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The (∗∗)-Haagerup Property for C^*-Algebras*

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Abstract Extending the notion of Haagerup property for finite von Neumann algebras to the general von Neumann algebras, the authors define and study the (∗∗)-Haagerup property for C^*-algebras in this paper. They first give an answer to Suzuki's question (2013), and then obtain several results of (∗∗)-Haagerup property parallel to those of Haagerup property for C^*-algebras. It is proved that a nuclear unital C^*-algebra with a faithful tracial state always has the (∗∗)-Haagerup property. Some heredity results concerning the (∗∗)-Haagerup property are also proved.

Keywords C^*-algebras, Von Neumann algebras, Haagerup property
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1 Introduction

Recall that a discrete group Γ is said to have the Haagerup property if there is a net $\{\Phi_n\}$ of positive definite functions, each of which vanishes at infinity and the net converges to 1 pointwise. This definition is motivated by the work of Haagerup [8] where he proved that free groups have such a property. It is well known that if we replace the condition "vanishes at infinity" in the above definition by "has a finite support", then this is equivalent to amenability. Hence, the Haagerup property is considered as a weak version of amenability. The Haagerup property has been intensively studied in the literature. On the one hand, the Haagerup property is satisfied by many important non-amenable groups, including free groups, the Coxeter groups, and so on. It is also known that the Haagerup property is equivalent to several other important properties, such as the a-T-menabilty introduced by Gromov [7]. On the other hand, the groups with the Haagerup property do not have the (relative) property (T), which is a rigidity property of discrete groups. In many situations, a group with the (relative) property (T) is essentially hard to study, so at least for the group with the Haagerup property, essential difficulties would not arise. We refer to the book [3] for a comprehensive account of this subject.

A similar Haagerup property has been considered for finite von Neumann algebras in [4–5, 9, 11]. Suppose that $\mathcal{M}$ is a finite von Neumann algebra with a normal faithful tracial state τ. We say that $\mathcal{M}$ has the Haagerup property with respect to τ (shortly, $\mathcal{M}$ has the Haagerup property for von Neumann algebras) if there is a net $\{\Phi_i\}_{i\in I}$ of unital completely positive

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normal maps from $\mathcal{M}$ to itself such that (1) $\tau \circ \Phi_i \leq \tau$, (2) each Φ_i defines a compact operator $\widetilde{\Phi_i}$ on $L^2(\mathcal{M},\tau)$, and (3) $\|\Phi_i(A) - A\|_\tau = \tau((\Phi_i(A)-A)^*(\Phi_i(A)-A))^{\frac{1}{2}} \to 0$ for all $A \in \mathcal{M}$. We note that condition (1) in the definition guarantees that each Φ_i defines a bound linear map $\widetilde{\Phi_i}$ on $L^2(\mathcal{M},\tau)$. Jolissaint [9] proved that the definition does not depend on the choice of faithful normal tracial states. It was shown by Choda [4] that a discrete group Γ has the Haagerup property if and only if its group von Neumann algebra $L(\Gamma)$ has the Haagerup property for von Neumann algebras.

Recently, Dong [6] introduced a notion of the Haagerup property for unital C^*-algebras admitting a faithful tracial state, by imitating the case of von Neumann algebras. The definition of the Haagerup property for unital C^*-algebras is the same as above except for the assumption that τ is only a faithful tracial state and $\{\Phi_i\}_{i\in I}$ are only unital completely positive maps. Dong [6] showed that a discrete group Γ has the Haagerup property if and only if its reduced group C^*-algebra $C_r^*(\Gamma)$ has the Haagerup property with respect to the canonical tracial state. Suzukai [13] provided an example, which showed that the Haagerup property for C^*-algebras strictly depends on the choice of faithful tracial states. He also proved that a nuclear C^*-algebra with a faithful tracial state always has the Haagerup property for C^*-algebras.

Because the Haagerup property for C^*-algebras strictly depends on the choice of faithful tracial states, this paper is an attempt to find a hopefully better definition of the Haagerup property for C^*-algebras by looking at the Haagerup property for the biduals of C^*-algebras (such property is thus called the $(**)$-Haagerup property) such that such a definition does not depend on the choice of faithful tracial states. In this paper, we will show that a nuclear unital C^*-algebra with a faithful tracial state always has the $(**)$-Haagerup property. We also show, if Γ is an amenable countable discrete group and unital separable C^*-algebra $\mathcal{A}$ has the $(**)$-Haagerup property, then the reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the $(**)$-Haagerup property. Moreover, we will give an answer to the following open question (see [13, Question 6.2]):

Does the Haagerup property for C^*-algebras pass to a C^*-subalgebra? That is, let C^*-algebra $\mathcal{A}$ have the Haagerup property with respect to a faithful tracial state τ, and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. Then does $\mathcal{B}$ have the Haagerup property with respect to $\tau|_{\mathcal{B}}$?

2 The $(**)$-Haagerup Property for C^*-Algebras

Definition 2.1 *Let $\mathcal{A}$ be a unital C^*-algebra and τ be a faithful tracial state on $\mathcal{A}$. We say that $\mathcal{A}$ has the $(**)$-Haagerup property with respect to τ (shortly, $\mathcal{A}$ has the $(**)$-Haagerup property) if $\mathcal{A}^{**}$ has the Haagerup property for von Neumann algebras.*

Remark 2.1 (1) $\mathcal{A}^{**}$ is the bidual (the universal enveloping von Neumann algebra) of $\mathcal{A}$, and we refer the readers to the books [10] and [14].

(2) If τ is a faithful tracial state on $\mathcal{A}$, then the biduals map $\tau^{**} : \mathcal{A}^{**} \to \mathbb{C}^{**} = \mathbb{C}$ is a normal faithful tracial state on $\mathcal{A}^{**}$.

(3) Since the Haagerup property for von Neumann algebras does not depend on the choice of faithful normal tracial states, our definition of the $(**)$-Haagerup property for C^*-algebras does not depend on the choice of faithful tracial states.

Suzuki [13] gave an infinite-dimensional example of C^*-algebra which has both the Haagerup property with respect to some faithful tracial state and the property (T).

Example (see [13, Example 4.15]) Let $n \geq 3$, and on the group algebra $\mathbb{C}[\mathrm{SL}_n(\mathbb{Z})]$ of $\mathrm{SL}_n(\mathbb{Z})$, define the C^*-seminorm $\|x\|_{\mathrm{fin}}$ as follows:

$$\|x\|_{\mathrm{fin}} := \sup\{\|\pi(x)\| : \pi \text{ is a finite representation of } L_n(\mathbb{Z})\}.$$

Then define the C^*-algebra $C^*_{\mathrm{fin}}(\mathrm{SL}_n(\mathbb{Z}))$ as the completion of $\mathbb{C}[\mathrm{SL}_n(\mathbb{Z})]$ with respect to the norm $\|\cdot\|_{\mathrm{fin}}$. Suzuki [13, Theorem 4.18] proved that there exist two faithful tracial states τ_1 and τ_2 such that $C^*_{\mathrm{fin}}(\mathrm{SL}_n(\mathbb{Z}))$ has the Haagerup property with respect to τ_1 and $C^*_{\mathrm{fin}}(\mathrm{SL}_n(\mathbb{Z}))$ does not have the Haagerup property with respect to τ_2. Hence the Haagerup property for C^*-algebras does depend on the choice of faithful tracial states.

Remark 2.2 (1) If a unital C^*-algebra $\mathcal{A}$ has the Haagerup property with respect to faithful tracial state τ, then $\mathcal{A}$ has the $(**)$-Haagerup property. Indeed, if there exists a net of unital completely positive maps $\Phi_i : \mathcal{A} \to \mathcal{A}$ satisfying three conditions in the definition of the Haagerup property for C^*-algebras, then the biduals maps $\Phi_i^{**} : \mathcal{A}^{**} \to \mathcal{A}^{**}$ are unital completely positive normal maps, and it is easy to prove that Φ_i^{**} satisfy three conditions in the definition of the Haagerup property for von Neumann algebras. Hence $\mathcal{A}^{**}$ has the Haagerup property with respect to normal faithful tracial state τ^{**}, that is, $\mathcal{A}$ has the $(**)$-Haagerup property. But if $\mathcal{A}$ has the $(**)$-Haagerup property, then $\mathcal{A}$ does not have the Haagerup property with some faithful tracial state in general. Indeed, by [13, Theorem 4.18] there exist two faithful tracial states τ_1 and τ_2 such that $C^*_{\mathrm{fin}}(\mathrm{SL}_n(\mathbb{Z}))$ $(n \geq 3)$ has the Haagerup property with respect to τ_1 (hence $C^*_{\mathrm{fin}}(\mathrm{SL}_n(\mathbb{Z}))$ has the $(**)$-Haagerup property) and $C^*_{\mathrm{fin}}(\mathrm{SL}_n(\mathbb{Z}))$ does not have the Haagerup property with respect to τ_2.

(2) It is well known (see [9, Theorem 2.3]) that the Haagerup property for von Neumann algebras can pass to every von Neumann subalgebra. Hence, for von Neumann algebra $\mathcal{M}$, if $\mathcal{M}$ has the $(**)$-Haagerup property, then $\mathcal{M}$ also has Haagerup property for von Neumann algebras. By Remark 2.2(1), $\mathcal{M}$ has the $(**)$-Haagerup property if and only if $\mathcal{M}$ has the Haagerup property for von Neumann algebras.

(3) Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ. If $\mathcal{A}$ has the $(**)$-Haagerup property and there exists a τ-preserving conditional expectation E from $\mathcal{A}^{**}$ to $\mathcal{A}$, then $\mathcal{A}$ has the Haagerup property with respect to τ. Indeed, since $\mathcal{A}$ has the $(**)$-Haagerup property, there exists a net $\{\Phi_i\}_{i\in I}$ of unital completely positive normal maps on $\mathcal{A}^{**}$ satisfying three conditions in the definition of the Haagerup property for von Neumann algebras. Let $\Psi_i = E \circ \Phi_i|_{\mathcal{A}}$. Then it is easy to prove that $\{\Psi_i\}_{i\in I}$ is a net of unital completely positive maps on $\mathcal{A}$ satisfying three conditions in the definition of the Haagerup property for C^*-algebras. Hence $\mathcal{A}$ has the Haagerup property with respect to τ.

In 2013, Suzuki gave the following open question.

Question 2.1 (see [13, Question 6.2]) Does the Haagerup property for C^*-algebras pass to a C^*-subalgebra? That is, let C^*-algebra $\mathcal{A}$ have the Haagerup property with respect to a faithful tracial state τ, and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. Then does $\mathcal{B}$ have the Haagerup property with respect to $\tau|_{\mathcal{B}}$?

Note first that this is true if $\mathcal{A}$ is nuclear (see [13, Corollary 3.7]). Note also that Question 2.1 has a positive answer in the context of the von Neumann algebras. The reason why we can prove this for von Neumann algebras is that we can always construct a trace-preserving condition expectation (see [2, Lemma 1.5.11]). But in the context of the C^*-algebras and, we can not construct a condition expectation in general, even if we do not consider the condition about the trace. For example, let $\mathcal{A}$ be a nuclear C^*-algebra, and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ which is not nuclear. Then there is no condition expectation from $\mathcal{A}$ to $\mathcal{B}$, because any range of a conditional expectation on a nuclear C^*-algebra is nuclear.

Now we give an example, which shows that Question 2.1 does not have a postive answer in general. By [13, Theorem 4.18] there exist two faithful tracial states τ_1 and τ_2 such that $C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z}))$ has the Haagerup property with respect to τ_1 and $C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z}))$ does not have the Haagerup property with respect to τ_2. By Remark 2.2(1), we have that $(C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z})))^{**}$ has the Haagerup property with respect to τ_1^{**}. Since the Haagerup property for von Neumann algebras does not depend on the choice of faithful normal tracial states, so $(C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z})))^{**}$ has the Haagerup property with respect to τ_2^{**}. $C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z}))$ is a C^*-subalgebra of $(C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z})))^{**}$, but $C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z}))$ does not have the Haagerup property with respect to $\tau_2^{**}|_{C^*_{\text{fin}}(\text{SL}_n(\mathbb{Z}))} = \tau_2$.

3 Some Main Results

Theorem 3.1 *Let $\mathcal{A}$ be a unital nuclear C^*-algebra and τ be a faithful tracial state on $\mathcal{A}$. Then $\mathcal{A}$ has the $(**)$-Haagerup property.*

Proof Since the $(**)$-Haagerup property for C^*-algebras is weaker than the Haagerup property for C^*-algebras (see Remark 2.2(1)), by [13, Theorem 3.6] it is easy to get the theorem. Now we give a simple proof. It is mentioned in [9] that the injective finite von Neumann algebras have the Haagerup property for von Neumann algebras. Indeed, it is a deep and by now classical result that injective von Neumann algebras are semidiscrete. It then follows from [12, Proposition 4.6] that injective von Neumann algebras which admit a faithful normal trace state have the Haagerup property for von Neumann algebras. It is well known that $\mathcal{A}$ is nuclear if and only if A^{**} is injective (see [1, IV. 3.1.5]). The proof of the theorem is finished.

In a way similar to [13, Corollary 3.7], we also have the following corollary.

Corollary 3.1 (1) *Let $\mathcal{A}$ be a unital exact C^*-algebra with a faithful amenable tracial state. Then $\mathcal{A}$ has the $(**)$-Haagerup property.*

(2) *Let $\mathcal{A}$ be a unital residually finite-dimensional C^*-algebra with a faithful tracial state. Then $\mathcal{A}$ has the $(**)$-Haagerup property.*

Now we gather some heredity results concerning the $(**)$-Haagerup property.

Theorem 3.2 *Let $\mathcal{A}$ be a unital C^*-algebra and τ be a faithful tracial state on $\mathcal{A}$. Let $\mathcal{B}$ and $\mathcal{C}$ be unital C^*-algebras. Assume that $\mathcal{B}$ and $\mathcal{C}$ have the $(**)$-Haagerup property.*

(1) *If $1_{\mathcal{A}} \in \mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, and $\mathcal{A}$ has the $(**)$-Haagerup property, then $\mathcal{B}$ has the $(**)$-Haagerup property.*

(2) *Assume that there exists a sequence of unital C^*-algebras $\mathcal{A}_n$, each of which has the $(**)$-Haagerup property with respect to faithful tracial states τ_n, and assume that for every n there exist unital completely positive maps $S_n : \mathcal{A} \to \mathcal{A}_n$ and $T_n : \mathcal{A}_n \to \mathcal{A}$ such that $\tau_n \circ S_n \leq \tau$, $\tau \circ T_n \leq \tau_n$, and such that $\|T_n \circ S_n(A) - A\|_\tau \to 0$ as $n \to \infty$. Then $\mathcal{A}$ has the $(**)$-Haagerup property.*

(3) *If $\mathcal{A}$ has the $(**)$-Haagerup property, then for each $n \in \mathbf{N}$, $M_n(\mathcal{A})$ has the $(**)$-Haagerup property.*

(4) *The spatial tensor product $\mathcal{B} \otimes \mathcal{C}$ has the $(**)$-Haagerup property.*

(5) *The direct sum $\mathcal{B} \oplus \mathcal{C}$ has the $(**)$-Haagerup property.*

Proof (1) and (2) follow from [9, Theorem 2.3(i) and (ii)]. By [6, Lemma 2.4] and the fact $(M_n(\mathcal{A}))^{**} = M_n(\mathcal{A}^{**})$, we get (3). By [9, Theorem 2.3(iii)] and the fact $(\mathcal{B} \otimes \mathcal{C})^{**} = \mathcal{B}^{**} \overline{\otimes} \mathcal{C}^{**}$, where $\overline{\otimes}$ is the von Neumann algebra tensor product, we can prove (4). By [13, Theorem 3.12(1)] and the fact $(\mathcal{B} \oplus \mathcal{C})^{**} = \mathcal{B}^{**} \oplus \mathcal{C}^{**}$, we can prove (5).

Theorem 3.3 *Let Γ be an amenable countable discrete group and $\mathcal{A}$ be a unital separable C^*-algebra with a faithful tracial state τ. If $\mathcal{A}$ has the $(**)$-Haagerup property, then the reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the $(**)$-Haagerup property, where α is a τ-preserving action of Γ.*

Proof Since Γ is amenable, there exists a sequence of finite sets $F_n \subseteq \Gamma$ such that

$$\max_{s \in E} \frac{|sF_n \triangle F_n|}{|F_n|} \to 0$$

for all finite sets $E \subseteq \Gamma$. We define $\varphi_n : \mathcal{A} \rtimes_{\alpha,r} \Gamma \to \mathcal{A} \otimes M_{F_n}(\mathbb{C})$ such that

$$\varphi_n(a\lambda_\Gamma(s)) = \sum_{p \in F_n \cap sF_n} \alpha_p^{-1}(a) \otimes e_{p,s^{-1}p}$$

and $\psi_n : \mathcal{A} \otimes M_{F_n}(\mathbb{C}) \to \mathcal{A} \rtimes_{\alpha,r} \Gamma$ such that

$$\psi_n(a \otimes e_{p,q}) = \frac{1}{|F_n|} \alpha_p(a) \lambda_\Gamma(pq^{-1}).$$

Dong [6, Theorem 2.5] showed that φ_n, ψ_n are unital completely positive maps such that $\tau_n \circ \varphi_n \leq \tau'$, $\tau' \circ \psi_n \leq \tau_n$, where τ_n and τ' are the induced traces of τ on $\mathcal{A} \otimes M_{F_n}(\mathbb{C})$ and $\mathcal{A} \rtimes_{\alpha,r} \Gamma$, respectively, and such that $\|\psi_n \circ \varphi_n(x) - x\|_{\tau'} \to 0$ for all $x \in \mathcal{A} \rtimes_{\alpha,r} \Gamma$. Since $\mathcal{A}$ has the $(**)$-Haagerup property, it follows from Theorem 3.2(3) that $\mathcal{A} \otimes M_{F_n}(\mathbb{C})$ has the $(**)$-Haagerup property. By Theorem 3.2(2), we get that $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the $(**)$-Haagerup property.

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