Chin. Ann. Math.
37B(3), 2016, 373–384
DOI: 10.1007/s11401-016-0966-4

Chinese Annals of Mathematics, Series B

A Riemann-Hilbert Approach to the Harry-Dym Equation on the Line*

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Abstract In this paper, the authors consider the Harry-Dym equation on the line with decaying initial value. They construct the solution of the Harry-Dym equation via the solution of a 2×2 matrix Riemann-Hilbert problem in the complex plane. Further, one-cusp soliton solution is expressed in terms of the Riemann-Hilbert problem.

Keywords Harry-Dym equation, Riemann-Hilbert problem, Initial-value problem, One-cusp soliton solution
2000 MR Subject Classification 17B40, 17B50

1 Introduction

The following nonlinear partial differential equation

$$q_t - 2\Big(\frac{1}{\sqrt{1+q}}\Big)_{xxx} = 0 \tag{1.1}$$

is known as the Harry-Dym equation (see [1]). This equation was obtained by Harry Dym and Martin Kruskal as an evolution equation solvable by a spectral problem based on the string equation instead of the Schrödinger equation. The Harry-Dym equation plays an important role in the study of the Saffman-Taylor problem which describes the motion of a two-dimensional interface between a viscous and a nonviscous fluid (see [2]). The Harry-Dym equation shares many of the properties typical of the soliton equations. It is a completely integrable equation which can be solved by the inverse scattering transform (see [3]). It has a bi-Hamiltonian structure (see [4]), an infinite number of conservation laws and infinitely many symmetries (see [5]), and has reciprocal Backlund transformations to the KdV equation (see [6]). The Harry-Dym equation has been solved by different methods such as the inversing scattering method (see [3]), the Bäcklund transformation technique (see [7]), and the straightforward method (see [8]). Especially, Wadati obtained the one-cusp soliton solution (see [3])

$$\begin{aligned} &q(x,t) = \tan h^{-4}(\kappa x - 4\kappa^3 t + \kappa x_0 + \varepsilon_+) - 1, \\ &\varepsilon_+ = \frac{1}{\kappa}[1 + \tan h(\kappa x - 4\kappa^3 t + \kappa x_0 + \varepsilon_+)] \end{aligned}$$

by using inverse scattering transformation.

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*This work was supported by the National Natural Science Foundation of China (No. 11271079).

The main aim of this paper is to develop the inversing scattering method, based on a Riemann-Hilbert problem for solving nonlinear integrable systems, which has further developed and applied many equations with initial value problems on the line (see [9–11]) and initial boundary value problems on the half line (see [12–17]). In this paper, we consider the initial value problem of the Harry-Dym equation

$$\begin{aligned} &q_t - 2\Big(\frac{1}{\sqrt{1+q}}\Big)_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ &q(x,0) = q_0(x), \end{aligned} \tag{1.2}$$

where the $q_0(x)$ is a smoothly real-valued function and decays as $|x| \to \infty$. The organization of the paper is as follows. In the following Section 2, we perform the spectral analysis of the associated Lax pair for the Harry-Dym equation. In Section 3, we formulate the main Riemann-Hilbert problem associated with the initial value problem (1.2). In Section 4, we obtain the one-cusp soliton solution in terms of the Riemann-Hilbert problem, which has a similar, but not the same, form constructed by the inverse scattering method (see [3]).

2 Spectral Analysis

2.1 A Lax pair

In general, the matrix Riemann-Hilbert problem is defined in the λ plane and has explicit (x,t) dependence, while for the Harry-Dym equation (1.2), we need to construct a new matrix Riemann-Hilbert problem with explicit (y,t) dependence, where $y(x,t)$ is a function unknown from the initial value condition. For this purpose, we make a transformation

$$\rho = \sqrt{1+q},$$

and (1.2) can be expressed by

$$(\rho^2)_t - 2\Big(\frac{1}{\rho}\Big)_{xxx} = 0.$$

Then the initial value problem (1.2) is transformed into

$$\begin{aligned} &(\rho^2)_t - 2\Big(\frac{1}{\rho}\Big)_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ &\rho(x,0) = \rho_0(x) = \sqrt{1+q_0(x)}, \\ &\rho_0(x) \to 1, \quad |x| \to \infty. \end{aligned} \tag{2.1}$$

It was shown that (1.2) admits the following Lax pair (see [3]):

$$\begin{cases} \psi_{xx} = -\lambda^2(1+q)\psi, \\ \psi_t = 2\lambda^2\Big[\dfrac{2}{\sqrt{1+q}}\psi_x - \Big(\dfrac{1}{\sqrt{1+q}}\Big)_x\psi\Big]. \end{cases} \tag{2.2}$$

Making a transformation

$$\rho = \sqrt{1+q}, \quad \varphi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix},$$

then the Lax pair (2.2) can be written in the matrix form

$$\begin{cases}\varphi_x = M\varphi,\\ \varphi_t = N\varphi,\end{cases} \tag{2.3}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ -\lambda^2\rho^2 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -2\lambda^2\Big(\dfrac{1}{\rho}\Big)_x & 4\lambda^2\dfrac{1}{\rho} \\ -4\lambda^4\rho - 2\lambda^2\Big(\dfrac{1}{\rho}\Big)_{xx} & 2\lambda^2\Big(\dfrac{1}{\rho}\Big)_x \end{pmatrix}.$$

Further, by the gauge transformations

$$\phi = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda\rho} & 0 \\ 0 & \dfrac{1}{\sqrt{\lambda\rho}} \end{pmatrix} \varphi,$$

we have

$$\begin{cases}\phi_x + \mathrm{i}\lambda\rho\sigma_3\phi = U\phi,\\ \phi_t + \mathrm{i}\Big(\lambda\dfrac{1}{\rho}\Big(\dfrac{1}{\rho}\Big)_{xx} + 4\lambda^3\Big)\sigma_3\phi = V\phi,\end{cases} \tag{2.4}$$

where

$$U(x,t) = \frac{1}{2}\frac{\rho_x}{\rho}\sigma_2, \quad V(x,t,\lambda) = -\lambda\frac{1}{\rho}\Big(\frac{1}{\rho}\Big)_{xx}\sigma_1 - 2\lambda^2\Big(\frac{1}{\rho}\Big)_x\sigma_2.$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is clear that as $|x| \to \infty$, $U(x,t) \to 0$ and $V(x,t,\lambda) \to 0$. We define a real-valued function $y(x,t)$ by

$$y(x,t) = x + \int_x^\infty (1 - \rho(\xi,t))\mathrm{d}\xi.$$

It is obvious that

$$y_x = \rho(x,t), \quad y_t = -\int_x^\infty \rho_t(\xi,t)\mathrm{d}\xi.$$

The conservation law

$$\rho_t - \Big(-\frac{1}{2}\Big(\Big(\frac{1}{\rho}\Big)_x\Big)^2 + \frac{1}{\rho}\Big(\frac{1}{\rho}\Big)_{xx}\Big)_x = 0$$

implies that

$$y_t = -\frac{1}{2}\Big(\Big(\frac{1}{\rho}\Big)_x\Big)^2 + \frac{1}{\rho}\Big(\frac{1}{\rho}\Big)_{xx}.$$

Extending the column vector ϕ to be a 2×2 matrix and letting

$$\mu = \phi\exp(\mathrm{i}\lambda y(x,t)\sigma_3 + 4\mathrm{i}\lambda^3 t\sigma_3),$$

then μ solves

$$\begin{cases}\mu_x + \mathrm{i}\lambda y_x[\sigma_3, \mu] = \widetilde{U}\mu,\\ \mu_t + \mathrm{i}(\lambda y_t + 4\lambda^3)[\sigma_3, \mu] = \widetilde{V}\mu,\end{cases} \tag{2.5}$$

which can be written in the full derivative form

$$\mathrm{d}(\mathrm{e}^{\mathrm{i}(y(x,t)x+4\lambda^3 t)\widehat{\sigma_3}}\mu) = \mathrm{e}^{\mathrm{i}(y(x,t)x+4\lambda^3 t)\widehat{\sigma_3}}(\widetilde{U}\mathrm{d}x + \widetilde{V}\mathrm{d}t)\mu,$$

where

$$\begin{aligned}
&\widetilde{U} = U,\\
&\widetilde{V} = -\frac{1}{2}i\lambda\Big(\Big(\frac{1}{\rho}\Big)_x\Big)^2\sigma_3 - \lambda\frac{1}{\rho}\Big(\frac{1}{\rho}\Big)_{xx}\sigma_1 - 2\lambda^2\Big(\frac{1}{\rho}\Big)_x\sigma_2,
\end{aligned}$$

and $[\sigma_3, \mu] = \sigma_3\mu - \mu\sigma_3$. As $|x| \to \infty$, $\widetilde{V} \to 0$. The lax pair in (2.5) is very convenient for dedicated solutions via the integral Volterra equation, which is also what we study in the following paper.

Remark 2.1 By the representation of M, N and U, V in (2.3) and (2.4) respectively, we find that ψ_x, ψ_t and ϕ_x, ϕ_t have no singularity in $\lambda = 0$. Therefore, ϕ has no real singularity in $\lambda = 0$.

2.2 Eigenfunctions

We define two eigenfunctions $\mu_\pm$ of (2.5) as the solutions of the following two Volterra integral equations in the (x, t) plane:

$$\begin{aligned}
\mu(x,t,\lambda) = I + \int_{(x^*,t^*)}^{(x,t)} &\mathrm{e}^{-[\mathrm{i}\lambda(y(x,t)-y(x',t))+4\mathrm{i}\lambda^3(t-\tau)]\widehat{\sigma_3}}(\widetilde{U}(x',t)\mu(x',t,\lambda)\mathrm{d}x'\\
&+ \widetilde{V}(x',\tau,\lambda)\mu(x',\tau,\lambda))\mathrm{d}\tau,
\end{aligned} \tag{2.6}$$

where I is a 2×2 identity matrix, and $\widehat{\sigma_3}$ acts on a 2×2 matrix A by $\widehat{\sigma_3}A = \sigma_3 A\sigma_3$. Since the integrated expression is independent of the path of integration, we choose the particular initial points of integration to be parallel to the x-axis and obtain that for μ_+ and μ_-,

$$\begin{aligned}
\mu_+(x,t,\lambda) &= I - \int_x^\infty \mathrm{e}^{-\mathrm{i}\lambda(y(x,t)-y(x',t))\widehat{\sigma_3}}\widetilde{U}(x',t)\mu_+(x',t,\lambda)\mathrm{d}x',\\
\mu_-(x,t,\lambda) &= I + \int_{-\infty}^x \mathrm{e}^{-\mathrm{i}\lambda(y(x,t)-y(x',t))\widehat{\sigma_3}}\widetilde{U}(x',t)\mu_-(x',t,\lambda)\mathrm{d}x'.
\end{aligned} \tag{2.7}$$

Define the following sets:

$$\begin{aligned}
D_1 &= \{\lambda \in C \mid \mathrm{Im}\lambda > 0\},\\
D_2 &= \{\lambda \in C \mid \mathrm{Im}\lambda < 0\}.
\end{aligned}$$

Since for any fixed t, $y_x = \rho(x,t) > 0$, $y(x,t)$ is an increasing function of x for fixed t. As $x - x' < 0$, $y(x,t) - y(x',t) < 0$; as $x - x' > 0$, $y(x,t) - y(x',t) > 0$. We can deduce that the second column vectors of μ_+ and μ_- are bounded and analytic for $\lambda \in C$ provided that λ belongs to D_1 and D_2, respectively. We denote these vectors with superscripts (1), (2) to indicate the domains of their boundedness. Then

$$\mu_+ = (\mu_+^{(2)}, \mu_+^{(1)}), \quad \mu_- = (\mu_-^{(1)}, \mu_-^{(2)}).$$

For any x and t, the following conditions are satisfied:

$$(\mu_-^{(1)}, \mu_+^{(1)}) = I + O\Big(\frac{1}{\lambda}\Big), \quad \lambda \to \infty,\ \lambda \in D_1,$$
$$(\mu_+^{(2)}, \mu_-^{(2)}) = I + O\Big(\frac{1}{\lambda}\Big), \quad \lambda \to \infty,\ \lambda \in D_2,$$
$$\mu_\pm = I + O\Big(\frac{1}{\lambda}\Big), \quad \lambda \to \infty.$$

2.3 Spectral functions

For $\lambda \in \mathbb{R}$, the eigenfunctions μ_+, μ_- being the solution of the system of differential equations (2.5) are related by a matrix independent of (x,t). We define the spectral function by

$$\mu_+(x,t,\lambda) = \mu_-(x,t,\lambda)\mathrm{e}^{-\mathrm{i}(\lambda y(x,t)+4\lambda^3 t)\widehat{\sigma_3}} s(\lambda). \tag{2.8}$$

From (2.5), we get

$$\det(\mu_\pm(x,t,\lambda)) = 1. \tag{2.9}$$

Since $\overline{\widetilde{U}}(x,t) = -\widetilde{U}(x,t)$, the $\mu_\pm(x,t,\lambda)$ have the relations:

$$\begin{cases} \mu_{\pm 11}(x,t,\lambda) = \overline{\mu_{\pm 22}(x,t,\overline{\lambda})}, \quad \mu_{\pm 21}(x,t,\lambda) = \overline{\mu_{\pm 12}(x,t,\overline{\lambda})}, \\ \mu_{\pm 11}(x,t,-\lambda) = \mu_{\pm 22}(x,t,\lambda), \quad \mu_{\pm 12}(x,t,-\lambda) = \mu_{\pm 21}(x,t,\lambda). \end{cases} \tag{2.10}$$

The spectral function $s(\lambda)$ can be written as

$$s(\lambda) = \begin{pmatrix} \overline{a(\overline{\lambda})} & b(\lambda) \\ \overline{b(\overline{\lambda})} & a(\lambda) \end{pmatrix}, \tag{2.11}$$

$$s(\lambda) = I - \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda y(x',0)\widehat{\sigma_3}} \widetilde{U}(x',0)\mu_+(x',0,\lambda)\mathrm{d}x', \quad \mathrm{Im}\lambda = 0. \tag{2.12}$$

From the (2.9), $\det(s(\lambda)) = 1$. Equations (2.8)–(2.9) imply that $a(\lambda)$ and $b(\lambda)$ have the following properties:

(1) $a(\lambda)$ is analytic in D_1 and continuous for $\lambda \in \overline{D}_1$.
(2) $b(\lambda)$ is continuous for $\lambda \in \mathbb{R}$.
(3) $a(\lambda)\overline{a(\overline{\lambda})} - b(\lambda)\overline{b(\overline{\lambda})} = 1,\ \lambda \in \mathbb{R}$.
(4) $a(\lambda) = 1 + O\big(\frac{1}{\lambda}\big),\ \lambda \to \infty,\ \lambda \in D_1$.
(5) $b(\lambda) = O\big(\frac{1}{\lambda}\big),\ \lambda \to \infty,\ \lambda \in \mathbb{R}$.

2.4 Residue conditions

We assume that $a(\lambda)$ has N simple zeros $\{\lambda_j\}_{j=1}^N$ in the upper half plane. These eigenvalues are purely imaginary. The second column of (2.8) is

$$\mu_+^{(1)} = b(\lambda)\mu_-^{(1)}\mathrm{e}^{-2\mathrm{i}(\lambda y(x,t)+4\lambda^3 t)} + \mu_-^{(2)} a(\lambda). \tag{2.13}$$

For (2.9) and (2.13), it yields

$$a(\lambda) = \det(\mu_-^{(1)}, \mu_+^{(1)}),$$

where we have used that both sides are well defined and analytic in D_1 to extend the above relation to $\overline{D}_1$. Hence, if $a(\lambda_j)=0$, the $\mu_-^{(1)},\mu_+^{(1)}$ are linearly dependent vectors for each x and t, i.e., there exist constants $b_j\neq 0$ such that

$$\mu_-^{(1)}=b_j\mathrm{e}^{2\mathrm{i}(\lambda_j y(x,t)+4\lambda_j^3 t)}\mu_+^{(1)},\quad x\in\mathbb{R},\ t>0.$$

Recalling the symmetries in (2.10), we find

$$\mu_-^{(2)}=\overline{b}_j\mathrm{e}^{-2\mathrm{i}(\overline{\lambda}_j y(x,t)+4\overline{\lambda}_j^3 t)}\mu_+^{(2)},\quad x\in\mathbb{R},\ t>0.$$

Consequently, the residues of $\frac{\mu_-^{(1)}}{a}$ and $\frac{\mu_-^{(2)}}{a(\overline{\lambda})}$ at λ_j and $\overline{\lambda}_j$ are

$$\begin{aligned}
&\operatorname*{Res}_{\lambda=\lambda_j}\frac{\mu_-^{(1)}(x,t,\lambda)}{a(\lambda)}=C_j\mathrm{e}^{2\mathrm{i}(\lambda_j y(x,t)+4\lambda_j^3 t)}\mu_+^{(2)}(x,t,\lambda_j),\quad j=1,\cdots,N,\\
&\operatorname*{Res}_{k=\overline{\lambda}_j}\frac{\mu_-^{(2)}(x,t,\lambda)}{\overline{a(\overline{\lambda})}}=\overline{C}_j\mathrm{e}^{-2\mathrm{i}(\overline{\lambda_j} y(x,t)+4\overline{\lambda}_j^3 t)}\mu_+^{(1)}(x,t,\overline{\lambda}_j),\quad j=1,\cdots,N,
\end{aligned}$$

where $C_j=\frac{b_j}{\dot{a}(k_j)}$, $\dot{a}(k)=\frac{\mathrm{d}a}{\mathrm{d}k}$.

Remark 2.2 There is the relation of $\mu_\pm$ that the $s(\lambda)$ is the scattering matrix for the one-dimensional Schödinger equation:

$$W_{yy}+\lambda^2 W=f(y)W$$

via the Liouville transformation:

$$\begin{aligned}
&y=x+\int_x^\infty(1-\rho(\xi,0))\mathrm{d}\xi,\quad W(y,\lambda)=\psi(y,\lambda)\rho_0(y),\\
&\rho_0(y)=\rho_0(x),\quad f(y)=\frac{1}{2}(\rho_{0yy}\rho_0^{-1}-\frac{1}{2}\rho_{0y}^2\rho_0^{-2}).
\end{aligned}$$

Therefore, in terms of the spectral problem of the Schrödinger equation, we deduce that $a(\lambda)$ has only pure imaginary part of simple poles in the upper plane.

3 The Riemann-Hilbert Problem

3.1 A Riemann-Hilbert problem for (x,t)

We now solve the initial value problem for (2.1) on the line, and the solution can be expressed in terms of a 2×2 matrix Riemann-Hilbert problem. Let $M(x,t,\lambda)$ be defined by

$$M_+=\Big(\frac{\mu_-^{(1)}}{a(\lambda)},\mu_+^{(1)}\Big),\quad \lambda\in D_1;\quad M_-=\Big(\mu_+^{(2)},\frac{\mu_-^{(2)}}{\overline{a(\overline{\lambda})}}\Big),\quad \lambda\in D_2,\tag{3.1}$$

and let the M satisfy the jump condition:

$$M_+(x,t,\lambda)=M_-(x,t,\lambda)J(x,t,\lambda),\quad \mathrm{Im}\lambda=0,$$

where

$$J(x,t,\lambda)=\begin{pmatrix}\dfrac{1}{a(\lambda)\overline{a(\bar\lambda)}} & \dfrac{b(\lambda)}{\overline{a(\bar\lambda)}}\mathrm{e}^{-2\mathrm{i}(\lambda y(x,t)+4\lambda^3t)}\\ -\dfrac{\overline{b(\bar\lambda)}}{a(\lambda)}\mathrm{e}^{2\mathrm{i}(\lambda y(x,t)+4\lambda^3t)} & 1\end{pmatrix},\quad \mathrm{Im}\lambda=0. \tag{3.2}$$

These definitions imply

$$\det M(x,t,\lambda)=1 \tag{3.3}$$

and

$$M(x,t,\lambda)=I+O\Big(\frac{1}{\lambda}\Big),\quad \lambda\to\infty. \tag{3.4}$$

This contour of the Riemann-Hilbert problem is the real axis.

The jump matrix $J(x,t,\lambda)$, and the spectral $a(\lambda)$ and $b(\lambda)$ are dependent on the $y(x,t)$, while $y(x,t)$ doesn't involve initial data. Therefore, this Riemann-Hilbert problem can not be formulated in terms of initial data alone. In order to overcome this problem, we will reconstruct a new jump matrix by changing

$$(x,t)\to(y,t),\quad y=y(x,t),$$

where y is a new scale. Then we can transform this Riemann-Hilbert problem into the Riemann-Hilbert problem parametrized by (y,t).

3.2 A Riemann-Hilbert problem for (y,t)

Theorem 3.1 *Let $q_0(x)$, $x\in\mathbb{R}$ be a smooth function and decay as $|x|\to\infty$. Moreover $1+q_0(x)>0$. Define the $\widetilde{U}_0,\rho_0$ and $y_0(x)$ as follows*:

$$\widetilde{U}_0(x)=\frac{1}{2}\frac{\rho_{0x}(x)}{\rho_0(x)}\sigma_2,\quad \rho_0(x)=\sqrt{1+q_0(x)},$$
$$y_0(x)=x+\int_x^{\infty}(1-\rho_0(\xi))\mathrm{d}\xi.$$

Let $\mu_+(x,0,\lambda)$ and $\mu_-(x,0,\lambda)$ be the unique solution of the Volterra linear integral equation (2.5) evaluated at $t=0$ with $\widetilde{U}_0(x,0)=\widetilde{U}_0(x)$, $\rho_0(x)=\rho(x,0)$ and $y_0(x)=y(x,0)$. Define $a(\lambda)$, $b(\lambda)$, C_j by

$$\begin{pmatrix}b(\lambda)\\ a(\lambda)\end{pmatrix}=[s(\lambda)]_2,\quad s(\lambda)=I-\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\lambda y_0(x')\widehat{\sigma_3}}\widetilde{U}_0(x')\mu_+(x',0,\lambda)\mathrm{d}x',\quad \mathrm{Im}\lambda=0 \tag{3.5}$$

and

$$[\mu_-(x,0,\lambda_j)]_1=\dot{a}(\lambda_j)C_j\mathrm{e}^{2\mathrm{i}\lambda_j y_0(x)}[\mu_+(x,0,\lambda_j)]_2,\quad j=1,\cdots,N, \tag{3.6}$$

here and here after $([A]_1\ [A]_2)$ denotes the first (second) column of a 2×2 matrix A. We assume that $a(\lambda)$ has N simple zeros $\{\lambda_j\}_{j=1}^N$ in the upper half plane and is pure imaginary. Then

(1) *$a(\lambda)$ is defined for $k\in\overline{D}_1$ and analytic in D_1.*

(2) $b(\lambda)$ *is defined for* $\lambda\in\mathbb{R}$.

(3) $a(\lambda)\overline{a(\overline{\lambda})}-b(\lambda)\overline{b(\overline{\lambda})}=1,\ \lambda\in\mathbb{R}$.

(4) $a(\lambda)=1+O\big(\frac{1}{\lambda}\big),\ \lambda\to\infty,\ \lambda\in D_1$.

(5) $b(\lambda)=O\big(\frac{1}{\lambda}\big),\ \lambda\to\infty,\ \lambda\in\mathbb{R}$.

Suppose that there exists a uniquely solution $q(x,t)$ *of* (1.2) *with initial data* $q_0(x)$ *such that* $\rho_0(x)=\sqrt{1+q_0(x)}$ *has sufficient smoothness and decays for* $t>0$. *Then* $q(x,t)$ *is given in the parametric form by*

$$q(x(y,t),t)=\mathrm{e}^{8\int_y^{+\infty}m(y',t)\mathrm{d}y'}-1 \tag{3.7}$$

and the function $x(y,t)$ *is defined by*

$$x(y,t)=y+\int_{-\infty}^{y}(\mathrm{e}^{-4\int_{y'}^{\infty}m(\xi,t)\mathrm{d}\xi}-1)\mathrm{d}y', \tag{3.8}$$

where $m(y,t)=-\mathrm{i}\lim\limits_{\lambda\to\infty}(\lambda M(y,t,\lambda))_{12}$, *and* $M(y,t,\lambda)$ *is the unique solution of the following Riemann-Hilbert problem*:

(1)

$$M(y,t,\lambda)=\begin{cases}M_-(y,t,\lambda), & \lambda\in D_2,\\ M_+(y,t,\lambda), & \lambda\in D_1\end{cases}$$

is a sectionally meromorphic function.

(2)

$$M_+(y,t,\lambda)=M_-(y,t,\lambda)J^{(y)}(y,t,\lambda),\quad \mathrm{Im}\lambda=0,$$

where $J^{(y)}(y,t,\lambda)$ *is defined by*

$$J^{(y)}(y,t,\lambda)=\begin{pmatrix}\dfrac{1}{a(\lambda)\overline{\overline{a(\overline{\lambda})}}} & \dfrac{b(\lambda)}{\overline{\overline{a(\overline{\lambda})}}}\mathrm{e}^{-2\mathrm{i}(\lambda y+4\lambda^3t)}\\ -\dfrac{\overline{b(\overline{\lambda})}}{a(\lambda)}\mathrm{e}^{2\mathrm{i}(\lambda y+4\lambda^3t)} & 1\end{pmatrix},\quad \mathrm{Im}\lambda=0. \tag{3.9}$$

(3)

$$M(y,t,\lambda)=I+O\Big(\frac{1}{\lambda}\Big),\quad \lambda\to\infty. \tag{3.10}$$

(4) *The possible simple poles of the first column of* $M_+(y,t,\lambda)$ *occur at* $\lambda=\lambda_j$, $j=1,\cdots,N$, *and the possible simple poles of the second column of* $M_-(y,t,\lambda)$ *occur at* $\lambda=\overline{\lambda}_j$, $j=1,\cdots,N$. *The associated residues are given by*

$$\operatorname*{Res}_{\lambda=\lambda_j}[M(y,t,\lambda)]_1=C_j\mathrm{e}^{2\mathrm{i}(\lambda_jy+4\lambda_j^3t)}[M(y,t,\lambda_j)]_2,\quad j=1,\cdots,N, \tag{3.11}$$

$$\operatorname*{Res}_{\lambda=\overline{\lambda}_j}[M(y,t,\lambda)_2=\overline{C}_j\mathrm{e}^{-2\mathrm{i}(\overline{\lambda}_jy+4\overline{\lambda}_j^3t)}[M(y,t,\overline{\lambda}_j)]_1,\quad j=1,\cdots,N. \tag{3.12}$$

Proof Assume that $\mu(x,t)$ is the solution of equation (2.5), and its asymptotic expansion is

$$\mu(x,t,\lambda)=I+\frac{\mu^{(1)}(x,t)}{\lambda}+\frac{\mu^{(2)}(x,t)}{\lambda^2}+\frac{\mu^{(3)}(x,t)}{\lambda^3}+O\Big(\frac{1}{\lambda^4}\Big),\quad \lambda\to\infty$$

into the x-part of (2.5), where $\mu^{(1)}(x,t)$, $\mu^{(2)}(x,t)$ and $\mu^{(3)}(x,t)$ are 2×2 matrices, dependent on x,t. By considering the terms of $O(1)$, We get

$$4\mu_{12}^{(1)}(x,t) = -\frac{\rho_x(x,t)}{\rho^2(x,t)}. \tag{3.13}$$

By construction of the new Riemann-Hilbert problem about (y,t,λ), we can deduce that

$$\mu_{12}^{(1)}(x,t) = -\mathrm{i}\lim_{\lambda\to\infty}(\lambda M(y,t,\lambda))_{12} = m(y,t). \tag{3.14}$$

Then

$$-\frac{1}{4}\frac{\rho_x(x,t)}{\rho^2(x,t)} = m(y,t). \tag{3.15}$$

(3.13) can be expressed in terms of $y=y(x,t)$. Indeed, using $\frac{\mathrm{d}y}{\mathrm{d}x}=\rho$, then (3.15) becomes

$$-\frac{1}{4}\frac{\rho_y}{\rho} = m(y,t). \tag{3.16}$$

As $|y|\to\infty$, $\rho(y,t)\to 1$, by the evaluation of (3.16), we get

$$\rho(y,t) = \mathrm{e}^{4\int_y^{+\infty} m(y',t)\mathrm{d}y'}.$$

Therefore

$$q(x,t) = \mathrm{e}^{8\int_y^{+\infty} m(y',t)\mathrm{d}y'} - 1.$$

As $|x|\to\infty$, $|y|\to\infty$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\rho>0$,

$$x = y + \int_{-\infty}^{y}(\mathrm{e}^{-4\int_{y'}^{+\infty} m(\xi,t)\mathrm{d}\xi} - 1)\mathrm{d}y'.$$

Remark 3.1 It follows from the symmetries (2.10) that the solution $M(y,t,\lambda)$ of the Riemann-Hilbert problem in Theorem 3.1 has the symmetries:

$$\begin{cases} M_{11}(y,t,\lambda) = \overline{M_{22}(y,t,\overline{\lambda})}, & M_{21}(y,t,\lambda) = \overline{M_{12}(y,t,\overline{\lambda})}, \\ M_{11}(y,t,-\lambda) = M_{22}(y,t,\lambda), & M_{12}(y,t,-\lambda) = M_{21}(y,t,\lambda). \end{cases} \tag{3.17}$$

4 Soliton Solution

The solitons correspond to the spectral data $\{a(\lambda), b(\lambda), C_j\}$ for which $b(\lambda)$ vanishes identically. In this case, the jump matrix $J^{(y)}(y,t,\lambda)$ in the (3.9) is the identity matrix and the Riemann-Hilbert problem of Theorem 3.1 consists of finding a meromorphic function $M(y,t,\lambda)$ satisfying (3.10) and the residue conditions (3.11)–(3.12). From (3.10)–(3.11), we get

$$[M(y,t,\lambda)]_1 = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{N}\frac{C_j}{\lambda-\lambda_j}\mathrm{e}^{2\mathrm{i}(\lambda_j y + 4\lambda_j^3 t)}[M(y,t,\lambda_j)]_2. \tag{4.1}$$

For the symmetries (3.17), (4.1) can be written as

$$\begin{pmatrix}\overline{M_{22}(y,t,\overline{\lambda})}\\ \overline{M_{12}(y,t,\overline{\lambda})}\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{N}\frac{C_j}{\lambda-\lambda_j}\mathrm{e}^{2\mathrm{i}(\lambda_j y + 4\lambda_j^3 t)}\begin{pmatrix}M_{12}(y,t,\lambda_j)\\ M_{22}(y,t,\lambda_j)\end{pmatrix}. \tag{4.2}$$

Let $\overline{\lambda} = \overline{\lambda}_n$, (4.2) becomes

$$\begin{pmatrix} \overline{M_{22}(y,t,\lambda_n)} \\ \overline{M_{12}(y,t,\lambda_n)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{C_j}{\overline{\lambda}_n - \lambda_j} \mathrm{e}^{2\mathrm{i}(\lambda_j y + 4\lambda_j^3 t)} \begin{pmatrix} M_{12}(y,t,\lambda_j) \\ M_{22}(y,t,\lambda_j) \end{pmatrix}, \quad n = 1, \cdots, N. \tag{4.3}$$

Solving this algebraic system for $M_{12}(y,t,\lambda_j)$, $M_{22}(y,t,\lambda_j)$, $n = 1, \cdots, N$, and substituting them into (4.1) provide an explicit expression for the $[M(y,t,\lambda)]_1$. In terms of the symmetries (3.17), we can get $M_{12}(y,t,\lambda)$, which solves the Riemann-Hilbert problem. Then

$$-\mathrm{i} \lim_{\lambda \to \infty} (\lambda M(y,t,\lambda))_{12} = m(y,t) = -\mathrm{i} \sum_{j=1}^{N} C_j \mathrm{e}^{2\mathrm{i}(\lambda_j y + 4\lambda_j^3 t)} M_{12}(y,t,\lambda_j).$$

Therefore, the N soliton solution $q(x,t)$ is expressed by the (3.7).

4.1 The one-soliton solution

In this section, we derive an explicit formula for the one-soliton solution, which arises when $a(\lambda)$ has a pure imaginary λ_1 of simple zero. Letting $N = 1$ in (4.3), from the the symmetries of (2.10), we can deduce that $a(\lambda_1) = \overline{a(-\overline{\lambda}_1)} = 0$, and then $\lambda_1 = -\overline{\lambda}_1$ and $\dot{a}(\lambda_1) = \overline{\dot{a}(-\overline{\lambda}_1)}$. Since the b_1 is a real constant, we find that $C_1 = -\overline{C_1}$, and thus C_1 is pure imaginary. Making use of the symmetries of (3.17), we can obtain

$$\begin{aligned} \overline{M_{22}(y,t,\lambda_1)} &= 1 + \frac{C_1}{\overline{\lambda}_1 - \lambda_1} \mathrm{e}^{2\mathrm{i}(\lambda_1 y + 4\lambda_1^3 t)} M_{12}(y,t,\lambda_1), \\ \overline{M_{12}(y,t,\lambda_1)} &= \frac{C_1}{\overline{\lambda}_1 - \lambda_1} \mathrm{e}^{2\mathrm{i}(\lambda_1 y + 4\lambda_1^3 t)} M_{22}(y,t,\lambda_1). \end{aligned}$$

Then

$$\overline{M_{22}(y,t,\lambda_1)} = \frac{(\overline{\lambda}_1 - \lambda_1)^2}{(\overline{\lambda}_1 - \lambda_1)^2 + |C_1|^2 \mathrm{e}^{2\mathrm{i}(\lambda_1 y + 4\lambda_1^3 t)} \mathrm{e}^{-2\mathrm{i}(\overline{\lambda}_1 y + 4\overline{\lambda}_1^3 t)}}.$$

Substituting this result into (4.3), we get

$$M_{12}(y,t,\lambda) = \frac{\overline{C_1}(\overline{\lambda}_1 - \lambda_1)^2}{(\lambda - \overline{\lambda}_1)[(\overline{\lambda}_1 - \lambda_1)^2 \mathrm{e}^{2\mathrm{i}(\overline{\lambda}_1 y + 4\overline{\lambda}_1^3 t)} + |C_1|^2 \mathrm{e}^{2\mathrm{i}(\lambda_1 y + 4\lambda_1^3 t)}]}. \tag{4.4}$$

Let $\lambda_1 = \mathrm{i}\varepsilon$, $\varepsilon > 0$, and in order to conveniently study the properties of the one soliton solution, we choose $C_1 = \pm 2\mathrm{i}\varepsilon$. When $C_1 = -2\mathrm{i}\varepsilon$, substituting both parameters into (4.4), it comes into being that

$$M_{12}(y,t,\lambda) = \frac{2\mathrm{i}\varepsilon \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}}{(\lambda + \mathrm{i}\varepsilon)[1 - \mathrm{e}^{-4(\varepsilon y - 4\varepsilon^3 t)}]}. \tag{4.5}$$

Then

$$-\mathrm{i} \lim_{\lambda \to \infty} (\lambda M(y,t,\lambda))_{12} = -(\operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)})_y,$$

where the $\operatorname{arctanh} x$ is the inverse function of $\tanh x$. Furthermore,

$$\begin{aligned} \int_y^\infty m(y',t) \mathrm{d}y' &= -\mathrm{i} \int_y^\infty \lim_{\lambda \to \infty} (\lambda M(y',t,\lambda))_{12} \mathrm{d}y' \\ &= -\int_y^\infty (\operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)})_{y'} \mathrm{d}y' \\ &= \operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}. \end{aligned} \tag{4.6}$$

The solution $q(x,t)$ in (3.7) can be transformed into

$$q(x,t) = \mathrm{e}^{8\operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}} - 1. \tag{4.7}$$

Letting $\alpha(y,t) = \mathrm{e}^{\operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}}$, we find that $\mathrm{Ln}\alpha(y,t) = \operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}$, and then

$$\tanh(\ln\alpha(y,t)) = \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)},$$

i.e.,

$$\frac{\mathrm{e}^{\ln\alpha(y,t)} - \mathrm{e}^{-\ln\alpha(y,t)}}{\mathrm{e}^{\ln\alpha(y,t)} + \mathrm{e}^{-\ln\alpha(y,t)}} = \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}.$$

We deduce

$$\alpha^2(y,t) = -\tanh^{-1}(-\varepsilon y + 4\varepsilon^3 t).$$

(4.7) can be written as

$$q(x,t) = (\mathrm{e}^{\operatorname{arctanh} \mathrm{e}^{-2(\varepsilon y - 4\varepsilon^3 t)}})^8 - 1 = \tanh^{-4}(-\varepsilon y + 4\varepsilon^3 t) - 1. \tag{4.8}$$

Substituting y with x, (4.8) becomes

$$q(x,t) = \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x,t)) - 1, \tag{4.9}$$

where $\gamma(x,t) = \int_x^\infty (1 - \rho(\xi,t))\mathrm{d}\xi$ and $\rho(x,t) = \tanh^{-2}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x,t))$. Then (4.9) can be varied as $(1+q(x,t))^{\frac{1}{2}} - 1 = \cosh^2(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x,t))$, and hence the one soliton solution $q(x,t)$ has a singularity at the peak of the soliton, the so-called cusp soliton.

When $\lambda_1 = \mathrm{i}\varepsilon$ and $C_1 = 2\mathrm{i}\varepsilon$, the corresponding one soliton solution $q(x,t)$ of (1.2) can be expressed as

$$q(x,t) = \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x,t)) - 1, \tag{4.10}$$

where $\gamma(x,t) = \int_x^\infty (1 - \rho(\xi,t))\mathrm{d}\xi$, $\rho(x,t) = \tanh^{-2}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x,t))$.

Remark 4.1 In this paper, we use the Riemann-Hilbert approach to obtain the solution $q(x,t)$ of (1.2) expressed by (4.9)–(4.10). While [3] applies the inverse scattering method to get the solution $q(x,t)$. If $\varepsilon = \kappa$ (κ in [3], to the one soliton solution, when $C_1 = -2\mathrm{i}\varepsilon$, the expression of the solution in both papers is similar, identical with $-\varepsilon x + 4\varepsilon^3 t$ in the

$$\tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x,t))$$

and $\kappa x - 4\kappa^3 t$ in the $\tanh^{-4}(\kappa x - 4\kappa^3 t - \kappa x_0 + \varepsilon_+)$ in [3]). There is a different point about the expression of the one soliton solution in the two papers, i.e., one is dependent of the $-\varepsilon\gamma(x,t)$ of x and the other is $-\kappa x_0 + \varepsilon_+$ of x.

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