1$ $(\exists\ \gamma_j \neq 0, n_1+1 \leq j \leq n_1+n_2, \lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k, 1 \leq i \leq n_1+n_2)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1,\cdots,\theta_{n_1},\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},1,0$
$\gamma_1,\cdots,\gamma_{n_1},0,\cdots,0,1,1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k+1)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
$\gamma_1,\cdots,\gamma_{n_1+n_2},1,1$ $(\exists\ \gamma_j \neq 0, n_1+1 \leq j \leq n_1+n_2, \lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k+1, 1 \leq i \leq n_1+n_2)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1,\cdots,\theta_{n_1},\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},0,1$

Let $B_{12}^1(n_1,n_2,m) = \{\lambda \in B_{12}(n_1,n_2,m) \mid \lambda(c_{m-1}) = \sum_{i=1}^{n_1}\theta_i e_i + e_{n_1+n_2+1}\}$ and $B_{12}^2(n_1,n_2,m) = B_{12}(n_1,n_2,m) \setminus B_{12}^1(n_1,n_2,m)$. Then $|B_{12}^1(n_1,n_2,m)| = 2^{2n_1+n_2-1}|B_{12}(n_1,n_2,m-2)|$ and $|B_{12}^2(n_1,n_2,m)| = 2^{n_1-1}|B_{12}(n_1,n_2,m-1)|$. With the similar argument above, we get $|B_{12}(n_1,n_2,m)| = 2^{n_1-1}|B_{12}(n_1,n_2,m-1)| + 2^{2n_1+n_2-1}|B_{12}(n_1,n_2,m-2)|$, $|B_{12}(n_1,n_2,3)| = 2^{n_1+n_2-1}\cdot 2^{n_1-1} = 2^{2n_1+n_2-2}$, and $|B_{12}(n_1,n_2,4)| = 3\cdot 2^{3n_1+n_2-3}$. So

$$f_2(n_1,n_2,m) = |B_{12}(n_1,n_2,m)|.$$

(3) $B_{13}(n_1,n_2,m) = \left\{\lambda \in B_1(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+2} + \sum_{i=1}^{n_1}\delta_i e_i\right\}$.

Similarly, we have that the number of $\delta_i = 1$ is odd for n_2 odd and the number of $\delta_i = 1$ is even for n_2 even. The coefficients γ_i and θ_j appear in Table 3.

Let $B_{13}^1(n_1,n_2,m) = \left\{\lambda \in B_{13}(n_1,n_2,m) \mid \lambda(c_{m-1}) = \sum_{i=1}^{n_1+n_2}\theta_i e_i + e_{n_1+n_2+1}\right.$ or $\sum_{i=1}^{n_1+n_2}\theta_i e_i + e_{n_1+n_2+1} + e_{n_1+n_2+2}$ $\left.(\exists\ \theta_j \neq 0, n_1+1 \leq j \leq n_1+n_2)\right\}$ and $B_{13}^2(n_1,n_2,m) = B_{13}(n_1,n_2,m) \setminus B_{13}^1(n_1,n_2,m)$. Then $|B_{13}^1(n_1,n_2,m)| = 2^{2n_1+n_2-1}|B_{13}(n_1,n_2,m-2)|$ and $|B_{13}^2(n_1,n_2,m)| = 2^{n_1-1}|B_{13}(n_1,n_2,m-1)|$. We have $|B_{13}(n_1,n_2,m)| = 2^{n_1-1}|B_{13}(n_1,n_2,m-1)| + 2^{2n_1+n_2-1}|B_{13}(n_1,n_2,m-2)|$, $|B_{13}(n_1,n_2,3)| = 2^{2n_1-2}$ and $|B_{13}(n_1,n_2,4)| = 2^{3n_1+n_2-2} + 2^{3n_1-3}$. So

$$f_3(n_1,n_2,m) = |B_{13}(n_1,n_2,m)|.$$

Table 3

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1,\cdots,\gamma_{n_1},0,\cdots,0,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$\theta_1,\cdots,\theta_{n_1+n_2},1,0$ $(\exists\ \theta_j\neq 0, n_1+1\leq j\leq n_1+n_2, \lvert\{i\mid\theta_i\neq 0\}\rvert=2k, 1\leq i\leq n_1+n_2)$
	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1)$
	$\theta_1,\cdots,\theta_{n_1+n_2},1,1$ $(\exists\ \theta_j\neq 0, n_1+1\leq j\leq n_1+n_2, \lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1, 1\leq i\leq n_1+n_2)$
$\gamma_1,\cdots,\gamma_{n_1},0,\cdots,0,1,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k+1)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$

(4) $B_{14}(n_1,n_2,m)=\big\{\lambda\in B_1(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}+e_{n_1+n_2+2}+\sum\limits_{i=1}^{n_1}\delta_ie_i\big\}$.

The number of $\delta_i=1$ is even for n_2 odd and the number of $\delta_i=1$ is odd for n_2 even. The coefficients γ_i and θ_j are in Table 4.

Table 4

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1,\cdots,\gamma_{n_1},0,\cdots,0,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1)$
$\gamma_1,\cdots,\gamma_{n_1},0,\cdots,0,1,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k+1)$	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$\theta_1,\cdots,\theta_{n_1+n_2},1,0$ $(\exists\ \theta_j\neq 0, n_1+1\leq j\leq n_1+n_2(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k, 1\leq i\leq n_1+n_2)$
	$\theta_1,\cdots,\theta_{n_1},0,\cdots,0,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1)$
	$\theta_1,\cdots,\theta_{n_1+n_2},0,1$ $(\exists\ \theta_j\neq 0, n_1+1\leq j\leq n_1+n_2(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1, 1\leq i\leq n_1+n_2)$

Let $B_{14}^1(n_1,n_2,m)=\{\lambda\in B_{14}(n_1,n_2,m)\mid\lambda(c_{m-1})=\sum\limits_{i=1}^{n_1+n_2}\theta_ie_i+e_{n_1+n_2+1}$ or $\sum\limits_{i=1}^{n_1+n_2}\theta_ie_i+e_{n_1+n_2+2}$ $(\exists\,\theta_j\neq 0, n_1+1\leq j\leq n_1+n_2)\}$ and $B_{14}^2(n_1,n_2,m)=B_{14}(n_1,n_2,m)\backslash B_{14}^1(n_1,n_2,m)$. So, $|B_{14}^1(n_1,n_2,m)|=2^{2n_1+n_2-1}|B_{14}(n_1,n_2,m-2)|$ and $|B_{14}^2(n_1,n_2,m)|=2^{n_1-1}|B_{14}(n_1,n_2,m-1)|$. Then $|B_{14}(n_1,n_2,m)|=2^{n_1-1}|B_{14}(n_1,n_2,m-1)|+2^{2n_1+n_2-1}|B_{14}(n_1,n_2,m-2)|$, $|B_{14}(n_1,n_2,3)|=2^{2n_1-2}$ and $|B_{14}(n_1,n_2,4)|=2^{3n_1+n_2-3}+2^{3n_1-2}$. So

$$f_4(n_1,n_2,m)=|B_{14}(n_1,n_2,m)|.$$

Thus $|B_1(n_1,n_2,m)|=\sum\limits_{i=1}^{4}|B_{1i}(n_1,n_2,m)|=f_1(n_1,n_2,m)+f_2(n_1,n_2,m)+f_3(n_1,n_2,m)+f_4(n_1,n_2,m)$.

Case 2 Calculation of $|B_2(n_1,n_2,m)|$.

(1) $B_{21}(n_1,n_2,m)=\big\{\lambda\in B_2(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}\big\}$.

If n_2 is even, then

$$|B_{21}(n_1,n_2,m)|=0.$$

If n_2 is odd, we consider the values of $\lambda(c_m)$ and $\lambda(c_{m-1})$ listed in Table 5.

Table 5

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1, \cdots, \gamma_{n_1+n_2}, 0, 1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k)$	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$
$\gamma_1, \cdots, \gamma_{n_1+n_2}, 1, 1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k+1)$	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1, \cdots, \theta_{n_1+n_2}, 0, 1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$

We get $|B_{21}(n_1, n_2, m)| = 2^{n_1+n_2-1}|B_{21}(n_1, n_2, m-1)| + 2^{2n_1+2n_2-1}|B_{21}(n_1, n_2, m-2)|$, $|B_{21}(n_1, n_2, 3)| = 2^{n_1+2n_2-2} - 2^{n_1+n_2-1}$ and $|B_{21}(n_1, n_2, 4)| = 3 \cdot 2^{2n_1+3n_2-3} - 3 \cdot 2^{2n_1+2n_2-2}$. So $f_5(n_1, n_2, m) = |B_{21}(n_1, n_2, m)|$.

(2) $B_{22}(n_1, n_2, m) = \big\{\lambda \in B_2(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+1}\big\}$.

If n_2 is odd, then $|B_{22}(n_1, n_2, m)| = 0$. If n_2 is even, we consider the values of $\lambda(c_m)$ and $\lambda(c_{m-1})$ in Table 6.

Table 6

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1, \cdots, \gamma_{n_1+n_2}, 0, 1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k)$	$0, \cdots, 0, 1, 0$
$\gamma_1, \cdots, \gamma_{n_1+n_2}, 1, 1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k+1)$	$0, \cdots, 0, 1, 0$

So $|B_{22}(n_1, n_2, m)| = 2^{n_1+n_2}|B_{22}(n_1, n_2, m-2)|$, $|B_{22}(n_1, n_2, 3)| = 2^{n_1+2n_2-2} - 2^{n_1+n_2-1}$ and $|B_{22}(n_1, n_2, 4)| = 2^{n_1+2n_2-1} - 2^{n_1+n_2}$. Thus $f_6(n_1, n_2, m) = |B_{22}(n_1, n_2, m)|$.

(3) $B_{23}(n_1, n_2, m) = \big\{\lambda \in B_2(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+2}\big\}$.

If n_2 is odd, then $|B_{23}(n_1, n_2, m)| = 0$. If n_2 is even, we consider the values of $\lambda(c_m)$ and $\lambda(c_{m-1})$ in Table 7.

Table 7

$\lambda(c_m)$	$\lambda(c_{m-1})$	$\lambda(c_{m-2})$
$0, \cdots, 0, 1$	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$	$0, \cdots, 0, 1$
$0, \cdots, 0, 1$	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$	$0, \cdots, 0, 1$

So $|B_{23}(n_1, n_2, m)| = 2^{n_1+n_2}|B_{23}(n_1, n_2, m-2)|$, $|B_{23}(n_1, n_2, 3)| = 0$ and $|B_{23}(n_1, n_2, 4)| = 2^{n_1+2n_2-1} - 2^{n_1+n_2}$. Thus $f_7(n_1, n_2, m) = |B_{23}(n_1, n_2, m)|$.

(4) $B_{24}(n_1, n_2, m) = \big\{\lambda \in B_2(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+1} + e_{n_1+n_2+2}\big\}$.

If n_2 is even, then $|B_{24}(n_1, n_2, m)| = 0$. If n_2 is odd, then $\lambda(c_m) = e_{n_1+n_2+2}, \lambda(c_{m-1}) = e_{n_1+n_2+1}$. Thus $|B_{24}(n_1, n_2, m)| = |B_{24}(n_1, n_2, m-2)|$, $|B_{24}(n_1, n_2, 3)| = 0$ and $|B_{24}(n_1, n_2, 4)| = 2^{n_2-1} - 1$. So $f_8(n_1, n_2, m) = |B_{24}(n_1, n_2, m)|$.

Thus $|B_2(n_1, n_2, m)| = \sum\limits_{i=1}^{4} |B_{2i}(n_1, n_2, m)| = f_5(n_1, n_2, m) + f_6(n_1, n_2, m) + f_7(n_1, n_2, m) + f_8(n_1, n_2, m)$.

Case 3 Calculation of $|B_3(n_1, n_2, m)|$.

Because n_1 is odd, $|B_3(n_1, n_2, m)| = 0$.

Case 4 Calculation of $|B_4(n_1, n_2, m)|$.

(1) $B_{41}(n_1, n_2, m) = \big\{\lambda \in B_4(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i}\big\}$.

If n_2 is even, then $|B_{41}(n_1,n_2,m)|=0$. If n_2 is odd, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 8.

Table 8

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},0,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1)$
$\gamma_1,\cdots,\gamma_{n_1+n_2},0,1$ $(\exists\ \gamma_i\neq 0, 1\le i\le n_1, \lvert\{j\mid\gamma_j\neq 0\}\rvert=2k, 1\le j\le n_1+n_2)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$\gamma_1,\cdots,\gamma_{n_1},\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},1,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k+1)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},0,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
$\gamma_1,\cdots,\gamma_{n_1+n_2},1,1$ $(\exists\ \gamma_i\neq 0, 1\le j\le n_1, \lvert\{j\mid\gamma_j\neq 0\}\rvert=2k+1, 1\le i\le n_1+n_2)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$
	$\gamma_1,\cdots,\gamma_{n_1},\theta_{n_1+1},\cdots,\theta_{n_1+n_2},0,1$

Let $B_{41}^1(n_1,n_2,m)=\big\{\lambda\in B_4(n_1,n_2,m)\mid\lambda(c_{m-1})=\sum\limits_{i=1}^{n_1}\theta_i e_i+e_{n_1+n_2+1}\big\}$ and $B_{41}^2(n_1,n_2,m)=B_{41}(n_1,n_2,m)\setminus B_{41}^1(n_1,n_2,m)$. Then $|B_{41}^1(n_1,n_2,m)|=2^{n_1+2n_2-1}|B_{41}(n_1,n_2,m-2)|$ and $|B_{41}^2(n_1,n_2,m)|=2^{n_2-1}|B_{41}(n_1,n_2,m-1)|$. So $|B_{41}(n_1,n_2,m)|=2^{n_2-1}|B_{41}(n_1,n_2,m-1)|+2^{n_1+2n_2-1}|B_{41}(n_1,n_2,m-2)|$, $|B_{41}(n_1,n_2,3)|=2^{n_1+n_2-1}\cdot 2^{n_2-1}=2^{n_1+2n_2-2}$ and

$$|B_{41}(n_1,n_2,4)|=3\cdot 2^{n_1+3n_2-3}.$$

So $f_9(n_1,n_2,m)=|B_{41}(n_1,n_2,m)|$.

(2) $B_{42}(n_1,n_2,m)=\big\{\lambda\in B_4(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}\big\}$.

If n_2 is odd, then $|B_{42}(n_1,n_2,m)|=0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 9.

Table 9

$\lambda(c_m)$	$\lambda(c_{m-1})$
$\gamma_1,\cdots,\gamma_{n_1+n_2},0,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k)$	$0,\cdots,0,1,0$
$\gamma_1,\cdots,\gamma_{n_1+n_2},1,1$ $(\lvert\{i\mid\gamma_i\neq 0\}\rvert=2k+1)$	$0,\cdots,0,1,0$

We get $|B_{42}(n_1,n_2,m)|=2^{n_1+n_2}|B_{42}(n_1,n_2,m-2)|$, $|B_{42}(n_1,n_2,3)|=2^{n_1+2n_2-2}$ and $|B_{42}(n_1,n_2,4)|=2^{n_1+2n_2-1}$. So $f_{10}(n_1,n_2,m)=|B_{42}(n_1,n_2,m)|$.

(3) $B_{43}(n_1,n_2,m)=\big\{\lambda\in B_4(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+2}\big\}$.

Table 10

$\lambda(c_m)$	$\lambda(c_{m-1})$	$\lambda(c_{m-2})$
$0,\cdots,0,1$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k)$	$0,\cdots,0,1$
$0,\cdots,0,1$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$ $(\lvert\{i\mid\theta_i\neq 0\}\rvert=2k+1)$	$0,\cdots,0,1$

If n_2 is odd, then $|B_{43}(n_1,n_2,m)|=0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 10.

We have $|B_{43}(n_1,n_2,m)|=2^{n_2}|B_{43}(n_1,n_2,m-2)|$, $|B_{43}(n_1,n_2,3)|=0$ and $|B_{43}(n_1,n_2,4)|=2^{2n_2}-1$. So $f_{11}(n_1,n_2,m)=|B_{43}(n_1,n_2,m)|$.

(4) $B_{44}(n_1,n_2,m)=\big\{\lambda\in B_4(n_1,n_2,m)\mid \lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}+e_{n_1+n_2+2}\big\}$.

If n_2 is even, then $|B_{44}(n_1,n_2,m)|=0$. If n_2 is odd, then $\lambda(c_m)=e_{n_1+n_2+2}$, $\lambda(c_{m-1})=e_{n_1+n_2+1}$. We have $|B_{44}(n_1,n_2,m)|=|B_{44}(n_1,n_2,m-2)|$, $|B_{44}(n_1,n_2,3)|=0$ and $|B_{44}(n_1,n_2,4)|=2^{n_2-1}$. So $f_{12}(n_1,n_2,m)=|B_{44}(n_1,n_2,m)|$.

Thus $|B_4(n_1,n_2,m)|=\sum\limits_{i=1}^{4}|B_{4i}(n_1,n_2,m)|=f_9(n_1,n_2,m)+f_{10}(n_1,n_2,m)+f_{11}(n_1,n_2,m)+f_{12}(n_1,n_2,m)$.

Case 5 Calculation of $|B_5(n_1,n_2,m)|$.

Because n_1 is odd, $|B_5(n_1,n_2,m)|=0$.

Case 6 Calculation of $|B_6(n_1,n_2,m)|$.

In this case, the number of $\varepsilon_i=1$ is odd.

(1) $B_{61}(n_1,n_2,m)=\{\lambda\in B_6(n_1,n_2,m)\mid \lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}\}$.

If n_2 is even, then $|B_{61}(n_1,n_2,m)|=0$. If n_2 is odd, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 11.

Table 11

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},0,1$ $(\lvert\{i\mid\gamma_i\neq0\}\rvert=2k)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq0\}\rvert=2k)$
	$\theta_1,\cdots,\theta_{n_1+n_2},1,0$ $(\exists\ \theta_i\neq0,1\le i\le n_1,\lvert\{j\mid\theta_j\neq0\}\rvert=2k,1\le j\le n_1+n_2)$
	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$ $(\lvert\{i\mid\theta_i\neq0\}\rvert=2k+1)$
	$\theta_1,\cdots,\theta_{n_1+n_2},1,1$ $(\exists\ \theta_i\neq0,1\le i\le n_1,\lvert\{j\mid\theta_j\neq0\}\rvert=2k+1,1\le j\le n_1+n_2)$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},1,1$ $(\lvert\{i\mid\gamma_i\neq0\}\rvert=2k+1)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i\mid\theta_i\neq0\}\rvert=2k)$
	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},0,1$ $(\lvert\{i\mid\theta_i\neq0\}\rvert=2k)$

Let $B^1_{61}(n_1,n_2,m)=\big\{\lambda\in B_{61}(n_1,n_2,m)\mid \lambda(c_{m-1})=\sum\limits_{i=1}^{n_1+n_2}\theta_ie_i+e_{n_1+n_2+1}$ or $\sum\limits_{i=1}^{n_1+n_2}\theta_ie_i+e_{n_1+n_2+1}+e_{n_1+n_2+2}\ (\exists\ \theta_i\neq0,1\le i\le n_1)\big\}$, and $B^2_{61}(n_1,n_2,m)=B_{61}(n_1,n_2,m)\setminus B^1_{61}(n_1,n_2,m)$. We have $|B^1_{61}(n_1,n_2,m)|=2^{n_1+2n_2-1}|B_{61}(n_1,n_2,m-2)|$ and $|B^2_{61}(n_1,n_2,m)|=2^{n_2-1}|B_{61}(n_1,n_2,m-1)|$. Then $|B_{61}(n_1,n_2,m)|=2^{n_2-1}|B_{61}(n_1,n_2,m-1)|+2^{n_1+2n_2-1}|B_{61}(n_1,n_2,m-2)|$, $|B_{61}(n_1,n_2,3)|=2^{2n_2-2}$ and $|B_{61}(n_1,n_2,4)|=2^{n_1+3n_2-2}+2^{3n_2-3}$. So $f_{13}(n_1,n_2,m)=|B_{61}(n_1,n_2,m)|$.

Table 12

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},0,1$ $(\lvert\{i\mid\gamma_i\neq0\}\rvert=2k)$	$0,\cdots,0,1,0$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},1,1$ $(\lvert\{i\mid\gamma_i\neq0\}\rvert=2k+1)$	$0,\cdots,0,1,0$

(2) $B_{62}(n_1,n_2,m)=\big\{\lambda\in B_6(n_1,n_2,m)\mid \lambda(b_{n_2+1})=\sum\limits_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}\big\}$.

If n_2 is odd, then $|B_{62}(n_1,n_2,m)|=0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 12.

We get $|B_{62}(n_1,n_2,m)| = 2^{n_2}|B_{62}(n_1,n_2,m-2)|$, $|B_{62}(n_1,n_2,3)| = 2^{2n_2-2}$ and $|B_{62}(n_1,n_2,4)| = 2^{2n_2-1}$. So $f_{14}(n_1,n_2,m) = |B_{62}(n_1,n_2,m)|$.

(3) $B_{63}(n_1,n_2,m) = \{\lambda \in B_6(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+2}\}$.

If n_2 is odd, then $|B_{63}(n_1,n_2,m)| = 0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 13.

Table 13

$\lambda(c_m)$	$\lambda(c_{m-1})$	$\lambda(c_{m-2})$
$0,\cdots,0,1$	$\theta_1,\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$	$0,\cdots,0,1$
$0,\cdots,0,1$	$\theta_1,\cdots,\theta_{n_1+n_2},1,1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$	$0,\cdots,0,1$

We have $|B_{63}(n_1,n_2,m)| = 2^{n_1+n_2}|B_{63}(n_1,n_2,m-2)|$, $|B_{63}(n_1,n_2,3)| = 0$ and $|B_{63}(n_1,n_2,4)| = 2^{n_1+2n_2-1}$. So $f_{15}(n_1,n_2,m) = |B_{43}(n_1,n_2,m)|$.

(4) $B_{64}(n_1,n_2,m) = \{\lambda \in B_6(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+1} + e_{n_1+n_2+2}\}$.

If n_2 is even, then $|B_{64}(n_1,n_2,m)| = 0$. If n_2 is odd, then $\lambda(c_m) = e_{n_1+n_2+2}, \lambda(c_{m-1}) = e_{n_1+n_2+1}$. So, $|B_{64}(n_1,n_2,m)| = |B_{64}(n_1,n_2,m-2)|$, $|B_{64}(n_1,n_2,3)| = 0$ and $|B_{64}(n_1,n_2,4)| = 2^{n_2-1}$. By definition, $f_{12}(n_1,n_2,m) = |B_{64}(n_1,n_2,m)|$.

Thus $|B_6(n_1,n_2,m)| = \sum_{i=1}^{4} |B_{6i}(n_1,n_2,m)| = f_{13}(n_1,n_2,m) + f_{14}(n_1,n_2,m) + f_{15}(n_1,n_2,m) + f_{12}(n_1,n_2,m)$.

Case 7 Calculation of $|B_7(n_1,n_2,m)|$.

(1) $B_{71}(n_1,n_2,m) = \{\lambda \in B_7(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum_{i=1}^{n_2} e_{n_1+i}\}$.

If n_2 is even, then $|B_{71}(n_1,n_2,m)| = 0$. If n_2 is odd, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 14.

Table 14

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},0,1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},1,1$ $(\lvert\{i \mid \gamma_i \neq 0\}\rvert = 2k+1)$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$
	$\theta_1,\cdots,\theta_{n_1+n_2},1,0$ $(\exists\ \theta_i \neq 0, 1 \le i \le n_1, \lvert\{j \mid \theta_j \neq 0\}\rvert = 2k, 1 \le j \le n_1+n_2)$
	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},0,1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$
	$\theta_1,\cdots,\theta_{n_1+n_2},0,1$ $(\exists\ \theta_i \neq 0, 1 \le i \le n_1, \lvert\{j \mid \theta_j \neq 0\}\rvert = 2k+1, 1 \le j \le n_1+n_2)$

Let $B_{71}^1(n_1,n_2,m) = \{\lambda \in B_7(n_1,n_2,m) \mid \lambda(c_{m-1}) = \sum_{i=1}^{n_1+n_2} \theta_i e_i + e_{n_1+n_2+1}$ or $\sum_{i=1}^{n_1+n_2} \theta_i e_i + e_{n_1+n_2+2}$ $(\exists\, \theta_i \neq 0, 1 \le i \le n_1)\}$, and $B_{71}^2(n_1,n_2,m) = B_{71}(n_1,n_2,m) \backslash B_{71}^1(n_1,n_2,m)$. We have $|B_{71}^1(n_1,n_2,m)| = 2^{n_1+2n_2-1}|B_{71}(n_1,n_2,m-2)|$ and $|B_{71}^2(n_1,n_2,m)| = 2^{n_2-1}|B_{71}(n_1,n_2,m-1)|$. Then $|B_{71}(n_1,n_2,m)| = 2^{n_2-1}|B_{71}(n_1,n_2,m-1)| + 2^{n_1+2n_2-1}|B_{71}(n_1,n_2,m-2)|$, $|B_{71}(n_1,n_2,3)| = 2^{n_2-1}$, and $|B_{71}(n_1,n_2,4)| = 2^{n_1+2n_2-2} + 2^{2n_2-1}$. So $f_{16}(n_1,n_2,m) = |B_{71}(n_1,n_2,m)|$.

(2) $B_{72}(n_1,n_2,m) = \{\lambda \in B_7(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum_{i=1}^{n_2} e_{n_1+i} + \sum_{i=1}^{n_1} \delta_i e_i, \exists\, \delta_i \neq 0\}$.

The number of $\delta_i = 1$ is even for n_2 odd, and the number of $\delta_i = 1$ is odd for n_2 even. $\lambda(c_m) = e_{n_1+n_2+2}, \lambda(c_{m-1}) = e_{n_1+n_2+1}$. We have $|B_{72}(n_1,n_2,m)| = |B_{72}(n_1,n_2,m-2)|$ and

$|B_{72}(n_1,n_2,3)|=0$. Note that $|B_{72}(n_1,n_2,4)|=2^{n_1-1}-1$ for n_2 odd, and $|B_{72}(n_1,n_2,4)|=2^{n_1-1}$ for n_2 even. So $f_{17}(n_1,n_2,m)=|B_{72}(n_1,n_2,m)|$.

(3) $B_{73}(n_1,n_2,m)=\{\lambda\in B_7(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}\}$.

If n_2 is odd, then $|B_{73}(n_1,n_2,m)|=0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 15.

Table 15

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},0,1$ ($|\{i\mid\gamma_i\neq0\}|=2k$)	$0,\cdots,0,1,0$
$0,\cdots,0,\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2},1,1$ ($|\{i\mid\gamma_i\neq0\}|=2k+1$)	$0,\cdots,0,1,0$

We have $|B_{73}(n_1,n_2,m)|=2^{n_2}|B_{73}(n_1,n_2,m-2)|$, $|B_{73}(n_1,n_2,3)|=2^{n_2-1}$ and $|B_{73}(n_1,n_2,4)|=2^{n_2}$. So $f_{18}(n_1,n_2,m)=|B_{73}(n_1,n_2,m)|$.

(4) $B_{74}(n_1,n_2,m)=\{\lambda\in B_7(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}+\sum_{i=1}^{n_1}\delta_ie_i,\exists\ \delta_i\neq0\}$.

The number of $\delta_i=1$ is odd for n_2 odd, and the number of $\delta_i=1$ is even for n_2 even. $\lambda(c_m)=e_{n_1+n_2+2}$, $\lambda(c_{m-1})=e_{n_1+n_2+1}$. So, $|B_{74}(n_1,n_2,m)|=|B_{74}(n_1,n_2,m-2)|$, and $|B_{74}(n_1,n_2,3)|=0$. Note that $|B_{74}(n_1,n_2,4)|=2^{n_1-1}$ for n_2 odd, and $|B_{74}(n_1,n_2,4)|=2^{n_1-1}-1$ for n_2 even. So $f_{19}(n_1,n_2,m)=|B_{74}(n_1,n_2,m)|$.

(5) $B_{75}(n_1,n_2,m)=\{\lambda\in B_7(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+2}\}$.

If n_2 is odd, then $|B_{75}(n_1,n_2,m)|=0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 16. We have $|B_{75}(n_1,n_2,m)|=2^{n_2}|B_{75}(n_1,n_2,m-2)|$, $|B_{75}(n_1,n_2,3)|=0$, and $|B_{75}(n_1,n_2,4)|=2^{n_2}$. So $f_{20}(n_1,n_2,m)=|B_{75}(n_1,n_2,m)|$.

Table 16

$\lambda(c_m)$	$\lambda(c_{m-1})$	$\lambda(c_{m-2})$
$0,\cdots,0,1$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ ($|\{i\mid\theta_i\neq0\}|=2k$)	$0,\cdots,0,1$
$0,\cdots,0,1$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$ ($|\{i\mid\theta_i\neq0\}|=2k+1$)	$0,\cdots,0,1$

(6) $B_{76}(n_1,n_2,m)=\{\lambda\in B_7(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+2}+\sum_{i=1}^{n_1}\delta_ie_i,\lambda(b_{n_2+1})\neq\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+2}\}$.

The number of $\delta_i=1$ is odd for n_2 odd, and the number of $\delta_i=1$ is even for n_2 even. $\lambda(c_m)=e_{n_1+n_2+2}$, $\lambda(c_{m-1})=e_{n_1+n_2+1}$. We have $|B_{76}(n_1,n_2,m)|=|B_{76}(n_1,n_2,m-2)|$ and $|B_{76}(n_1,n_2,3)|=0$. Note that $|B_{76}(n_1,n_2,4)|=2^{n_1-1}$ for n_2 odd, and $|B_{76}(n_1,n_2,4)|=2^{n_1-1}-1$ for n_2 even. By definition, $f_{19}(n_1,n_2,m)=|B_{76}(n_1,n_2,m)|$.

(7) $B_{77}(n_1,n_2,m)=\{\lambda\in B_7(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}+e_{n_1+n_2+2}\}$.

If n_2 is even, then $|B_{77}(n_1,n_2,m)|=0$. If n_2 is odd, then $\lambda(c_m)=e_{n_1+n_2+2},\lambda(c_{m-1})=e_{n_1+n_2+1}$. So $|B_{77}(n_1,n_2,m)|=|B_{77}(n_1,n_2,m-2)|$, $|B_{77}(n_1,n_2,3)|=0$, and $|B_{77}(n_1,n_2,4)|=1$. So $f_{21}(n_1,n_2,m)=|B_{77}(n_1,n_2,m)|$.

(8) $B_{78}(n_1,n_2,m)=\{\lambda\in B_7(n_1,n_2,m)\mid\lambda(b_{n_2+1})=\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}+e_{n_1+n_2+2}+\sum_{i=1}^{n_1}\delta_ie_i,\lambda(b_{n_2+1})\neq\sum_{i=1}^{n_2}e_{n_1+i}+e_{n_1+n_2+1}+e_{n_1+n_2+2}\}$.

The number of $\delta_i = 1$ is even for n_2 odd, and the number of $\delta_i = 1$ is odd for n_2 even. $\lambda(c_m) = e_{n_1+n_2+2}$, $\lambda(c_{m-1}) = e_{n_1+n_2+1}$. So, $|B_{78}(n_1, n_2, m)| = |B_{78}(n_1, n_2, m-2)|$ and $|B_{78}(n_1, n_2, 3)| = 0$. Note that $|B_{78}(n_1, n_2, 4)| = 2^{n_1-1} - 1$ for n_2 odd, and $|B_{78}(n_1, n_2, 4)| = 2^{n_1-1}$ for n_2 even. By definition, $f_{17}(n_1, n_2, m) = |B_{78}(n_1, n_2, m)|$.

So, $|B_7(n_1, n_2, m)| = \sum\limits_{i=1}^{8} |B_{7i}(n_1, n_2, m)| = f_{16}(n_1, n_2, m) + 2f_{17}(n_1, n_2, m) + f_{18}(n_1, n_2, m) + 2f_{19}(n_1, n_2, m) + f_{20}(n_1, n_2, m) + f_{21}(n_1, n_2, m)$.

Case 8 Calculation of $|B_8(n_1, n_2, m)|$.

The number of $\varepsilon_i = 1$ is even.

(1) $B_{81}(n_1, n_2, m) = \big\{\lambda \in B_8(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i}\big\}$.

If n_2 is even, then $|B_{81}(n_1, n_2, m)| = 0$. If n_2 is odd, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 17.

Table 17

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0, \cdots, 0, \gamma_{n_1+1}, \cdots, \gamma_{n_1+n_2}, 0, 1$ $(\vert\{i \mid \gamma_i \neq 0\}\vert = 2k)$	$0, \cdots, 0, \theta_{n_1+1}, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\vert\{i \mid \theta_i \neq 0\}\vert = 2k)$
	$0, \cdots, 0, \theta_{n_1+1}, \cdots, \theta_{n_1+n_2}, 1, 1$ $(\vert\{i \mid \theta_i \neq 0\}\vert = 2k+1)$
$0, \cdots, 0, \gamma_{n_1+1}, \cdots, \gamma_{n_1+n_2}, 1, 1$ $(\vert\{i \mid \gamma_i \neq 0\}\vert = 2k+1)$	$0, \cdots, 0, \theta_{n_1+1}, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\vert\{i \mid \theta_i \neq 0\}\vert = 2k)$
	$\theta_1, \cdots, \theta_{n_1+n_2}, 1, 0$ $(\exists\ \theta_i \neq 0, 1 \leq i \leq n_1, (\vert\{j \mid \theta_j \neq 0\}\vert = 2k, 1 \leq j \leq n_1+n_2)$
	$0, \cdots, 0, \theta_{n_1+1}, \cdots, \theta_{n_1+n_2}, 0, 1$ $\vert\{i \mid \theta_i \neq 0\}\vert = 2k)$
	$\theta_1, \cdots, \theta_{n_1+n_2}, 0, 1$ $(\exists\ \theta_i \neq 0, 1 \leq i \leq n_1, \vert\{j \mid \theta_j \neq 0\}\vert = 2k, 1 \leq j \leq n_1+n_2)$

Let $B_{81}^1(n_1, n_2, m) = \big\{\lambda \in B_8(n_1, n_2, m) \mid \lambda(c_{m-1}) = \sum\limits_{i=1}^{n_1+n_2} \theta_i e_i + e_{n_1+n_2+1}$ or $\sum\limits_{i=1}^{n_1+n_2} \theta_i e_i + e_{n_1+n_2+2}$ $(\exists\ \theta_i \neq 0, 1 \leq i \leq n_1)\big\}$ and $B_{81}^2(n_1, n_2, m) = B_{81}(n_1, n_2, m) \setminus B_{81}^1(n_1, n_2, m)$. We have $|B_{81}^1(n_1, n_2, m)| = 2^{n_1+2n_2-1}|B_{81}(n_1, n_2, m-2)|$ and $|B_{81}^2(n_1, n_2, m)| = 2^{n_2-1}|B_{81}(n_1, n_2, m-1)|$. Thus $|B_{81}(n_1, n_2, m)| = 2^{n_2-1}|B_{81}(n_1, n_2, m-1)| + 2^{n_1+2n_2-1}|B_{81}(n_1, n_2, m-2)|$, $|B_{81}(n_1, n_2, 3)| = 2^{2n_2-2} - 2^{n_2-1}$ and $|B_{81}(n_1, n_2, 4)| = 2^{n_1+3n_2-3} + 2^{3n_2-2} - 2^{2n_2-1} - 2^{n_1+2n_2-2}$. So $f_{22}(n_1, n_2, m) = |B_{81}(n_1, n_2, m)|$.

(2) $B_{82}(n_1, n_2, m) = \big\{\lambda \in B_8(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+1}\big\}$.

If n_2 is odd, then $|B_{82}(n_1, n_2, m)| = 0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 18.

Table 18

$\lambda(c_m)$	$\lambda(c_{m-1})$
$0, \cdots, 0, \gamma_{n_1+1}, \cdots, \gamma_{n_1+n_2}, 0, 1$ $(\vert\{i \mid \gamma_i \neq 0\}\vert = 2k)$	$0, \cdots, 0, 1, 0$
$0, \cdots, 0, \gamma_{n_1+1}, \cdots, \gamma_{n_1+n_2}, 1, 1$ $(\vert\{i \mid \gamma_i \neq 0\}\vert = 2k+1)$	$0, \cdots, 0, 1, 0$

We have $|B_{82}(n_1, n_2, m)| = 2^{n_2}|B_{82}(n_1, n_2, m-2)|$, $|B_{82}(n_1, n_2, 3)| = 2^{2n_2-2} - 2^{n_2-1}$ and $|B_{82}(n_1, n_2, 4)| = 2^{2n_2-1} - 2^{n_2}$. So $f_{23}(n_1, n_2, m) = |B_{82}(n_1, n_2, m)|$.

(3) $B_{83}(n_1, n_2, m) = \big\{\lambda \in B_8(n_1, n_2, m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+2}\big\}$.

If n_2 is odd, then $|B_{83}(n_1, n_2, m)| = 0$. If n_2 is even, then $\lambda(c_m)$ and $\lambda(c_{m-1})$ have the possible values in Table 19.

Table 19

$\lambda(c_m)$	$\lambda(c_{m-1})$	$\lambda(c_{m-2})$
$0,\cdots,0,1$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,0$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k)$	$0,\cdots,0,1$
$0,\cdots,0,1$	$0,\cdots,0,\theta_{n_1+1},\cdots,\theta_{n_1+n_2},1,1$ $(\lvert\{i \mid \theta_i \neq 0\}\rvert = 2k+1)$	$0,\cdots,0,1$

Then $|B_{83}(n_1,n_2,m)| = 2^{n_2}|B_{83}(n_1,n_2,m-2)|$, $|B_{83}(n_1,n_2,3)| = 0$ and $|B_{83}(n_1,n_2,4)| = 2^{2n_2-1} - 2^{n_2}$. So $f_{24}(n_1,n_2,m) = |B_{83}(n_1,n_2,m)|$.

(4) $B_{84}(n_1,n_2,m) = \big\{\lambda \in B_8(n_1,n_2,m) \mid \lambda(b_{n_2+1}) = \sum\limits_{i=1}^{n_2} e_{n_1+i} + e_{n_1+n_2+1} + e_{n_1+n_2+2}\big\}$.

If n_2 is even, then $|B_{84}(n_1,n_2,m)| = 0$. If n_2 is odd, then $\lambda(c_m) = e_{n_1+n_2+2}$, $\lambda(c_{m-1}) = e_{n_1+n_2+1}$. We have $|B_{84}(n_1,n_2,m)| = |B_{84}(n_1,n_2,m-2)|$, $|B_{84}(n_1,n_2,3)| = 0$ and $|B_{84}(n_1,n_2,4)| = 2^{n_2-1} - 1$. By definition, $f_8(n_1,n_2,m) = |B_{84}(n_1,n_2,m)|$.

So, $|B_8(n_1,n_2,m)| = \sum\limits_{i=1}^{4} |B_{8i}(n_1,n_2,m)| = f_{22}(n_1,n_2,m) + f_{23}(n_1,n_2,m) + f_{24}(n_1,n_2,m) + f_8(n_1,n_2,m)$.

From $|B(n_1,n_2,m)| = \sum\limits_{i=1}^{8} |B_i(n_1,n_2,m)|$, we have

$$|B(n_1,n_2,m)| = \sum_{i\in I_1} f_i(n_1,n_2,m) + \sum_{i\in I_2} f_i(n_1,n_2,m),$$

where I_1={$1,2,\cdots,24$} and I_2={8,12,17,19}.

The proof is completed.

Remark 3.1 A direct calculation shows that $|B(3,1,3)| = 106$.

Theorem 3.2 *Suppose n_1 is odd, $n_2 \geq 2$, $n_2 \geq 1$, $n_1 \geq n_2$ and $m \geq 3$. Then the number of orientable characteristic functions on $\Delta^{n_1} \times \Delta^{n_2} \times P(m)$ is*

$$|O(n_1,n_2,m)| = \prod_{k=1}^{n_1+n_2+2} (2^{n_1+n_2+2} - 2^{k-1})\Big[\sum_{i\in I_1} f_i(n_1,n_2,m) + \sum_{i\in I_2} f_i(n_1,n_2,m)\Big],$$

where $I_1 = \{1,2,\cdots,24\}$ and $I_2 = \{8,12,17,19\}$.

Proof By Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}|O(n_1,n_2,m)| &= \prod_{k=1}^{n_1+n_2+2} (2^{n_1+n_2+2} - 2^{k-1})|B(n_1,n_2,m)| \\ &= \prod_{k=1}^{n_1+n_2+2} (2^{n_1+n_2+2} - 2^{k-1})\Big[\sum_{i\in I_1} f_i(n_1,n_2,m) + \sum_{i\in I_2} f_i(n_1,n_2,m)\Big],\end{aligned}$$

where I_1={$1,2,\cdots,24$} and I_2={8,12,17,19}.

4 Equivariant Homeomorphism Classes

In this section, we determine the number of equivariant homeomorphism classes of all orientable small covers over $\Delta^{n_1} \times \Delta^{n_2} \times P(m)$, which is denoted by $E_o(n_1,n_2,m)$.

Let φ denote the Euler's totient function, that is, $\varphi(1) = 1$ and $\varphi(N)$ for a positive integer $N(N \geq 2)$ is the number of positive integers both less than N and coprime to N. Recursive functions $f_i(n_1, n_2, m)$, $g_i(n_1, m)$ and $k_i(n_1, n_2, m)$ are listed in the appendixes. Then we have the following theorem.

Theorem 4.1 *Suppose that n_1 is odd, $n_1 \geq 2$, $n_2 \geq 1$, $m \geq 3$ and $n_1 \geq n_2$, then the number of equivariant homeomorphism classes of orientable small covers over $\Delta^{n_1} \times \Delta^{n_2} \times P(m)$ is*

(1) *for $n_1 > 2$, $n_2 = 1$ and $m = 3$ or $n_1 \geq 2$, $n_2 = 1$ and $m > 4$,*

$$\begin{aligned}E_o(n_1,1,m) = \frac{1}{(n_1+1)!4m}\prod_{i=1}^{n_1+3}(2^{n_1+3}-2^{i-1})\Big\{\sum_{k>1,k|m}\varphi\Big(\frac{m}{k}\Big)\Big[\sum_{i\in I_1} f_i(n_1,1,k) \\ + \sum_{i\in I_2} f_i(n_1,1,k) + \frac{1}{2^{n_1-1}}f_1(n_1,1,k)\Big] + \Big(\frac{m}{2}+\frac{1}{2^{n_1-1}}\Big)g_1(n_1,m) \\ + \frac{m}{2}\sum_{i\in I_3} g_i(n_1,m) + \frac{3m}{2}g_5(n_1,m) + mg_6(n_1,m) + mg_7(n_1,m)\Big\},\end{aligned}$$

where $I_1 = \{1, 2, \cdots, 24\}$, $I_2 = \{5, 8, 9, 12, 13, 16, 17, 19, 22\}$ and $I_3 = \{2, 3, 4\}$;

(2) *for $n_1 \geq 2$, $n_2 = 1$ and $m = 4$,*

$$\begin{aligned}E_o(n_1,1,4) = \frac{1}{48(n_1+1)!}\prod_{i=1}^{n_1+3}(2^{n_1+3}-2^{i-1})\Big\{\sum_{k>1,k|4}\varphi\Big(\frac{4}{k}\Big)\Big[\sum_{i\in I_1} f_i(n_1,1,k) \\ + \sum_{i\in I_2} f_i(n_1,1,k) + \frac{1}{2^{n_1-1}}f_1(n_1,1,k)\Big] + \Big(2+\frac{1}{2^{n_1-1}}\Big)g_1(n_1,4) \\ + 2\sum_{i\in I_3} g_i(n_1,4) + 6g_5(n_1,4) + 4g_6(n_1,4) + 4g_7(n_1,4)\Big\},\end{aligned}$$

where $I_1 = \{1, 2, \cdots, 24\}$, $I_2 = \{5, 8, 9, 12, 13, 16, 17, 19, 22\}$ and $I_3 = \{2, 3, 4\}$;

(3) *for $n_1 > n_2 = 2$ and $m > 3$ or $n_1 > n_2 > 2$ and $m \geq 3$,*

$$\begin{aligned}E_o(n_1,n_2,m) = \frac{1}{2m(n_1+1)!(n_2+1)!}\prod_{i=1}^{n_1+n_2+2}(2^{n_1+n_2+2}-2^{i-1})\Big\{\sum_{k>1,k|m}\varphi\Big(\frac{m}{k}\Big) \\ \cdot\Big[\sum_{i\in I_1} f_i(n_1,n_2,k) + \sum_{i\in I_2} f_i(n_1,n_2,k)\Big] \\ + \frac{m}{2}\Big[\sum_{i\in I_1} k_i(n_1,n_2,m) + \sum_{i\in I_2} k_i(n_1,n_2,m)\Big]\Big\},\end{aligned}$$

where $I_1 = \{1, 2, \cdots, 24\}$ and $I_2 = \{8, 12, 17, 19\}$;

(4) *for $n_1 > n_2 = 2$ and $m = 3$ or $n_1 = n_2 > 2$ and $m \geq 3$,*

$$\begin{aligned}E_o(n_1,n_2,m) = \frac{1}{4m(n_1+1)!(n_2+1)!}\prod_{i=1}^{n_1+n_2+2}(2^{n_1+n_2+2}-2^{i-1})\Big\{\sum_{k>1,k|m}\varphi\Big(\frac{m}{k}\Big) \\ \cdot\Big[\sum_{i\in I_1} f_i(n_1,n_2,k) + \sum_{i\in I_2} f_i(n_1,n_2,k)\Big] \\ + \frac{m}{2}\Big[\sum_{i\in I_1} k_i(n_1,n_2,m) + \sum_{i\in I_2} k_i(n_1,n_2,m)\Big]\Big\},\end{aligned}$$

where $I_1 = \{1, 2, \cdots, 24\}$ and $I_2 = \{8, 12, 17, 19\}$.

Proof According to Theorem 2.3, Lemma 2.1 and Burnside Lemma, we have

$$E_o(n_1,n_2,m)=\frac{1}{|\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))|}\sum_{g\in\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))}|O_g|,$$

where $O_g=\{\lambda\in O(n_1,n_2,m)\mid\lambda=\lambda\circ g\}$.

In order to determine $|O_g|$ for $g\in\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$, we exhibit a system of generators of the group $\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$.

Let x, y, s_i $(i=1,\cdots,n_1)$, t_j $(j=1,\cdots,n_2)$ and z_k $(k=1,2,3,4)$ be the elememts of $\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$ with the following properties:

(1) $x(c_i)=c_{i+1}$ $(i=1,2,\cdots,m-1)$, $x(c_m)=c_1$, $x(a_j)=a_j$ $(j=1,2,\cdots,n_1+1)$ and $x(b_k)=b_k$ $(k=1,2,\cdots,n_2+1)$.

(2) $y(c_i)=c_{m+1-i}$ $(i=1,2,\cdots,m)$, $y(c_m)=c_1$, $y(a_j)=a_j$ $(j=1,2,\cdots,n_1+1)$ and $y(b_k)=b_k$ $(k=1,2,\cdots,n_2+1)$.

(3) $s_i(a_1)=a_{i+1}$, $s_i(a_{i+1})=a_1$, $s_i(a_p)=a_p$ $(p\neq 1,i+1)$, $i=1,\cdots,n_1$, $s_i(b_j)=b_j$ $(j=1,\cdots,n_2+1)$ and $s_i(c_k)=c_k$ $(k=1,\cdots,m)$.

(4) $t_i(b_1)=b_{i+1}$, $t_i(b_{i+1})=b_1$, $t_i(b_q)=b_q$ $(q\neq 1,i+1)$, $i=1,\cdots,n_2$, $t_i(a_j)=a_j$ $(j=1,\cdots,n_1+1)$ and $t_i(c_k)=c_k$ $(k=1,\cdots,m)$.

(5) $z_1(a_i)=b_i$, $z_1(b_i)=a_i$ $(i=1,\cdots,n_1+1)$ and $z_1(c_k)=c_k$ $(k=1,\cdots,m)$ if $n_1=n_2$.

(6) $z_2(b_i)=c_i$, $z_2(c_i)=b_i$, and $z_2(a_k)=a_k$ $(k=1,\cdots,n_1+1)$ if $n_2=2$ and $m=3$.

(7) $z_3(a_i)=c_i$, $z_3(c_i)=a_i$ $(i=1,2,3)$ and $z_3(b_i)=b_i$ $(i=1,2)$ if $n_1=2$, $n_2=1$ and $m=3$.

(8) $z_4(b_1)=c_1$, $z_4(c_1)=b_1$, $z_4(b_2)=c_3$, $z_4(c_3)=b_2$, $z_4(c_2)=c_2$, $z_4(c_4)=c_4$ and $z_4(a_k)=a_k$ $(k=1,\cdots,n_1+1)$ for $n_1\geq 2$, $n_2=1$ and $m=4$.

Every $g\in\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$ can be expressed in the form $x^uy^v(\prod z_i)^w(\prod s_i)^\alpha(\prod t_j)^\beta$, $u\in\mathbb{Z}_m$, v,w,α and $\beta\in\mathbb{Z}_2$. The calculation of $|O_g|$ is divided into the following cases.

Case 1 $n_1>2$, $n_2=1$, and $m=3$ or $n_1\geq 2$, $n_2=1$ and $m>4$.

According to Lemma 2.3, $\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))=S_{n_1+1}\times\mathbb{Z}_2\times\mathcal{D}_m$ and $g\in\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$ can be written in the form $x^uy^vt_1^w(\prod s_i)^\alpha$, where $u\in\mathbb{Z}_m$, v,w and $\alpha\in\mathbb{Z}_2$.

Subcase 1.1 $g=x^u$.

Let $k=\gcd(u,m)$ (i.e., the greatest common divisor of u and m). Then all facets in $\mathcal{F}(n_1,n_2,m)$ are divided into k orbits under the action of g and each orbit contains $\frac{m}{k}$ facets. This means $k\neq 1$. An argument similar to that of $|B_i(n_1,n_2,m)|$ shows that $|O_g|=|O(n_1,1,k)|$ for $k>2$ and $|O_g|=\prod_{i=1}^{n_1+3}(2^{n_1+3}-2^{i-1})\big(\sum_{i\in I_1}f_i(n_1,1,k)+\sum_{i\in I_2}f_i(n_1,1,k)\big)$ for $k=2$, where I_1=$\{1,2,\cdots,24\}$ and $I_2=\{8,12,17,19\}$. For every $k>1$, there are exactly $\varphi\big(\frac{m}{k}\big)$ automorphisms of the form x^u, each of which divides all facets in $\mathcal{F}(n_1,n_2,m)$ into k orbits. Thus

$$\sum_{g=x^u}|O_g|=\sum_{k>1,k|m}\varphi\Big(\frac{m}{k}\Big)\prod_{i=1}^{n_1+3}(2^{n_1+3}-2^{i-1})\Big[\sum_{i\in I_1}f_i(n_1,1,k)+\sum_{i\in I_2}f_i(n_1,1,k)\Big],$$

where $I_1=\{1,2,\cdots,24\}$ and $I_2=\{8,12,17,19\}$.

Subcase 1.2 $g=x^ut_1$.

In this case, $\lambda(b_1)=\lambda(b_2)$ for $\lambda\in O_g$. An argument similar to that of $|B_i(n_1,n_2,m)|$ shows $\sum_{g=x^ut_1}|O_g|=\sum_{k>1,k|m}\varphi\big(\frac{m}{k}\big)\prod_{i=1}^{n_1+3}(2^{n_1+3}-2^{i-1})\big(\frac{1}{2^{n_1-1}}f_1(n_1,1,k)+f_5(n_1,1,k)+f_9(n_1,1,k)+$

$f_{13}(n_1,1,k)+f_{16}(n_1,1,k)+f_{22}(n_1,1,k)\big)$.

Subcase 1.3 $g=x^uy$, x^uyt_1, where m is odd, or u and m are even.

If $\lambda\in O_g$, then λ restricted to some adjacent facets has the same value, which contradicts the non-singularity condition. So $|O_g|=0$.

Subcase 1.4 $g=x^uy$, where u is odd and m is even.

Because $|O_{g_1}|=|O_{g_2}|$ for $g_1,g_2\in O_g$, suppose $g=x^{m-1}y$ (i.e., $g=yx$). Similarly to the proof of Theorem 3.1, let $X_i(n_1,1,m)=\{\lambda\mid\lambda\in B_i(n_1,1,m),\lambda(c_j)=\lambda(c_{m-j}),m$ is even, and $j=1,\cdots,m-1\}$, where $i=1,\cdots,8$. It is easy to show $|X_2(n_1,1,m)|=|X_3(n_1,1,m)|=|X_5(n_1,1,m)|=|X_8(n_1,1,m)|=0$.

(1) Calculation of $|X_1(n_1,1,m)|$.

Let $X_{1i}(n_1,1,m)=\{\lambda\mid\lambda\in B_{1i}(n_1,1,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\},\ i=1,2,3,4$. We have $|X_{11}(n_1,1,m)|=2^{n_1}|X_{11}(n_1,1,m-2)|+2^{2n_1+1}|X_{11}(n_1,1,m-4)|$. $|X_{11}(n_1,1,8)|=2^{4n_1+2}$, $|X_{11}(n_1,1,6)|=2^{3n_1+1}$. So $g_1(n_1,m)=|X_{11}(n_1,1,m)|$.

Similarly, $|X_{12}(n_1,1,m)|=2^{n_1-1}|X_{12}(n_1,1,m-2)|+2^{2n_1}|X_{12}(n_1,1,m-4)|$. $|X_{12}(n_1,1,8)|=2^{4n_1}$, $|X_{12}(n_1,1,6)|=2^{3n_1}$. So $g_2(n_1,m)=|X_{12}(n_1,1,m)|$.

$|X_{13}(n_1,1,m)|=2^{n_1-1}|X_{13}(n_1,1,m-2)|+2^{2n_1}|X_{13}(n_1,1,m-4)|$. $|X_{13}(n_1,1,8)|=3\cdot 2^{4n_1-1}$, $|X_{13}(n_1,1,6)|=2^{3n_1}$. So $g_3(n_1,m)=|X_{13}(n_1,1,m)|$.

$|X_{14}(n_1,1,m)|=2^{n_1-1}|X_{14}(n_1,1,m-2)|+2^{2n_1}|X_{14}(n_1,1,m-4)|$. $|X_{14}(n_1,1,8)|=3\cdot 2^{4n_1-2}$, $|X_{14}(n_1,1,6)|=2^{3n_1-1}$. So $g_4(n_1,m)=|X_{14}(n_1,1,m)|$.

$|X_1(n_1,1,m)|=\sum\limits_{i=1}^{4}|X_{1i}(n_1,1,m)|=g_1(n_1,m)+g_2(n_1,m)+g_3(n_1,m)+g_4(n_1,m)$.

(2) Calculation of $|X_4(n_1,1,m)|$.

Let $X_{4i}(n_1,1,m)=\{\lambda\mid\lambda\in B_{4i}(n_1,1,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\},\ i=1,2,3,4$. Then

$$\begin{aligned}&|X_{41}(n_1,1,m)|=|X_{42}(n_1,1,m)|=|X_{43}(n_1,1,m)|=0,\\&|X_{44}(n_1,1,m)|=|X_{44}(n_1,1,m-4)|,\\&|X_{44}(n_1,1,8)|=|X_{44}(n_1,1,6)|=1.\end{aligned}$$

So $g_5(n_1,m)=|X_{44}(n_1,1,m)|$. Thus $|X_4(n_1,1,m)|=\sum\limits_{i=1}^{4}|X_{4i}(n_1,1,m)|=g_5(n_1,m)$.

(3) Calculation of $|X_6(n_1,1,m)|$.

Let $X_{6i}(n_1,1,m)=\{\lambda\mid\lambda\in B_{6i}(n_1,1,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\},\ i=1,2,3,4$. Then

$$\begin{aligned}&|X_{61}(n_1,1,m)|=|X_{62}(n_1,1,m)|=|X_{63}(n_1,1,m)|=0,\\&|X_{64}(n_1,1,m)|=|X_{64}(n_1,1,m-4)|,\\&|X_{64}(n_1,1,8)|=|X_{64}(n_1,1,6)|=1.\end{aligned}$$

So $|X_6(n_1,1,m)|=|X_{64}(n_1,1,m)|=g_5(n_1,m)$.

(4) Calculation of $|X_7(n_1,1,m)|$.

Let $X_{7i}(n_1,1,m)=\{\lambda\mid\lambda\in B_{7i}(n_1,1,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\},\ i=1,\cdots,8$. Then

$$|X_{71}(n_1,1,m)|=|X_{73}(n_1,1,m)|=|X_{75}(n_1,1,m)|=0,$$

$$|X_{72}(n_1,1,m)| = |X_{72}(n_1,1,m-4)|,$$
$$|X_{72}(n_1,1,8)| = |X_{72}(n_1,1,6)| = 2^{n_1-1}-1.$$

So $g_6(n_1,m) = |X_{72}(n_1,1,m)|$.

$|X_{74}(n_1,1,m)| = |X_{74}(n_1,1,m-4)|$, $|X_{74}(n_1,1,8)| = |X_{74}(n_1,1,6)| = 2^{n_1-1}$. So $g_7(n_1,m) = |X_{74}(n_1,1,m)|$.

$|X_{76}(n_1,1,m)| = |X_{76}(n_1,1,m-4)|$, $|X_{76}(n_1,1,8)| = |X_{76}(n_1,1,6)| = 2^{n_1-1}$. So $g_7(n_1,m) = |X_{76}(n_1,1,m)|$.

$|X_{77}(n_1,1,m)| = |X_{77}(n_1,1,m-4)|$, $|X_{77}(n_1,1,8)| = |X_{77}(n_1,1,6)| = 1$. So $g_5(n_1,m) = |X_{77}(n_1,1,m)|$.

$|X_{78}(n_1,1,m)| = |X_{78}(n_1,1,m-4)|$, $|X_{78}(n_1,1,8)| = |X_{78}(n_1,1,6)| = 2^{n_1-1}-1$. So $g_6(n_1,m) = |X_{78}(n_1,1,m)|$.

Thus $|X_7(n_1,1,m)| = g_5(n_1,m) + 2g_6(n_1,m) + 2g_7(n_1,m)$.

We get

$$\sum_{i=1}^{8} |X_i(n_1,1,m)| = g_1(n_1,m) + g_2(n_1,m) + g_3(n_1,m) + g_4(n_1,m)$$
$$+ 3g_5(n_1,m) + 2g_6(n_1,m) + 2g_7(n_1,m).$$

By Burnside Lemma,

$$\sum_{\substack{g=x^u y \\ u \text{ is odd} \\ m \text{ is even}}} |O_g| = \frac{m}{2} \prod_{i=1}^{n_1+3} (2^{n_1+3} - 2^{i-1}) [g_1(n_1,m) + g_2(n_1,m) + g_3(n_1,m) + g_4(n_1,m)$$
$$+ 3g_5(n_1,m) + 2g_6(n_1,m) + 2g_7(n_1,m)].$$

Subcase 1.5 $g = x^u y t_1$, where u is odd and m is even.

If $\lambda \in O_g$, then $\lambda(b_1) = \lambda(b_2)$. A similar argument as in subcase 1.4 shows

$$\sum_{\substack{g=x^u y t_1 \\ u \text{ is odd} \\ m \text{ is even}}} |O_g| = \prod_{i=1}^{n_1+3} (2^{n_1+3} - 2^{i-1}) \frac{1}{2^{n_1-1}} g_1(n_1,m).$$

Subcase 1.6 $g = x^u y^v t_1^w (\prod s_i)$.

By the non-singularity condition, $|O_g| = 0$.

From Burnside Lemma and $|\mathrm{Aut}(\mathcal{F}(n_1,n_2,m))| = (n_1+1)!4m$, we get

$$E_o(n_1,1,m) = \frac{1}{(n_1+1)!4m} \prod_{i=1}^{n_1+3} (2^{n_1+3} - 2^{i-1}) \Big\{ \sum_{k>1, k|m} \varphi\Big(\frac{m}{k}\Big) \Big[\sum_{i \in I_1} f_i(n_1,1,k)$$
$$+ \sum_{i \in I_2} f_i(n_1,1,k) + \frac{1}{2^{n_1-1}} f_1(n_1,1,k) \Big] + \Big(\frac{m}{2} + \frac{1}{2^{n_1-1}}\Big) g_1(n_1,m)$$
$$+ \frac{m}{2} \sum_{i \in I_3} g_i(n_1,m) + \frac{3m}{2} g_5(n_1,m) + m g_6(n_1,m) + m g_7(n_1,m) \Big\},$$

where $I_1 = \{1,2,\cdots,24\}$, $I_2 = \{5,8,9,12,13,16,17,19,22\}$ and $I_3 = \{2,3,4\}$.

Case 2 $n_1 \geq 2$, $n_2 = 1$ and $m = 4$.

According to Lemma 2.3, $\mathrm{Aut}(\mathcal{F}(n_1, 1, 4)) = S_{n_1+1} \times (\mathbb{Z}_2)^3 \times S_3$. $g \in \mathrm{Aut}(\mathcal{F}(n_1, 1, 4))$ can be written in the form $x^u y^v (\prod s_i)^\alpha (\prod t_j)^\beta (z_4)^\gamma$, where $u \in \mathbb{Z}_m$, v, α, β and $\gamma \in \mathbb{Z}_2$.

Subcase 2.1 $g = x^u y^v (\prod s_i)^\alpha (\prod t_j)^\beta (z_4)^\gamma$, where $\alpha = 1$ or $\gamma = 1$.

By the non-singularity condition, $|O_g| = 0$.

Subcase 2.2 $g = x^u y^v (\prod t_j)^\beta$.

The calculation is similar to the subcases 1.1–1.5. We omit the details. So

$$\begin{aligned} E_o(n_1, 1, 4) = \frac{1}{48(n_1+1)!} \prod_{i=1}^{n_1+3} (2^{n_1+3} - 2^{i-1}) \Big\{ \sum_{k>1, k|4} \varphi\Big(\frac{4}{k}\Big) \Big[\sum_{i \in I_1} f_i(n_1, 1, k) \\ + \sum_{i \in I_2} f_i(n_1, 1, k) + \frac{1}{2^{n_1-1}} f_1(n_1, 1, k) \Big] + \Big(2 + \frac{1}{2^{n_1-1}}\Big) g_1(n_1, 4) \\ + 2 \sum_{i \in I_3} g_i(n_1, 4) + 6 g_5(n_1, 4) + 4 g_6(n_1, 4) + 4 g_7(n_1, 4) \Big\}, \end{aligned}$$

where $I_1 = \{1, 2, \cdots, 24\}$, $I_2 = \{5, 8, 9, 12, 13, 16, 17, 19, 22\}$ and $I_3 = \{2, 3, 4\}$.

Case 3 $n_1 > n_2 > 2$ and $m \geq 3$, or $n_1 > n_2 = 2$ and $m > 3$.

According to Lemma 2.3, $\mathrm{Aut}(\mathcal{F}(n_1, n_2, m)) = S_{n_1+1} \times S_{n_2+1} \times \mathcal{D}_m$. $g \in \mathrm{Aut}(\mathcal{F}(n_1, n_2, m))$ can be written in the form $x^u y^v (\prod s_i)^\alpha (\prod t_j)^\beta$, where $u \in \mathbb{Z}_m$, v, α and $\beta \in \mathbb{Z}_2$.

Subcase 3.1 $g = x^u$.

By an argument similar to that of the subcase 1.1, we have

$$\sum_{g=x^u} |O_g| = \sum_{k>1, k|m} \varphi\Big(\frac{m}{k}\Big) \prod_{i=1}^{n_1+n_2+2} (2^{n_1+n_2+2} - 2^{i-1}) \Big[\sum_{i \in I_1} f_i(n_1, n_2, k) + \sum_{i \in I_2} f_i(n_1, n_2, k) \Big],$$

where I_1=$\{1, 2, \cdots, 24\}$ and $I_2 = \{8, 12, 17, 19\}$.

Subcase 3.2 $g = x^u y$, where m is odd, or u and m are both even.

By the non-singularity condition, $|O_g| = 0$.

Subcase 3.3 $g = x^u y$, where u is odd and m is even.

Without loss of generality, suppose $g = x^{m-1} y$ (i.e., $g = yx$). Similar to the discussion in the subcase 1.4, let $X_i(n_1, n_2, m) = \{\lambda \mid \lambda \in B_i(n_1, n_2, m), \lambda(c_j) = \lambda(c_{m-j}), j = 1, \cdots, m-1\}$, where $i = 1, \cdots, 8$. It is easy to show $|X_3(n_1, n_2, m)| = |X_5(n_1, n_2, m)| = 0$.

(1) Calculation of $|X_1(n_1, n_2, m)|$.

Let $X_{1i}(n_1, n_2, m) = \{\lambda \mid \lambda \in B_{1i}(n_1, n_2, m), \lambda(c_j) = \lambda(c_{m-j}), j = 1, \cdots, m-1\}$, $i = 1, 2, 3, 4$. We have the following result.

$|X_{11}(n_1, n_2, m)| = 2^{n_1+n_2-1} |X_{11}(n_1, n_2, m-2)| + 2^{2n_1+2n_2-1} |X_{11}(n_1, n_2, m-4)|$. $|X_{11}(n_1, n_2, 4)| = 2^{2n_1+n_2-1}$, $|X_{11}(n_1, n_2, 6)| = 2^{3n_1+2n_2-1}$. So $k_1(n_1, n_2, m) = |X_{11}(n_1, n_2, m)|$.

$|X_{12}(n_1, n_2, m)| = 2^{n_1-1} |X_{12}(n_1, n_2, m-2)| + 2^{2n_1+n_2-1} |X_{12}(n_1, n_2, m-4)|$. $|X_{12}(n_1, n_2, 4)| = 2^{2n_1-1}$, $|X_{12}(n_1, n_2, 6)| = 2^{3n_1+n_2-1}$. So $k_2(n_1, n_2, m) = |X_{12}(n_1, n_2, m)|$.

$|X_{13}(n_1, n_2, m)| = 2^{n_1-1} |X_{13}(n_1, n_2, m-2)| + 2^{2n_1+n_2-1} |X_{13}(n_1, n_2, m-4)|$. $|X_{13}(n_1, n_2, 4)| = 2^{2n_1+n_2-1}$, $|X_{13}(n_1, n_2, 6)| = 2^{3n_1+n_2-1}$. So $k_3(n_1, n_2, m) = |X_{13}(n_1, n_2, m)|$.

$|X_{14}(n_1, n_2, m)| = 2^{n_1-1} |X_{14}(n_1, n_2, m-2)| + 2^{2n_1+n_2-1} |X_{14}(n_1, n_2, m-4)|$. $|X_{14}(n_1, n_2, 4)| = 2^{2n_1-1}$, $|X_{14}(n_1, n_2, 6)| = 2^{3n_1-1}$. So $k_4(n_1, n_2, m) = |X_{14}(n_1, n_2, m)|$.

$|X_1(n_1, n_2, m)| = k_1(n_1, n_2, m) + k_2(n_1, n_2, m) + k_3(n_1, n_2, m) + k_4(n_1, n_2, m)$.

(2) Calculation of $|X_2(n_1,n_2,m)|$.

Let $X_{2i}(n_1,n_2,m)=\{\lambda\mid\lambda\in B_{2i}(n_1,n_2,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\}$, $i=1,2,3,4$. We have $|X_{21}(n_1,n_2,m)|=2^{n_1+n_2-1}|X_{21}(n_1,n_2,m-2)|+2^{2n_1+2n_2-1}|X_{21}(n_1,n_2,m-4)|$. $|X_{21}(n_1,n_2,4)|=2^{n_1+2n_2-1}-2^{n_1+n_2}$, $|X_{21}(n_1,n_2,6)|=2^{2n_1+3n_2-1}-2^{2n_1+2n_2}$. So $k_5(n_1,n_2,m)=|X_{21}(n_1,n_2,m)|$.

$|X_{22}(n_1,n_2,m)|=2^{n_1+n_2}|X_{22}(n_1,n_2,m-4)|$. $|X_{22}(n_1,n_2,4)|=2^{n_2-1}-1$, $|X_{22}(n_1,n_2,6)|=2^{n_1+2n_2-1}-2^{n_1+n_2}$. So $k_6(n_1,n_2,m)=|X_{22}(n_1,n_2,m)|$.

$|X_{23}(n_1,n_2,m)|=2^{n_1+n_2}|X_{23}(n_1,n_2,m-4)|$. $|X_{23}(n_1,n_2,4)|=2^{n_1+2n_2-1}-2^{n_1+n_2}=|X_{23}(n_1,n_2,6)|$. So $k_7(n_1,n_2,m)=|X_{23}(n_1,n_2,m)|$.

$|X_{24}(n_1,n_2,m)|=|X_{24}(n_1,n_2,m-4)|$. $|X_{24}(n_1,n_2,4)|=2^{n_2-1}-1=|X_{24}(n_1,n_2,6)|$. So $k_8(n_1,n_2,m)=|X_{24}(n_1,n_2,m)|$.

$|X_2(n_1,n_2,m)|=k_5(n_1,n_2,m)+k_6(n_1,n_2,m)+k_7(n_1,n_2,m)+k_8(n_1,n_2,m)$.

(3) Calculation of $|X_4(n_1,n_2,m)|$.

Let $X_{4i}(n_1,n_2,m)=\{\lambda\mid\lambda\in B_{4i}(n_1,n_2,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\}$, $i=1,2,3,4$. We obtain $|X_{41}(n_1,n_2,m)|=2^{n_2-1}|X_{41}(n_1,n_2,m-2)|+2^{n_1+2n_2-1}|X_{41}(n_1,n_2,m-4)|$. $|X_{41}(n_1,n_2,4)|=2^{2n_2-1}$, $|X_{41}(n_1,n_2,6)|=2^{n_1+3n_2-1}$. So $k_9(n_1,n_2,m)=|X_{41}(n_1,n_2,m)|$.

$|X_{42}(n_1,n_2,m)|=2^{n_1+n_2}|X_{42}(n_1,n_2,m-4)|$. $|X_{42}(n_1,n_2,4)|=2^{n_2-1}$, $|X_{42}(n_1,n_2,6)|=2^{n_1+2n_2-1}$. So $k_{10}(n_1,n_2,m)=|X_{42}(n_1,n_2,m)|$.

$|X_{43}(n_1,n_2,m)|=2^{n_2}|X_{43}(n_1,n_2,m-4)|$. $|X_{43}(n_1,n_2,4)|=2^{2n_2-1}=|X_{43}(n_1,n_2,6)|$. So $k_{11}(n_1,n_2,m)=|X_{43}(n_1,n_2,m)|$.

$|X_{44}(n_1,n_2,m)|=|X_{44}(n_1,n_2,m-4)|$. $|X_{44}(n_1,n_2,4)|=2^{n_2-1}=|X_{44}(n_1,n_2,6)|$. So $k_{12}(n_1,n_2,m)=|X_{44}(n_1,n_2,m)|$.

$|X_4(n_1,n_2,m)|=k_9(n_1,n_2,m)+k_{10}(n_1,n_2,m)+k_{11}(n_1,n_2,m)+k_{12}(n_1,n_2,m)$.

(4) Calculation of $|X_6(n_1,n_2,m)|$.

Let $X_{6i}(n_1,n_2,m)=\{\lambda\mid\lambda\in B_{6i}(n_1,n_2,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\}$, $i=1,2,3,4$. We have $|X_{61}(n_1,n_2,m)|=2^{n_2-1}|X_{61}(n_1,n_2,m-2)|+2^{n_1+2n_2-1}|X_{61}(n_1,n_2,m-4)|$. $|X_{61}(n_1,n_2,4)|=2^{n_1+2n_2-1}$, $|X_{61}(n_1,n_2,6)|=2^{n_1+3n_2-1}$. So $k_{13}(n_1,n_2,m)=|X_{61}(n_1,n_2,m)|$.

$|X_{62}(n_1,n_2,m)|=2^{n_2}|X_{62}(n_1,n_2,m-4)|$. $|X_{62}(n_1,n_2,4)|=2^{n_2-1}$, $|X_{62}(n_1,n_2,6)|=2^{2n_2-1}$. So $k_{14}(n_1,n_2,m)=|X_{62}(n_1,n_2,m)|$.

$|X_{63}(n_1,n_2,m)|=2^{n_1+n_2}|X_{63}(n_1,n_2,m-4)|$. $|X_{63}(n_1,n_2,4)|=2^{n_1+2n_2-1}=|X_{63}(n_1,n_2,6)|$. So $k_{15}(n_1,n_2,m)=|X_{63}(n_1,n_2,m)|$.

$|X_{64}(n_1,n_2,m)|=|X_{64}(n_1,n_2,m-4)|$. $|X_{64}(n_1,n_2,4)|=2^{n_2-1}-1=|X_{64}(n_1,n_2,6)|$. So $k_{12}(n_1,n_2,m)=|X_{64}(n_1,n_2,m)|$.

$|X_6(n_1,n_2,m)|=k_{12}(n_1,n_2,m)+k_{13}(n_1,n_2,m)+k_{14}(n_1,n_2,m)+k_{15}(n_1,n_2,m)$.

(5) Calculation of $|X_7(n_1,n_2,m)|$.

Let $X_{7i}(n_1,n_2,m)=\{\lambda\mid\lambda\in B_{7i}(n_1,n_2,m),\lambda(c_j)=\lambda(c_{m-j}),j=1,\cdots,m-1\}$, $i=1,\cdots,8$. We have $|X_{71}(n_1,n_2,m)|=2^{n_2-1}|X_{71}(n_1,n_2,m-2)|+2^{n_1+2n_2-1}|X_{71}(n_1,n_2,m-4)|$. $|X_{71}(n_1,n_2,4)|=2^{n_2}$, $|X_{71}(n_1,n_2,6)|=2^{2n_2}$. So $k_{16}(n_1,n_2,m)=|X_{71}(n_1,n_2,m)|$.

$|X_{72}(n_1,n_2,m)|=|X_{72}(n_1,n_2,m-4)|$. For n_2 odd, $|X_{72}(n_1,n_2,4)|=|X_{72}(n_1,n_2,6)|=2^{n_1-1}-1$; for n_2 even, $|X_{72}(n_1,n_2,4)|=|X_{72}(n_1,n_2,6)|=2^{n_1-1}$. So $k_{17}(n_1,n_2,m)=|X_{72}(n_1,n_2,m)|$.

$|X_{73}(n_1,n_2,m)|=2^{n_2}|X_{73}(n_1,n_2,m-4)|$. $|X_{73}(n_1,n_2,4)|=1$, $|X_{73}(n_1,n_2,6)|=2^{n_2}$. So $k_{18}(n_1,n_2,m)=|X_{73}(n_1,n_2,m)|$.

$|X_{74}(n_1,n_2,m)| = |X_{74}(n_1,n_2,m-4)|$. For n_2 odd, $|X_{74}(n_1,n_2,4)| = |X_{74}(n_1,n_2,6)| = 2^{n_1-1}$; for n_2 even, $|X_{74}(n_1,n_2,4)| = |X_{74}(n_1,n_2,6)| = 2^{n_1-1}-1$. So $k_{19}(n_1,n_2,m) = |X_{74}(n_1,n_2,m)|$.

$|X_{75}(n_1,n_2,m)| = 2^{n_2}|X_{75}(n_1,n_2,m-4)|$. $|X_{75}(n_1,n_2,4)| = |X_{75}(n_1,n_2,6)| = 2^{n_2}$. So $k_{20}(n_1,n_2,m) = |X_{75}(n_1,n_2,m)|$.

$|X_{76}(n_1,n_2,m)| = |X_{76}(n_1,n_2,m-4)|$. For n_2 odd, $|X_{76}(n_1,n_2,4)| = |X_{76}(n_1,n_2,6)| = 2^{n_1-1}$; for n_2 even, $|X_{76}(n_1,n_2,4)| = |X_{76}(n_1,n_2,6)| = 2^{n_1-1}-1$. So $k_{19}(n_1,n_2,m) = |X_{76}(n_1,n_2,m)|$.

$|X_{77}(n_1,n_2,m)| = |X_{77}(n_1,n_2,m-4)|$. $|X_{77}(n_1,n_2,4)| = |X_{77}(n_1,n_2,6)| = 1$. So $k_{21}(n_1,n_2,m) = |X_{77}(n_1,n_2,m)|$.

$|X_{78}(n_1,n_2,m)| = |X_{78}(n_1,n_2,m-4)|$. For n_2 odd, $|X_{78}(n_1,n_2,4)| = |X_{78}(n_1,n_2,6)| = 2^{n_1-1}-1$; for n_2 even, $|X_{78}(n_1,n_2,4)| = |X_{78}(n_1,n_2,6)| = 2^{n_1-1}$. So $k_{17}(n_1,n_2,m) = |X_{78}(n_1,n_2,m)|$.

Thus $|X_7(n_1,n_2,m)| = k_{16}(n_1,n_2,m) + 2k_{17}(n_1,n_2,m) + k_{18}(n_1,n_2,m) + 2k_{19}(n_1,n_2,m) + k_{20}(n_1,n_2,m) + k_{21}(n_1,n_2,m)$.

(6) Calculation of $|X_8(n_1,n_2,m)|$.

Let $X_{8i}(n_1,n_2,m) = \{\lambda \mid \lambda \in B_{8i}(n_1,n_2,m), \lambda(c_j) = \lambda(c_{m-j}), j = 1,\cdots,m-1\}$, $i = 1,2,3,4$. We have $|X_{81}(n_1,n_2,m)| = 2^{n_2-1}|X_{81}(n_1,n_2,m-2)| + 2^{n_1+2n_2-1}|X_{81}(n_1,n_2,m-4)|$. $|X_{81}(n_1,n_2,4)| = 2^{2n_2-1}-2^{n_2}$, $|X_{81}(n_1,n_2,6)| = 2^{3n_2-1}-2^{2n_2}$. So $k_{22}(n_1,n_2,m) = |X_{81}(n_1,n_2,m)|$.

$|X_{82}(n_1,n_2,m)| = 2^{n_2}|X_{82}(n_1,n_2,m-4)|$. $|X_{82}(n_1,n_2,4)| = 2^{n_2-1}-1$, $|X_{82}(n_1,n_2,6)| = 2^{2n_2-1}-2^{n_2}$. So $k_{23}(n_1,n_2,m) = |X_{82}(n_1,n_2,m)|$.

$|X_{83}(n_1,n_2,m)| = 2^{n_2}|X_{83}(n_1,n_2,m-4)|$. $|X_{83}(n_1,n_2,4)| = 2^{2n_2-1}-2^{n_2} = |X_{83}(n_1,n_2,6)|$. So $k_{24}(n_1,n_2,m) = |X_{83}(n_1,n_2,m)|$.

$|X_{84}(n_1,n_2,m)| = |X_{84}(n_1,n_2,m-4)|$. $|X_{84}(n_1,n_2,4)| = 2^{n_2-1}-1 = |X_{84}(n_1,n_2,6)|$. So $k_8(n_1,n_2,m) = |X_{84}(n_1,n_2,m)|$.

$|X_8(n_1,n_2,m)| = k_{22}(n_1,n_2,m) + k_{23}(n_1,n_2,m) + k_{24}(n_1,n_2,m) + k_8(n_1,n_2,m)$.

$$\sum_{i=1}^{8}|X_i(n_1,n_2,m)| = \sum_{i\in I_1}k_i(n_1,n_2,m) + \sum_{i\in I_2}k_i(n_1,n_2,m),$$

where $I_1 = \{1,2,\cdots,24\}$ and $I_2 = \{8,12,17,19\}$.

By Burnside Lemma,

$$\sum_{\substack{g=x^u y\\ u \text{ is odd}\\ m \text{ is even}}}|O_g| = \frac{m}{2}\prod_{i=1}^{n_1+n_2+2}(2^{n_1+n_2+2}-2^{i-1})\Big[\sum_{i\in I_1}k_i(n_1,n_2,m) + \sum_{i\in I_2}k_i(n_1,n_2,m)\Big],$$

where $I_1 = \{1,2,\cdots,24\}$ and $I_2 = \{8,12,17,19\}$.

Subcase 3.4 $g = x^u y^v (\prod s_i)^\alpha (\prod t_j)^\beta$, where $\alpha = 1$ or $\beta = 1$.

By the non-singularity condition, $|O_g| = 0$.

From Burnside Lemma, we get

$$E_o(n_1,n_2,m)$$

$$= \frac{1}{2m(n_1+1)!(n_2+1)!} \prod_{i=1}^{n_1+n_2+2} (2^{n_1+n_2+2} - 2^{i-1}) \Big\{ \sum_{k>1, k|m} \varphi\Big(\frac{m}{k}\Big) \Big[\sum_{i\in I_1} f_i(n_1,n_2,k) + \sum_{i\in I_2} f_i(n_1,n_2,k) \Big] + \frac{m}{2} \Big[\sum_{i\in I_1} k_i(n_1,n_2,m) + \sum_{i\in I_2} k_i(n_1,n_2,m) \Big] \Big\},$$

where $I_1 = \{1, 2, \cdots, 24\}$ and $I_2 = \{8, 12, 17, 19\}$.

Case 4 $n_1 > n_2 = 2$ and $m = 3$ or $n_1 = n_2 > 2$ and $m \geq 3$.

According to Lemma 2.3, $\mathrm{Aut}(\mathcal{F}(n_1,n_2,m)) = S_{n_1+1} \times S_{n_2+1} \times \mathbb{Z}_2 \times \mathcal{D}_m$. $g \in \mathrm{Aut}(\mathcal{F}(n_1,n_2,m))$ can be written in the form $x^u y^v (\prod s_i)^\alpha (\prod t_j)^\beta (\prod z_k)^\gamma$, where $u \in \mathbb{Z}_m$, v, α, β and $\gamma \in \mathbb{Z}_2$.

Subcase 4.1 $g = x^u y^v (\prod s_i)^\alpha (\prod t_j)^\beta (\prod z_k)^\gamma$, where $\alpha = 1$ or $\beta = 1$ or $\gamma = 1$.

By the non-singularity condition, $|O_g| = 0$.

Subcase 4.2 $g = x^u$, $x^u y$.

The calculation is similar to the subcases 3.1–3.3. We omit the details. So

$$\begin{aligned} E_o(n_1,n_2,m) = &\frac{1}{4m(n_1+1)!(n_2+1)!} \prod_{i=1}^{n_1+n_2+2} (2^{n_1+n_2+2} - 2^{i-1}) \\ &\cdot \Big\{ \sum_{k>1, k|m} \varphi\Big(\frac{m}{k}\Big) \Big[\sum_{i\in I_1} f_i(n_1,n_2,k) + \sum_{i\in I_2} f_i(n_1,n_2,k) \Big] \\ &+ \frac{m}{2} \Big[\sum_{i\in I_1} k_i(n_1,n_2,m) + \sum_{i\in I_2} k_i(n_1,n_2,m) \Big] \Big\}, \end{aligned}$$

where $I_1 = \{1, 2, \cdots, 24\}$ and $I_2 = \{8, 12, 17, 19\}$.

The proof is completed.

Remark 4.1 A direct calculation shows that $E_o(3,1,3) = 8679444480$.

Appendix 1

Suppose that n_1 is odd, $n_1 \geq 2$, $n_2 \geq 1$, $n_1 \geq n_2$ and $m \geq 2$. We list recursive functions f_i as follows:

(1) $f_1(n_1,n_2,2) = 2^{n_1-1}, f_1(n_1,n_2,3) = 2^{2n_1+n_2-2}, f_1(n_1,n_2,4) = 3\cdot 2^{3n_1+2n_2-3}$, and for $m \geq 5, f_1(n_1,n_2,m) = 2^{n_1+n_2-1} f_1(n_1,n_2,m-1) + 2^{2n_1+2n_2-1} f_1(n_1,n_2,m-2)$.

(2) $f_2(n_1,n_2,2) = 2^{n_1-1}, f_2(n_1,n_2,3) = 2^{2n_1+n_2-2}, f_2(n_1,n_2,4) = 3\cdot 2^{3n_1+n_2-3}$, and for $m \geq 5, f_2(n_1,n_2,m) = 2^{n_1-1} f_2(n_1,n_2,m-1) + 2^{2n_1+n_2-1} f_2(n_1,n_2,m-2)$.

(3) $f_3(n_1,n_2,2) = 2^{n_1-1},\ f_3(n_1,n_2,3) = 2^{2n_1-2}, f_3(n_1,n_2,4) = 2^{3n_1+n_2-2} + 2^{3n_1-3}$, and for $m \geq 5,\ f_3(n_1,n_2,m) = 2^{n_1-1} f_3(n_1,n_2,m-1) + 2^{2n_1+n_2-1} f_3(n_1,n_2,m-2)$.

(4) $f_4(n_1,n_2,2) = 2^{n_1-1}, f_4(n_1,n_2,3) = 2^{2n_1-2}, f_4(n_1,n_2,4) = 2^{3n_1+n_2-3} + 2^{3n_1-2}$, and for $m \geq 5, f_4(n_1,n_2,m) = 2^{n_1-1} f_4(n_1,n_2,m-1) + 2^{2n_1+n_2-1} f_4(n_1,n_2,m-2)$.

(5) For n_2 even, $f_5(n_1,n_2,m) = 0$; for n_2 odd, $f_5(n_1,n_2,2) = 2^{n_2-1} - 1,\ f_5(n_1,n_2,3) = 2^{n_1+2n_2-2} - 2^{n_1+n_2-1},\ f_5(n_1,n_2,4) = 3\cdot 2^{2n_1+3n_2-3} - 3\cdot 2^{2n_1+2n_2-2}$, and for $m \geq 5, f_5(n_1,n_2,m) = 2^{n_1+n_2-1} f_5(n_1,n_2,m-1) + 2^{2n_1+2n_2-1} f_5(n_1,n_2,m-2)$.

(6) For n_2 odd, $f_6(n_1,n_2,m) = 0$; for n_2 even, $f_6(n_1,n_2,2) = 2^{n_2-1} - 1,\ f_6(n_1,n_2,3) = 2^{n_1+2n_2-2} - 2^{n_1+n_2-1}, f_6(n_1,n_2,4) = 2^{n_1+2n_2-1} - 2^{n_1+n_2}$, and for $m \geq 5, f_6(n_1,n_2,m) = 2^{n_1+n_2} f_6(n_1,n_2,m-2)$.

(7) For n_2 odd, $f_7(n_1,n_2,m) = 0$; for n_2 even, $f_7(n_1,n_2,2) = 2^{n_2-1} - 1,\ f_7(n_1,n_2,3) = 0,\ f_7(n_1,n_2,4) = 2^{n_1+2n_2-1} - 2^{n_1+n_2}$, and for $m \geq 5, f_7(n_1,n_2,m) = 2^{n_1+n_2} f_7(n_1,n_2,m-2)$.

(8) For n_2 even, $f_8(n_1,n_2,m)=0$; for n_2 odd, $f_8(n_1,n_2,2)=2^{n_2-1}-1$, $f_8(n_1,n_2,3)=0$, $f_8(n_1,n_2,4)=2^{n_2-1}-1$, and for $m\geq 5, f_8(n_1,n_2,m)=f_8(n_1,n_2,m-2)$.

(9) For n_2 even, $f_9(n_1,n_2,m)=0$; for n_2 odd, $f_9(n_1,n_2,2)=2^{n_2-1}, f_9(n_1,n_2,3)=2^{n_1+2n_2-2}, f_9(n_1,n_2,4)=3\cdot 2^{n_1+3n_2-3}$, and for $m\geq 5, f_9(n_1,n_2,m)=2^{n_2-1}f_9(n_1,n_2,m-1)+2^{n_1+2n_2-1}f_9(n_1,n_2,m-2)$.

(10) For n_2 odd, $f_{10}(n_1,n_2,m)=0$; for n_2 even, $f_{10}(n_1,n_2,2)=2^{n_2-1}$, $f_{10}(n_1,n_2,3)=2^{n_1+2n_2-2}, f_{10}(n_1,n_2,4)=2^{n_1+2n_2-1}$, and for $m\geq 5, f_{10}(n_1,n_2,m)=2^{n_1+n_2}f_{10}(n_1,n_2,m-2)$.

(11) For n_2 odd, $f_{11}(n_1,n_2,m)=0$; for n_2 even, $f_{11}(n_1,n_2,2)=2^{n_2-1}$, $f_{11}(n_1,n_2,3)=0, f_{11}(n_1,n_2,4)=2^{2n_2}-1$, and for $m\geq 5, f_{11}(n_1,n_2,m)=2^{n_2}f_{11}(n_1,n_2,m-2)$.

(12) For n_2 even, $f_{12}(n_1,n_2,m)=0$; for n_2 odd, $f_{12}(n_1,n_2,2)=2^{n_2-1}$, $f_{12}(n_1,n_2,3)=0$, $f_{12}(n_1,n_2,4)=2^{n_2-1}$, and for $m\geq 5, f_{12}(n_1,n_2,m)=f_{12}(n_1,n_2,m-2)$.

(13) For n_2 even, $f_{13}(n_1,n_2,m)=0$; for n_2 odd, $f_{13}(n_1,n_2,2)=2^{n_2-1}$, $f_{13}(n_1,n_2,3)=2^{2n_2-2}$, $f_{13}(n_1,n_2,4)=2^{n_1+3n_2-2}+2^{3n_2-3}$, and for $m\geq 5, f_{13}(n_1,n_2,m)=2^{n_2-1}f_{13}(n_1,n_2,m-1)+2^{n_1+2n_2-1}f_{13}(n_1,n_2,m-2)$.

(14) For n_2 odd, $f_{14}(n_1,n_2,m)=0$; for n_2 even, $f_{14}(n_1,n_2,2)=2^{n_2-1}$, $f_{14}(n_1,n_2,3)=2^{2n_2-2}, f_{14}(n_1,n_2,4)=2^{2n_2-1}$, and for $m\geq 5, f_{14}(n_1,n_2,m)=2^{n_2}f_{14}(n_1,n_2,m-2)$.

(15) For n_2 odd, $f_{15}(n_1,n_2,m)=0$; for n_2 even, $f_{15}(n_1,n_2,2)=2^{n_2-1}$, $f_{15}(n_1,n_2,3)=0, f_{15}(n_1,n_2,4)=2^{n_1+2n_2-1}$, and for $m\geq 5, f_{15}(n_1,n_2,m)=2^{n_1+n_2}f_{15}(n_1,n_2,m-2)$.

(16) For n_2 even, $f_{16}(n_1,n_2,m)=0$; for n_2 odd, $f_{16}(n_1,n_2,2)=1, f_{16}(n_1,n_2,3)=2^{n_2-1}$, $f_{16}(n_1,n_2,4)=2^{n_1+2n_2-2}+2^{2n_2-1}$, and for $m\geq 5, f_{16}(n_1,n_2,m)=2^{n_2-1}f_{16}(n_1,n_2,m-1)+2^{n_1+2n_2-1}f_{16}(n_1,n_2,m-2)$.

(17) For n_2 odd, $f_{17}(n_1,n_2,2)=2^{n_1-1}-1, f_{17}(n_1,n_2,3)=0$, $f_{17}(n_1,n_2,4)=2^{n_1-1}-1$, and for $m\geq 5, f_{17}(n_1,n_2,m)=f_{17}(n_1,n_2,m-2)$; for n_2 even, $f_{17}(n_1,n_2,2)=2^{n_1-1}, f_{17}(n_1,n_2,3)=0, f_{17}(n_1,n_2,4)=2^{n_1-1}$, and for $m\geq 5, f_{17}(n_1,n_2,m)=f_{17}(n_1,n_2,m-2)$.

(18) For n_2 odd, $f_{18}(n_1,n_2,m)=0$; for n_2 even, $f_{18}(n_1,n_2,2)=1$, $f_{18}(n_1,n_2,3)=2^{n_2-1}$, $f_{18}(n_1,n_2,4)=2^{n_2}$, and for $m\geq 5, f_{18}(n_1,n_2,m)=2^{n_2}f_{18}(n_1,n_2,m-2)$.

(19) For n_2 even, $f_{19}(n_1,n_2,2)=2^{n_1-1}-1$, $f_{19}(n_1,n_2,3)=0$, $f_{19}(n_1,n_2,4)=2^{n_1-1}-1$, and for $m\geq 5, f_{19}(n_1,n_2,m)=f_{19}(n_1,n_2,m-2)$; for n_2 odd, $f_{19}(n_1,n_2,2)=2^{n_1-1}$, $f_{19}(n_1,n_2,3)=0, f_{19}(n_1,n_2,4)=2^{n_1-1}$, and for $m\geq 5, f_{19}(n_1,n_2,m)=f_{19}(n_1,n_2,m-2)$.

(20) For n_2 odd, $f_{20}(n_1,n_2,m)=0$; for n_2 even, $f_{20}(n_1,n_2,2)=1, f_{20}(n_1,n_2,3)=0$, $f_{20}(n_1,n_2,4)=2^{n_2}$, and for $m\geq 5, f_{20}(n_1,n_2,m)=2^{n_2}f_{20}(n_1,n_2,m-2)$.

(21) For n_2 even, $f_{21}(n_1,n_2,m)=0$; for n_2 odd, $f_{21}(n_1,n_2,2)=1, f_{21}(n_1,n_2,3)=0$, $f_{21}(n_1,n_2,4)=1$, and for $m\geq 5, f_{21}(n_1,n_2,m)=f_{21}(n_1,n_2,m-2)$.

(22) For n_2 even, $f_{22}(n_1,n_2,m)=0$; for n_2 odd, $f_{22}(n_1,n_2,2)=2^{n_2-1}-1, f_{22}(n_1,n_2,3)=2^{2n_2-2}-2^{n_2-1}, f_{22}(n_1,n_2,4)=2^{n_1+3n_2-3}+2^{3n_2-2}-2^{2n_2-1}-2^{n_1+2n_2-2}$, and for $m\geq 5$, $f_{22}(n_1,n_2,m)=2^{n_2-1}f_{22}(n_1,n_2,m-1)+2^{n_1+2n_2-1}f_{22}(n_1,n_2,m-2)$.

(23) For n_2 odd, $f_{23}(n_1,n_2,m)=0$; for n_2 even, $f_{23}(n_1,n_2,2)=2^{n_2-1}-1, f_{23}(n_1,n_2,3)=2^{2n_2-2}-2^{n_2-1}, f_{23}(n_1,n_2,4)=2^{2n_2-1}-2^{n_2}$, and for $m\geq 5, f_{23}(n_1,n_2,m)=2^{n_2}f_{23}(n_1,n_2,m-2)$.

(24) For n_2 odd, $f_{24}(n_1,n_2,m)=0$; for n_2 even, $f_{24}(n_1,n_2,2)=2^{n_2-1}-1, f_{24}(n_1,n_2,3)=0, f_{24}(n_1,n_2,4)=2^{2n_2-1}-2^{n_2}$, and for $m\geq 5, f_{24}(n_1,n_2,m)=2^{n_2}f_{24}(n_1,n_2,m-2)$.

Appendix 2

Suppose positive integers $n_1\geq 2$ and $m\geq 3$. We list recursive functions g_i.

For m odd, $g_i(n_1,m)=0$ $(1\leq i\leq 7)$. For m even, g_i is defined as follows:

(1) $g_1(n_1,4)=2^{2n_1}$, $g_1(n_1,6)=2^{3n_1+1}$, and for $m\geq 8$, $g_1(n_1,m)=2^{n_1}g_1(n_1,m-2)+2^{2n_1+1}g_1(n_1,m-4)$.

(2) $g_2(n_1,4)=2^{2n_1-1}$, $g_2(n_1,6)=2^{3n_1}$, and for $m\geq 8$, $g_2(n_1,m)=2^{n_1-1}g_2(n_1,m-2)+2^{2n_1}g_2(n_1,m-4)$.

(3) $g_3(n_1,4)=2^{2n_1}$, $g_3(n_1,6)=2^{3n_1}$, and for $m\geq 8$, $g_3(n_1,m)=2^{n_1-1}g_3(n_1,m-2)+2^{2n_1}g_3(n_1,m-4)$.

(4) $g_4(n_1,4)=2^{2n_1-1}$, $g_4(n_1,6)=2^{3n_1-1}$, and for $m\geq 8$, $g_4(n_1,m)=2^{n_1-1}g_4(n_1,m-2)+2^{2n_1}g_4(n_1,m-4)$.

(5) $g_5(n_1,4)=g_5(n_1,6)=1$, and for $m\geq 8$, $g_5(n_1,m)=g_5(n_1,m-4)$.

(6) $g_6(n_1,4)=g_6(n_1,6)=2^{n_1-1}-1$, and for $m\geq 8$, $g_6(n_1,m)=g_6(n_1,m-4)$.

(7) $g_7(n_1,4)=g_7(n_1,6)=2^{n_1-1}$, and for $m\geq 8$, $g_7(n_1,m)=g_7(n_1,m-4)$.

Appendix 3

Suppose positive integers $n_1\geq 2$, $n_2\geq 1$ and $m\geq 3$. We list recursive functions k_i.

For m odd, $k_i(n_1,n_2,m)=0$ $(1\leq i\leq 24)$. For m even, k_i is defined as follows:

(1) $k_1(n_1,n_2,4)=2^{2n_1+n_2-1}, k_1(n_1,n_2,6)=2^{3n_1+2n_2-1}$, and for $m\geq 8$, $k_1(n_1,n_2,m)=2^{n_1+n_2-1}k_1(n_1,n_2,m-2)+2^{2n_1+2n_2-1}k_1(n_1,n_2,m-4)$.

(2) $k_2(n_1,n_2,4)=2^{2n_1-1}, k_2(n_1,n_2,6)=2^{3n_1+n_2-1}$, and for $m\geq 8$, $k_2(n_1,n_2,m)=2^{n_1-1}k_2(n_1,n_2,m-2)+2^{2n_1+n_2-1}k_2(n_1,n_2,m-4)$.

(3) $k_3(n_1,n_2,4)=2^{2n_1+n_2-1}, k_3(n_1,n_2,6)=2^{3n_1+n_2-1}$, and for $m\geq 8$, $k_3(n_1,n_2,m)=2^{n_1-1}k_3(n_1,n_2,m-2)+2^{2n_1+n_2-1}k_3(n_1,n_2,m-4)$.

(4) $k_4(n_1,n_2,4)=2^{2n_1-1}, k_4(n_1,n_2,6)=2^{3n_1-1}$, and for $m\geq 8$, $k_4(n_1,n_2,m)=2^{n_1-1}k_4(n_1,n_2,m-2)+2^{2n_1+n_2-1}k_4(n_1,n_2,m-4)$.

(5) $k_5(n_1,n_2,4)=2^{n_1+2n_2-1}-2^{n_1+n_2}, k_5(n_1,n_2,6)=2^{2n_1+3n_2-1}-2^{2n_1+2n_2}$, and for $m\geq 8$, $k_5(n_1,n_2,m)=2^{n_1+n_2-1}k_5(n_1,n_2,m-2)+2^{2n_1+2n_2-1}k_5(n_1,n_2,m-4)$.

(6) $k_6(n_1,n_2,4)=2^{n_2-1}-1, k_6(n_1,n_2,6)=2^{n_1+2n_2-1}-2^{n_1+n_2}$, and for $m\geq 8$, $k_6(n_1,n_2,m)=2^{n_1+n_2}k_6(n_1,n_2,m-4)$.

(7) $k_7(n_1,n_2,4)=k_7(n_1,n_2,6)=2^{n_1+2n_2-1}-2^{n_1+n_2}$, and for $m\geq 8$, $k_7(n_1,n_2,m)=2^{n_1+n_2}k_7(n_1,n_2,m-4)$.

(8) $k_8(n_1,n_2,4)=k_6(n_1,n_2,6)=2^{n_2-1}-1$, and for $m\geq 8$, $k_8(n_1,n_2,m)=k_8(n_1,n_2,m-4)$.

(9) $k_9(n_1,n_2,4)=2^{2n_2-1}$, $k_9(n_1,n_2,6)=2^{n_1+3n_2-1}$, and for $m\geq 8$, $k_9(n_1,n_2,m)=2^{n_2-1}k_9(n_1,n_2,m-2)+2^{n_1+2n_2-1}k_9(n_1,n_2,m-4)$.

(10) $k_{10}(n_1,n_2,4)=2^{n_2-1}$, $k_{10}(n_1,n_2,6)=2^{n_1+2n_2-1}$, and for $m\geq 8$, $k_{10}(n_1,n_2,m)=2^{n_1+n_2}k_{10}(n_1,n_2,m-4)$.

(11) $k_{11}(n_1,n_2,4)=k_{11}(n_1,n_2,6)=2^{2n_2-1}$, for $m\geq 8$, $k_{11}(n_1,n_2,m)=2^{n_2}k_{11}(n_1,n_2,m-4)$.

(12) $k_{12}(n_1,n_2,4)=k_{12}(n_1,n_2,6)=2^{n_2-1}$, for $m\geq 8$, $k_{12}(n_1,n_2,m)=k_{12}(n_1,n_2,m-4)$.

(13) $k_{13}(n_1,n_2,4)=2^{n_1+2n_2-1}, k_{13}(n_1,n_2,6)=2^{n_1+3n_2-1}$, and for $m\geq 8$, $k_{13}(n_1,n_2,m)=2^{n_2-1}k_{13}(n_1,n_2,m-2)+2^{n_1+2n_2-1}k_{13}(n_1,n_2,m-4)$.

(14) $k_{14}(n_1,n_2,4)=2^{n_2-1}, k_{14}(n_1,n_2,6)=2^{2n_2-1}$, and for $m\geq 8$, $k_{14}(n_1,n_2,m)=2^{n_2}k_{14}(n_1,n_2,m-4)$.

(15) $k_{15}(n_1,n_2,4)=k_{15}(n_1,n_2,6)=2^{n_1+2n_2-1}$, and for $m\geq 8$, $k_{15}(n_1,n_2,m)=2^{n_1+n_2}k_{15}(n_1,n_2,m-4)$.

(16) $k_{16}(n_1,n_2,4)=2^{n_2}, k_{16}(n_1,n_2,6)=2^{2n_2}$, and for $m\geq 8$, $k_{16}(n_1,n_2,m)=2^{n_2-1}k_{16}(n_1,n_2,m-2)+2^{n_1+2n_2-1}k_{16}(n_1,n_2,m-4)$.

(17) $k_{17}(n_1, n_2, 4) = k_{17}(n_1, n_2, 6) = 2^{n_1-1} - 1$ with n_2 odd, $k_{17}(n_1, n_2, 4) = k_{17}(n_1, n_2, 6) = 2^{n_1-1}$ with n_2 even, and for $m \geq 8$, $k_{17}(n_1, n_2, m) = k_{17}(n_1, n_2, m-4)$.

(18) $k_{18}(n_1, n_2, 4) = 1, k_{18}(n_1, n_2, 6) = 2^{n_2}$, for $m \geq 8$, $k_{18}(n_1, n_2, m) = 2^{n_2} k_{18}(n_1, n_2, m-4)$.

(19) $k_{19}(n_1, n_2, 4) = k_{19}(n_1, n_2, 6) = 2^{n_1-1}$ with n_2 odd, $k_{19}(n_1, n_2, 4) = k_{19}(n_1, n_2, 6) = 2^{n_1-1} - 1$ with n_2 even, and for $m \geq 8$, $k_{19}(n_1, n_2, m) = k_{19}(n_1, n_2, m-4)$.

(20) $k_{20}(n_1, n_2, 4) = k_{18}(n_1, n_2, 6) = 2^{n_2}$, and for $m \geq 8$, $k_{20}(n_1, n_2, m) = 2^{n_2} k_{20}(n_1, n_2, m-4)$.

(21) $k_{21}(n_1, n_2, 4) = k_{21}(n_1, n_2, 6) = 1$, and for $m \geq 8$, $k_{21}(n_1, n_2, m) = k_{21}(n_1, n_2, m-4)$.

(22) $k_{22}(n_1, n_2, 4) = 2^{2n_2-1} - 2^{n_2}, k_{22}(n_1, n_2, 6) = 2^{3n_2-1} - 2^{2n_2}$, and for $m \geq 8$, $k_{22}(n_1, n_2, m) = 2^{n_2-1} k_{22}(n_1, n_2, m-2) + 2^{n_1+2n_2-1} k_{22}(n_1, n_2, m-4)$.

(23) $k_{23}(n_1, n_2, 4) = 2^{n_2-1} - 1, k_{23}(n_1, n_2, 6) = 2^{2n_2-1} - 2^{n_2}$, and for $m \geq 8$, $k_{23}(n_1, n_2, m) = 2^{n_2} k_{23}(n_1, n_2, m-4)$.

(24) $k_{24}(n_1, n_2, 4) = k_{24}(n_1, n_2, 6) = 2^{2n_2-1} - 2^{n_2}$, and for $m \geq 8$, $k_{24}(n_1, n_2, m) = 2^{n_2} k_{24}(n_1, n_2, m-4)$.

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References

[1] Alperin, J. L. and Bell, R. B., Groups and Representations, Springer-Verlag, Berlin, 1995.

[2] Cai, M., Chen, X. and Lü, Z., Small covers over prisms, *Topology Appl.*, **154**, 2007, 2228–2234.

[3] Chen, Y. and Wang, Y., Small covers over a product of simplices, *Filomat,* **27**, 2013, 777–787.

[4] Choi, S., The number of orientable small covers over cubes, *Proc. Japan Acad., Ser. A,* **86**, 2010, 97–100.

[5] Davis, M. and Januszkiewicz, T., Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.*, **62**, 1991, 417–451.

[6] Garrison, A. and Scott, R., Small covers of the dodecahedron and the 120-cell, *Proc. Amer. Math. Soc.*, **131**, 2003, 963–971.

[7] Lü, Z. and Masuda, M., Equivariant classification of 2-torus manifolds, *Colloq. Math.,* **115**, 2009, 171–188.

[8] Meng, Y., On closed manifolds and moment-angle complex in toric topology, Dissertation, Hebei Normal Univ., 2012.

[9] Nakayama, H. and Nishimura, Y., The orientability of small covers and coloring simple polytopes, *Osaka J. Math.*, **42**, 2005, 243–256.

[10] Ziegler, G. M., Lectures on Polytopes, Springer-Verlag, Berlin, 1994.