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Canonical Metrics on Generalized Cartan-Hartogs Domains*

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Abstract In this paper, the author considers a class of bounded pseudoconvex domains, i.e., the generalized Cartan-Hartogs domains $\Omega(\mu, m)$. The first result is that the natural Kähler metric $g^{\Omega(\mu,m)}$ of $\Omega(\mu, m)$ is extremal if and only if its scalar curvature is a constant. The second result is that the Bergman metric, the Kähler-Einstein metric, the Carathéodary metric, and the Koboyashi metric are equivalent for $\Omega(\mu, m)$.

Keywords Canonical metric, Extremal metric, Comparison theorem, Generalized Cartan-Hartogs domains
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1 Introduction

In order to find the canonical representant of a given Kähler class $[\omega]$ of a complex compact Kähler manifold (M, J), Calabi made a search in [3–4]. He introduced the notion of the extremal metric defined as the minimizer of the L^2-norm of the Ricci tensor. This notion is one of the generalizations of the Kähler-Einstein metric. Let s_g be the scalar curvature of the Kähler metric g. He proved that g is an extremal metric if and only if ∇s_g is a holomorphic vector field. From the Euler-Lagrange equation for s_g, the metrics with constant scalar curvatures, in particular the Kähler-Einstein metrics, are extremal. He also proved that some extremal metrics with non-constant scalar curvatures do exist.

The existence and uniqueness of the extremal metrics in some given Kähler classes have been studied (see [7, 13, 18]). The important relationship between the existence of extremal metrics and various stability notions of the corresponding polarized manifolds has also been deeply investigated (see [2, 11–12, 19–22, 24]). However, a complete understanding of the existence theory for extremal metrics is still missing. One can see some recent progress on the study of Calabi's extremal Kähler metrics in [23].

In general, the problem of finding extremal metrics is quite natural but difficult (see [27]). On a complete noncompact smooth surface, Chang [6] proved the existence of extremal metrics. On a strongly pseudoconvex Hartogs domain, Loi and Zedda [16] proved that the only extremal metric is the hyperbolic metric. On Cartan-Hartogs domains endowed with their natural Kähler metrics, Zedda [29] proved that they are extremal if and only if they are Kähler-Einstein. In this paper, we extend Zedda's result to $(\Omega(\mu, m), g^{\Omega(\mu,m)})$, i.e., the generalized Cartan-Hartogs domain endowed with a natural Kähler metric $g^{\Omega(\mu,m)}$, where all elements of the vector $\mu \in \mathbb{R}^m$

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are positive (see the definition in Section 3). This domain is a Hartogs domain over the product of irreducible bounded symmetric domains. In particular, $\Omega(\mu,1)$ is exactly the Cartan-Hartogs domain introduced by Yin, Roos [26]. The first result of the paper is the following two theorems.

Theorem 1.1 *Let $\Omega(\mu,m)\subset\mathbb{C}^n$ be the generalized Cartan-Hartogs domain given by* (3.1). *Then the Kähler metric $g^{\Omega(\mu,m)}$ in* (3.2) *is extremal if and only if its scalar curvature is a constant, i.e., $\mu=(\mu_1,\mu_2,\cdots,\mu_m)$ satisfies the equation $\sum\limits_{i=1}^{m}\big(n-\frac{\gamma_i}{\mu_i}\big)d_i=0$.*

Theorem 1.2 *The metric $g^{\Omega(\mu,m)}$ in* (3.2) *is Kähler-Einstein if and only if the parameter μ satisfies $\mu_i=\frac{\gamma_i}{n}$ for $i=1,2,\cdots,m$.*

As we know, the Bergman metric, the Kähler-Einstein metric, the Carathéodary metric, and the Koboyashi metric are four classical invariant metrics. It is interesting to study the comparison theorem among them. For the holomorphic homogeneous regular manifolds (also called the uniformly squeezing domains) introduced by Liu, Sun, Yau and Yeung independently, the four classical invariant metrics on a homogenous regular domain are equivalent (see [15, 28]). It is known that bounded homogeneous domains, bounded strongly convex domains, bounded domains which cover a compact Kähler manifold, Teichmüller spaces $T_{g,n}$ of hyperbolic Riemann surfaces of genus g with n punctures, strongly pseudoconvex domains with C^2 boundary, Cartan-Hartogs domains, and bounded convex domains are such domains (see [10, 14–15, 28]). In this paper, we prove that $\Omega(\mu,m)$ is also holomorphic homogeneous regular (with the uniform squeezing property). This implies our second result Theorem 4.2.

This paper is organized as follows. We start by recalling some notions and results for Cartan domains and holomorphic homogeneous regular domains (uniformly squeezing domains) in Section 2. By investigating the geometry of $(\Omega(\mu,m),g^{\Omega(\mu,m)})$, we obtain Theorems 1.1–1.2 in Section 3. Finally, we prove that $\Omega(\mu,m)$ is a holomorphic homogeneous regular domain in Section 4, which implies that the four classical metrics are equivalent.

2 Preliminaries

2.1 Cartan domains

In this section, we recall some results of the irreducible bounded symmetric domains which have been completely classified up to a biholomorphic isomorphism due to Cartan [5].

Let $M_{m,n}$ be the space of $m\times n$ complex matrices, I be the identity matrix, $\overline{z}$ be the conjugate matrix of z, and z^t be the transposed matrix of z. If a square matrix A is positive definite, then we denote it by $A>0$. The list of irreducible bounded symmetric domains and the corresponding generic norms is the following (see [17, Chapter 4]):

Type I ($1\le m\le n$): $D_{\rm I}=\{z\in M_{m,n}(\mathbb{C}):I-z\overline{z}^t>0\}$, $N(z,\zeta)=\det(I-z\overline{\zeta}^t)$.

Type II ($m=n\ge 5$): $D_{\rm II}=\{z\in D_{\rm I}:z=-z^t\}$, $N(z,\zeta)=\det(I+z\overline{\zeta})$.

Type III ($m=n\ge 2$): $D_{\rm III}=\{z\in D_{\rm I}:z=z^t\}$, $N(z,\zeta)=\det(I-z\overline{\zeta})$.

Type IV ($m\ge 5$): $D_{\rm IV}=\{z\in\mathbb{C}^m:1-2q(z,\overline{z})+|q(z,z)|^2>0,|q(z,\overline{z})|<1\}$, $N(z,\zeta)=1-q(z,\zeta)+q(z,z)q(\zeta,\zeta)$, where $q(z,\zeta)=\sum\limits_{j=1}^{m}z_j\zeta_j$.

Type V: $D_{\rm V}=\big\{z\in M_{2,1}(O_{\mathbb{C}}):1-(z\mid z)+(z^\sharp\mid z^\sharp)>0,2-(z\mid z)>0\big\}$, $N(z,\zeta)=1-(z\mid\zeta)+(z^\sharp\mid\zeta^\sharp)$.

Type VI: $D_{\mathrm{VI}} = \{z \in M_{3,3}(O_{\mathbb{C}}) : 1 - (z \mid z) + (z^\sharp \mid z^\sharp) - |\det z|^2 > 0,\ 3 - 2(z \mid z) + (z^\sharp \mid z^\sharp) > 0, 3 - (z \mid z) > 0\}$, $N(z,\zeta) = 1 - (z \mid \zeta) + (z^\sharp \mid \zeta^\sharp) - \det z \overline{\det \zeta}$.

Here $O_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{O}$ is complex 8 dimensional Cayley algebra. $M_{3,3}(O_{\mathbb{C}})$ is the space of 3×3 matrices with entries in the space $O_{\mathbb{C}}$ of octonions over $\mathbb{C}$, which are Hermitian with respect to the Cayley conjugation. $z^\sharp$ is the adjoint matrix in $M_{3,3}(O_{\mathbb{C}})$, $(z \mid \zeta)$ is the standard Hermitian product in $M_{3,3}(O_{\mathbb{C}})$, and $M_{2,1}(O_{\mathbb{C}})$ is a subspace of $M_{3,3}(O_{\mathbb{C}})$.

The domains of types I–IV are classical, while D_{V} and D_{VI} are the exceptional 16 and 27 dimensional domains. These domains are also called Cartan domains (also see the details in [1, Section 2]). The genus γ, the rank r, and the numerical invariants a and b for an irreducible bounded symmetric domain D have the following relation:

$$\gamma = (r-1)a + b + 2.$$

The parameters of those domains are given in Table 1.

Table 1 Parameters of Cartan domains

Type	Dimension d	Rank r	a	b	Genus γ
D_{I} $(1 \leqslant m \leqslant n)$	mn	m	0 or 2	$n-m$	$m+n$
D_{II} $(m \geqslant 5)$	$\frac{m(m-1)}{2}$	$[\frac{m}{2}]$	4	0 or 2	$2(m-1)$
D_{III} $(m \geqslant 2)$	$\frac{m(m+1)}{2}$	m	1	0	$m+1$
D_{IV} $(m \geqslant 4)$	m	2	$m-2$	0	m
D_{V}	16	2	6	4	12
D_{VI}	27	3	8	0	18

The connection between the generic norm $N(z,z)$ and the Bergman kernel $K(z,z)$ of the bounded symmetric domain D is

$$V(D)K(z,z) = N(z,z)^{-\gamma}, \tag{2.1}$$

where $V(D)$ is the volume of D. Let g^D be the Bergman metric, and then

$$\det g^D = \gamma^d N(z,z)^{-\gamma} \tag{2.2}$$

(see (13) in [25]).

In the following, we study the derivative of the generic norm at zero. Let $z = (z_1, z_2, \cdots, z_d)$ be the coordinate of the Cartan domain $D \subset \mathbb{C}^d$. Since the Cartan domain is a circular domain with its center at 0, we have

$$N(z,0) \equiv 1. \tag{2.3}$$

It turns out that

$$\frac{\partial N(z,z)}{\partial z_\alpha}\Big|_{z=0} = \frac{\partial N(z,z)}{\partial \overline{z}_\alpha}\Big|_{z=0} = 0, \tag{2.4}$$

$$\frac{\partial^2 N(z,z)}{\partial z_\alpha \partial z_\beta}\Big|_{z=0} = \frac{\partial^2 N(z,z)}{\partial \overline{z}_\alpha \partial \overline{z}_\beta}\Big|_{z=0} = 0 \tag{2.5}$$

for $1 \le \alpha, \beta \le d$. Since the Bergman metric

$$g^D = \Big(\frac{\partial^2 \log K(z,z)}{\partial z_\alpha \partial \overline{z}_\beta}\Big) = -\gamma\Big(\frac{\partial^2 N(z,z)}{\partial z_\alpha \partial \overline{z}_\beta}\Big), \tag{2.6}$$

noticing that $g^D|_{z=0} = \gamma I^{(d)}$ (see [17]), we have

$$\Big(\frac{\partial^2 N(z,z)}{\partial z_\alpha \partial \overline{z}_\beta}\Big)\Big|_{z=0} = -I^{(d)},$$

where $I^{(d)}$ is the $d \times d$ identity matrix. Let $z_\alpha = x_\alpha + \sqrt{-1}y_\alpha$, and then

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial \overline{z}_\alpha}, \quad \frac{\partial}{\partial y_\alpha} = \sqrt{-1}\Big(\frac{\partial}{\partial z_\alpha} - \frac{\partial}{\partial \overline{z}_\alpha}\Big).$$

It follows that the real Hessian of the general norm N at the origin is

$$\mathrm{Hess}(N)(0,0) = -2I^{(2d)}. \tag{2.7}$$

2.2 Squeezing function

The notion of the squeezing function introduced by Deng, Guan, and Zhang is useful for studying the geometric and analytic properties of bounded domains (see [9]). In this section, we will recall some properties of the squeezing function which will be used in Section 4.

Definition 2.1 (see [9]) *Let D be a bounded domain in $\mathbb{C}^n$. For $z \in D$ and an (open) holomorphic embedding $f : D \to B^n$ with $f(z) = 0$, define*

$$s_D(z,f) = \sup\{r \mid B^n(0,r) \subset f(D)\},$$

and the squeezing number $s_D(z)$ of D at z is defined as

$$s_D(z) = \sup_f \{s_D(z,f)\},$$

where the supremum is taken over all holomorphic embeddings $f : D \to B^n$ with $f(z) = 0$, B^n is the unit ball in $\mathbb{C}^n$, and $B^n(0,r)$ is the ball in $\mathbb{C}^n$ with center 0 and radius r. The function s_D is called the squeezing function of D.

The definition shows that the holomorphic homogenous regular domain is a bounded domain whose squeezing function is bounded below by a positive constant and the squeezing function is invariant under the biholomorphic transformation. From [9], we know that the squeezing function is continuous. Hence, the boundary behavior of a squeezing function is important for studying its boundedness.

Definition 2.2 (see [10]) *A point p is called a globally strongly convex boundary point of D if ∂D is C^2 smooth and strongly convex at p, and $D \cap T_p\partial D = p$, where $T_p\partial D$ is the tangent hyperplane of ∂D at p.*

Theorem 2.1 (see [10]) *Let $D \subset \mathbb{C}^n$ be a bounded domain. Assume that $p \in \partial D$ is a globally strongly convex boundary point of D. Then*

$$\lim_{z \to p} s_D(z) = 1.$$

By this theorem, if the boundedness of the squeezing function of a bounded domain depends on its globally strongly convex boundary points, then we know that it is a holomorphic homogeneous regular domain immediately, for example, bounded strongly pseudoconvex domains with C^2 smooth boundaries and Cartan-Hartogs domains. In Section 4, we will show that the generalized Cartan-Hartogs domain has the similar property.

3 The Geometry of $(\Omega(\mu, m), g^{\Omega(\mu,m)})$

Let $D \times \triangle = D_1 \times D_2 \times \cdots \times D_m \times \triangle$ be the product of finite Cartan domains $D_i \subset \mathbb{C}^{d_i}$ and the disc $\triangle \subset \mathbb{C}$. The coordinates of D_i and $\triangle$ are denoted by $z_i = (z_{i1}, z_{i2}, \cdots, z_{id_i})$ in $\mathbb{C}^{d_i}$ and ξ in $\mathbb{C}$ respectively. Set $d = \sum\limits_{i=1}^{m} d_i$ and write $(z, \xi) := (z_1, z_2, \cdots, z_m, \xi) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_m} \times \mathbb{C}$. The generalized Cartan-Hartogs domain $\Omega(\mu, m)$ associated to D is defined to be

$$\Omega(\mu, m) = \Big\{(z, \xi) \in D \times \triangle : |\xi|^2 < \prod_{i=1}^{m} N_i^{u_i}(z_i, z_i)\Big\}, \tag{3.1}$$

where m is a positive integer, μ is an m-vector with the positive real numbers μ_i as its i-element and $N_i(z_i, w_i)$ is the generic norm of D_i with the dimension $n = d + 1$.

In [25], Wang and Hao have computed the explicit form of the uniquely complete Kähler-Einstein metric in the case that $\mu = \big(\frac{\gamma_1}{n}, \cdots, \frac{\gamma_m}{n}\big)$. The Kähler potential is

$$\Phi(z, \xi) = -\log\Big(\prod_{i=1}^{m} N_i^{\mu_i}(z_i, z_i) - |\xi|^2\Big).$$

In this section, we consider the domain $\Omega(\mu, m)$ equipped with a natural Kähler metric $g^{\Omega(\mu,m)}$, i.e., in the neighbourhood of the origin, the Kähler form associated to $g^{\Omega(\mu,m)}$ is

$$\omega(\mu, m) = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\Phi(z, \xi), \tag{3.2}$$

where the Kähler potential

$$\Phi(z, \xi) = -\log\Big(\prod_{i=1}^{m} N_i^{\mu_i}(z_i, z_i) - |\xi|^2\Big).$$

By (2.1), we know Φ is a C^∞ strictly plurisubharmonic function on $\Omega(\mu, m)$. So $g^{\Omega(\mu,m)}$ is a Kähler metric. Actually, it is just the natural complete Kähler metric given by a defining function of the domain. This Kähler metric was constructed by Cheng and Yau on the strictly pseudoconvex domain with C^k, $k \geq 5$ boundary in $\mathbb{C}^n$ firstly (see [8]). Now, we will describe the case when $(\Omega(\mu, m), g^{\Omega(\mu,m)})$ is a complete extremal Kähler manifold.

Let $g^{\Omega(\mu,m)}$ also denote its metric matrix, i.e.,

$$g^{\Omega(\mu,m)} = \left(\begin{array}{c|c} g^{\Omega(\mu,m)}_{j\alpha,\overline{k\beta}} & g^{\Omega(\mu,m)}_{j\alpha,\overline{n}} \\ \hline g^{\Omega(\mu,m)}_{n,\overline{k\beta}} & g^{\Omega(\mu,m)}_{n,\overline{n}} \end{array}\right), \tag{3.3}$$

where

$$g^{\Omega(\mu,m)}_{j\alpha,\overline{k\beta}} = \frac{\partial^2\Phi(z,\xi)}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}, \quad g^{\Omega(\mu,m)}_{j\alpha,\overline{n}} = \frac{\partial^2\Phi(z,\xi)}{\partial z_{j\alpha}\partial\overline{\xi}}, \quad g^{\Omega(\mu,m)}_{n,\overline{k\beta}} = \frac{\partial^2\Phi(z,\xi)}{\partial \xi\partial\overline{z}_{k\beta}}.$$

Lemma 3.1 *The metric $g^{\Omega(\mu,m)}$ satisfies the following equation:*

$$\det(g^{\Omega(\mu,m)}) = \frac{1}{(H - |\xi|^2)^{n+1}}\prod_{i=1}^{m} u_i^{d_i} N_i(z_i, z_i)^{n\mu_i - \gamma_i}, \tag{3.4}$$

where $H=\prod_{i=1}^{m} N_i^{\mu_i}(z_i,z_i)$.

Proof For convenient, we define

$$H_{j\alpha}:=\frac{\partial H}{\partial z_{j\alpha}},\quad H_{j\alpha,\overline{k\beta}}:=\frac{\partial^2 H}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}$$

for $1\le j,k\le m$. By a straightforward computation, the metric

$$g^{\Omega(\mu,m)}=\frac{1}{(H-|\xi|^2)^2}\left(\begin{array}{c|c} H_{j\alpha}H_{\overline{k\beta}}-H_{j\alpha,\overline{k\beta}}(H-|\xi|^2) & -H_{j\alpha}\xi \\ \hline -\overline{\xi}H_{\overline{k\beta}} & H \end{array}\right), \tag{3.5}$$

where the upper left block is a $d\times d$ submatrix. Under the elementary transformations, the matrix can be transformed into

$$\frac{1}{(H-|\xi|^2)^2}\left(\begin{array}{c|c} (H-|\xi|^2)H\frac{H_{j\alpha}H_{\overline{k\beta}}-H_{j\alpha,\overline{k\beta}}H}{H^2} & -H_{j\alpha}\xi \\ \hline 0 & H \end{array}\right), \tag{3.6}$$

and

$$\frac{H_{j\alpha}H_{\overline{k\beta}}-H_{j\alpha,\overline{k\beta}}H}{H^2}=\begin{cases}-\frac{\partial^2\log N_j(z_j,z_j)^{\mu_j}}{\partial z_{j\alpha}\partial\overline{z}_{j\beta}}, & j=k,\\ 0, & j\ne k.\end{cases} \tag{3.7}$$

It implies that the upper left block of (3.6) is a block diagonal matrix.

Let g^{D_i} be the Bergman metric of D_i, and $g^{D_i}_{\alpha\overline{\beta}}$ be the (α,β)-entry of $g^{D_i}_B$. Then

$$\frac{\mu_i}{\gamma_i}g^{D_i}_{\alpha\overline{\beta}}=-\frac{\partial^2\log N_i(z_i,z_i)^{\mu_i}}{\partial z_{i\alpha}\partial\overline{z}_{i\beta}}.$$

Hence, we have

$$\det(g^{\Omega(\mu,m)})=\frac{H^n}{(H-|\xi|^2)^{n+1}}\prod_{i=1}^{m}\Big(\frac{\mu_i}{\gamma_i}\Big)^{d_i}\det(g^{D_i}_{\alpha\overline{\beta}}).$$

From (2.2) and the equation above, it follows that

$$\det(g^{\Omega(\mu,m)})=\frac{1}{(H-|\xi|^2)^{n+1}}\prod_{i=1}^{m}u_i^{d_i}N_i(z_i,z_i)^{n\mu_i-\gamma_i}. \tag{3.8}$$

We complete the proof.

By using the standard formula of Ricci tensor, we can obtain the following lemma directly.

Lemma 3.2 *The* Ricci *tensor of* $g^{\Omega(\mu,m)}$ *is*

$$\mathrm{Ric}_g=\sum_{i=1}^{m}\Big(\frac{\mu_i n}{\gamma_i}-1\Big)g^{D_i}-(n+1)g^{\Omega(\mu,m)}.$$

Corollary 3.1 $g^{\Omega(\mu,m)}$ *is Kähler-Einstein if and only if* $\mu_i=\frac{\gamma_i}{n}$ *for* $1\le i\le m$.

Let

$$g_{\Omega(\mu,m)} = \left(\begin{array}{c|c} g^{j\alpha,\overline{k\beta}}_{\Omega(\mu,m)} & g^{j\alpha,\overline{n}}_{\Omega(\mu,m)} \\ \hline g^{n,\overline{k\beta}}_{\Omega(\mu,m)} & g^{n,\overline{n}}_{\Omega(\mu,m)} \end{array}\right) \tag{3.9}$$

be the inverse matrix of $g^{\Omega(\mu,m)}$, where $1 \leq j,k \leq m$ and $1 \leq \alpha \leq d_j$, $1 \leq \beta \leq d_k$. Let $g_{D_i} = (g^{\alpha\overline{\beta}}_{D_i})$ be the inverse matrix of the Bergman metric matrix $g^{D_i} = (g^{D_i}_{\alpha\overline{\beta}})$. By (2.4),

$$g^{j\alpha,\overline{k\beta}}_{\Omega(\mu,m)} = \begin{cases} \frac{H-|\xi|^2}{H}\frac{\gamma_j}{\mu_j} g^{\alpha\overline{\beta}}_{D_j}, & j = k, \\ 0, & j \neq k. \end{cases} \tag{3.10}$$

This equation shows that the upper left block of (3.9) is a block diagonal matrix.

Let s_g be the scalar curvature of $g^{\Omega(\mu,m)}$, and we have

$$\begin{aligned} s_g &= \sum_{i=1}^{m} \sum_{i\alpha, i\beta = i1}^{id_i} \frac{H-|\xi|^2}{H}\frac{\gamma_i}{\mu_i} g^{\beta\overline{\alpha}}_{D_i}\Big(\frac{\mu_i n}{\gamma_i} - 1\Big) g^{D_i}_{\alpha\overline{\beta}} - (n+1)n \\ &= \sum_{i=1}^{m} \Big(n - \frac{\gamma_i}{\mu_i}\Big) d_i \frac{H-|\xi|^2}{H} - (n+1)n. \end{aligned}$$

Lemma 3.3 *The scalar curvature of $g^{\Omega(\mu,m)}$ is*

$$s_g = \sum_{i=1}^{m} \Big(n - \frac{\gamma_i}{\mu_i}\Big) d_i \frac{(H-|\xi|^2)}{H} - (n+1)n.$$

Corollary 3.2 *s_g is a constant if and only if $\sum\limits_{i=1}^{m} \big(n - \frac{\gamma_i}{\mu_i}\big) d_i = 0$.*

Now we study the extremal condition. Let (M, g) be an n-dimensional Kähler manifold, and $(z_1, \cdots, z_n)$ be the local coordinate in a neighborhood of $p \in M$. From [3], the extremal condition can be given by the following equation:

$$\frac{\partial}{\partial \overline{z}_\eta} \sum_{\beta=1}^{n} g^{\beta\overline{\alpha}} \frac{\partial \kappa}{\partial \overline{z}_\beta} = 0 \tag{3.11}$$

for all $\alpha, \eta = 1, \cdots, n$. By using (3.11), we can obtain our main result in this section.

Theorem 3.1 *The metric $g^{\Omega(\mu,m)}$ is extremal if and only if its scalar curvature s_g is a constant, i.e., $\sum\limits_{i=1}^{m} \big(n - \frac{\gamma_i}{\mu_i}\big) d_i = 0$.*

Proof Let $\tau = \sum\limits_{i=1}^{m} \big(n - \frac{\gamma_i}{\mu_i}\big) d_i$. By Calabi's result, we only need to prove that $g^{\Omega(\mu,m)}$ is not an extremal metric if $\tau \neq 0$. By Lemma 3.3,

$$\frac{\partial s_g}{\partial \overline{z}_{k\beta}} = \frac{\tau|\xi|^2 H_{\overline{k\beta}}}{H^2} = -\frac{\tau\xi(H-|\xi|^2)^2}{H^2} g^{\Omega(\mu,m)}_{n,\overline{k\beta}}, \quad \frac{\partial s_g}{\partial \overline{\xi}} = -\frac{\tau\xi}{H} = -\frac{\tau\xi(H-|\xi|^2)^2}{H^2} g^{\Omega(\mu,m)}_{n,\overline{n}}.$$

Let $A = g^{\Omega(\mu,m)}$ and $A_{n,\overline{j\alpha}}$ be the algebraic cofactor of $g_{n,\overline{j\alpha}}$. Let B be the upper left $d \times d$ block matrix of A, i.e., $B = (g^{\Omega(\mu,m)}_{j\alpha,\overline{k\beta}})$.

$$g^{j\alpha,\overline{n}}_{\Omega(\mu,m)} = \frac{\det A_{n,\overline{j\alpha}}}{\det(g^{\Omega(\mu,m)})}, \quad g^{n,\overline{n}}_{\Omega(\mu,m)} = \frac{\det B}{\det(g^{\Omega(\mu,m)})}.$$

In our case, the extremal condition (3.11) turns out to be

$$\begin{aligned}
0 &= \sum_{k=1}^{m}\sum_{\beta=1}^{d_k} g^{k\beta,\overline{n}}_{\Omega(\mu,m)} \frac{\partial s_g}{\partial \overline{z}_{k\beta}} + g^{n,\overline{n}}_{\Omega(\mu,m)} \frac{\partial s_g}{\partial \overline{\xi}} \\
&= -\sum_{k=1}^{m}\sum_{\beta=1}^{d_k} \frac{\det A_{n,\overline{k\beta}}}{\det(g^{\Omega(\mu,m)})} \frac{\tau\xi(H-|\xi|^2)^2}{H^2} g^{\Omega(\mu,m)}_{n,\overline{k\beta}} - \frac{\det B}{\det(g^{\Omega(\mu,m)})} \frac{\tau\xi(H-|\xi|^2)^2}{H^2} g^{\Omega(\mu,m)}_{n,\overline{n}} \\
&= -\frac{\tau\xi(H-|\xi|^2)^2}{\det(g^{\Omega(\mu,m)})H^2} \Big(\sum_{k=1}^{m}\sum_{\beta=1}^{d_k} \det A_{n,\overline{k\beta}} g^{\Omega(\mu,m)}_{n,\overline{k\beta}} + \det B g^{\Omega(\mu,m)}_{n,\overline{n}} \Big) \\
&= -\frac{\tau(H-|\xi|^2)^2}{H^2}.
\end{aligned}$$

Thus we know that $g^{\Omega(\mu,m)}$ is not extremal if $\tau \neq 0$.

Remark 3.1 In particular, $\Omega(\mu, 1)$ is the Cartan-Hartogs domain over an irreducible bounded symmetric domain D. We can obtain Loi and Zedda's results in [29]. Their results can be summarized as follows. Let γ be the genus of D and n be the total dimension of $\Omega(\mu, 1)$. Then

(1) $g^{\Omega(\mu,1)}$ is an extremal metric;
(2) $g^{\Omega(\mu,1)}$ is a Kähler-Einstein metric;
(3) the scalar curvature of $g^{\Omega(\mu,1)}$ is a constant;
(4) the parameter μ equals $\frac{\gamma}{n}$

are equivalent.

4 The Equivalence of Four Classical Metrics

In [10], Deng, Guan and Zhang proved that Cartan-Hartogs domains are holomorphic homogeneous regular domains (i.e., they have the uniform squeezing property). As a generalization, we will show that their method is also valid for the generalized Cartan-Hartogs domain defined in (3.1).

In the following, we investigate the boundary $\partial(\Omega(\mu, m))$ of $\Omega(\mu, m)$. By (3.1), it is easy to see that

$$\partial(\Omega(\mu, m)) = (\partial D \times \{0\}) \cup \partial_0(\Omega(\mu, m)), \tag{4.1}$$

where $\partial D \times \{0\} = \{(z, 0) : z \in \partial D\}$ and $\partial_0(\Omega(\mu, m)) = \{(z, \xi) \in D \times \mathbb{C} : \rho(z, \xi) = 0, \xi \neq 0\}$. Here $\rho(z, \xi) = |\xi|^2 - \prod_{i=1}^{m} N_i(z_i, z_i)^{\mu_i}$. Now we claim that ρ is a local defining function of $\Omega(\mu, m)$ at the boundary point $\widetilde{p} = (\widetilde{z}, \widetilde{\xi}) \in \partial_0(\Omega(\mu, m))$. In fact, let $V(\widetilde{z}) \subset D$ be a neighborhood of $\widetilde{z}$, and $\triangle(\widetilde{\xi}, r)$ be a disc with radius $r < |\widetilde{\xi}|$. Then the neighborhood $U(\widetilde{p}) = V(\widetilde{z}) \times \triangle(\widetilde{\xi}, r)$ of $\widetilde{p}$ satisfies

$$U(\widetilde{p}) \cap \Omega(\mu, m) = \{(z, \xi) \in U(\widetilde{p}) : \rho(z, \xi) < 0\}$$

and $d\rho(z,\xi) \neq 0$ for $(z,\xi) \in U(\widetilde{p})$. So the claim is true. Moveover, ρ is smooth by (2.1).

In [25], we have constructed a holomorphic automorphism subgroup G of $\mathrm{Aut}(\Omega(\mu,m))$. The domain $\Omega(\mu,m)$ has the following characteristic: For any fixed point $p=(z,\xi)\in\Omega(\mu,m)$, there always exists $\Psi \in G$ such that $\Psi(z,\xi)=(0,\xi^*)$ for some real number $\xi^* \in [0,1)$. Let G act on $\Omega(\mu,m)$ through the action $G\times\Omega(\mu,m)\to\Omega(\mu,m)$. The orbit of a point $p\in\Omega(\mu,m)$ is $\mathcal{O}_p=\{\Psi(p)\in\Omega(\mu,m)\colon \Psi\in G\}$. Thus the set of all the orbits can be written in the following form $\big\{\mathcal{O}_{(0,x)} : x\in[0,1)\big\}$. By the holomorphic invariant property of the squeezing function, we have

$$\inf_{0\le x<1} s_\Omega(0,x) \le s_\Omega(z,\xi) \le 1 \tag{4.2}$$

for $(z,\xi)\in\Omega(\mu,m)$.

Theorem 4.1 *$\Omega(\mu,m)$ is a holomorphic homogeneous regular domain.*

Proof We have proved that ρ is a local defining function of the boundary of $\Omega(\mu,m)$ at the boundary point $(0,1)\in\partial_0(\Omega(\mu,m))$. In view of (2.7), the real Hessian

$$\mathrm{Hess}\rho(0,1)=\left(\begin{array}{c|c} 2\mu_i I^{2(d_i)} & 0 \\ \hline 0 & 2I^{(2)} \end{array}\right), \tag{4.3}$$

where the upper left block is a $2d\times 2d$ submatrix. Let $\xi=u+\mathrm{i}v$. Note that $\triangle X(0,1)=2\frac{\partial}{\partial u}\neq 0$, and then the tangent hyperplane $T_{(0,1)}\partial(\Omega(\mu,m))=\{u=1\}$. It is clear that $\Omega(\mu,m)\cap\{u=1\}=\{(0,1)\}$, so $(0,1)$ is a globally strongly convex boundary point of $\Omega(\mu,m)$. By Theorem 2.1, $\lim\limits_{(z,\xi)\to(0,1)} s_\Omega(z,\xi)=1$. Furthermore, $s(x)=s_\Omega(0,x)$ is continuous in $[0,1]$. By (4.2), s_Ω is bounded blow by a positive number and $\Omega(\mu,m)$ is a holomorphic homogeneous regular domain.

From [15, 28], we know that homogeneous regular domains have many interesting properties. One of the properties is the equivalence of four classical metrics. So we have the following result.

Theorem 4.2 *The Bergman metric, the Kähler-Einstein metric, the Carathéodary metric, and the Koboyashi metric on $\Omega(\mu,m)$ are equivalent.*

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