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Embedding Generalized Petersen Graph in Books*

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Abstract A book embedding of a graph G consists of placing the vertices of G on a spine and assigning edges of the graph to pages so that edges in the same page do not cross each other. The page number is a measure of the quality of a book embedding which is the minimum number of pages in which the graph G can be embedded. In this paper, the authors discuss the embedding of the generalized Petersen graph and determine that the page number of the generalized Petersen graph is three in some situations, which is best possible.

Keywords Book embedding, Page number, Generalized Petersen graph
2000 MR Subject Classification 05C10

1 Introduction

A book consists of a spine which is just a line and some number of pages each of which is a half-plane with the spine as its boundary. A book embedding of a graph G consists of placing the vertices of G on the line in order and assigning edges of the graph to pages so that the edges assigned to the same page do not cross each other. The page number is a measure of the quality of a book embedding. It is the minimum number of pages in which the graph G can be embedded, and is denoted by $pn(G)$.

Ollmann [18] first introduced the page number problem, and the problem is NP-complete even if the order of nodes on the spine is fixed (see [3, 13]). The book embedding problem has been motivated by several areas of computer science such as sorting with parallel stacks, single-row routing, fault-tolerant processor arrays and turning machine graphs (see [3]). Embedding a graph in a book with the minimum number of pages has received much attention in the literature (see [3–10]). In [16], Berhart and Keinen proved the theorem: $pn(G) \leq 1$ if and only if G is outplanar and $pn(G) \leq 2$ if and only if G is a subgraph of a Hamiltonian planar graph. By the above, for a connected graph G which is neither an outplanar nor a subhamiltonian planar graph, we have $pn(G) \geq 3$.

The Petersen graph is one of the most famous graphs. The notation of the generalized Petersen graph is that given integers $n \geq 3$ and $k \in Z_n \backslash \{0\}$, the graph $P(n, k)$ is defined on the set $\{x_i, y_i \mid i \in Z_n\}$ of $2n$ vertices, with the adjacencies given by $x_i x_{i+k}, x_i y_i, y_i y_{i+1}$ for all

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$i \in Z_n$. In this notion, the Petersen graph is $P(5,2)$ (see Figure a), which can be embedded in a 3-page book (see Figure b).

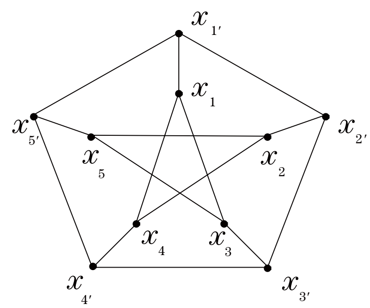


Figure a

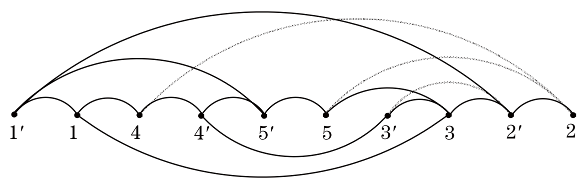


Figure b

From the definition of the generalized Petersen graph, $P(n,k) \cong P(n,n-k)$ for $k \le \lfloor \frac{n}{2} \rfloor$. Thus, we always assume $k \le \lfloor \frac{n}{2} \rfloor$. The graph $P(n,1)$ is called a Prism graph, which is an outplanar graph, so we assume $2 \le k \le \lfloor \frac{n}{2} \rfloor$.

Let $(n,k) = d$ be the greatest common denominator of n and k. For different parities of n and d, we give a complete description of the upper bounds of $pn(P(n,k))$, and in some situations, we obtain that $pn(P(n,k)) = 3$, which is best possible. We shall prove the following theorems.

Theorem 1.1 *If n and d are even, then $pn(P(n,k)) \leqslant 4$.*

Corollary 1.1 *If n is even and $k = 2$, then $pn(P(n,k)) = 3$.*

Theorem 1.2 *If n is even and d is odd, then $pn(P(n,k)) \leqslant 5$.*

Let $n = qk + r$ and $s = k - r$, where r is an integer less than k.

Theorem 1.3 *If n is odd and k is even, then $pn(P(n,k)) \leqslant 2s+1$. In particular, $pn(P(n,k)) \leqslant 2d+1$, if $s = d \neq 1$.*

Corollary 1.2 *If n is odd and $k = 2$, then $pn(P(n,k)) = 3$.*

Theorem 1.4 *If both n and k are odd, then $pn(P(n,k)) \leqslant \frac{k+1}{2} + 3$.*

Table 1

graph	generalized Petersen graph	page number
cubic symmetric graph (F(048)A)	$P(24,5)$	≤ 6
cubical graph	$P(4,1)$	1
Desargues graph	$P(10,3)$	3
dodecahedral graph	$P(10,2)$	3
Dürer graph	$P(6,2)$	3
Möbius-Kantor graph	$P(8,3)$	< 6
Nauru graph	$P(12,5)$	< 6
Petersen graph	$P(5,2)$	3
prism graph	$P(n,1)$	1

Some graphs are special cases of the generalized Petersen graph. For example, the Desargues graph is $P(10,3)$, the Möbius-Kantor graph is $P(8,3)$, and the prism graph is $P(n,1)$. We can know the page number of these graphs from the page number of the generalized Petersen graph (see Table 1).

2 Proofs of the Main Results

We assume $V(P(n,k)) = \{0,1,\cdots,n-1,0',1',\cdots,(n-1)'\}$, $E\cdots(P(n,k)\cdots) = \{ii' \mid i = 0,1,\cdots,n-1\} \cup \{i'(i+1)' \mid i = 0,1,\cdots,n-1\} \cup \{i(i+k) \mid i = 0,1,\cdots,n-1\} \pmod{n}$ and $V_1 \cup V_2 = V(P(n,k))$, where $V_1 = \{0,1,\cdots,(n-1)\}$ and $V_2 = \{0',1',\cdots,(n-1)'\}$ in the following. Each edge in $E\cdots(P(n,k)[V_2]\cdots)$ is called a 1-edge, and each edge in $E\cdots(P(n,k)[V_1]\cdots)$ is called a k-edge, where $P(n,k)[V_i]$ is the subgraph of $P(n,k)$ induced by the vertex set V_i, so $E(P(n,k)[V_i]) \subseteq E(P(n,k))$. $E(C_i)$ denotes the edge set containing all edges induced by the vertex set C_i, and $E[C_i,C_j]$ denotes the edge set containing all edges from C_i to C_j.

For $P(n,k)$, we lay out V_1 in the spine by ordering β, and use β^{-1} to denote the reverse ordering of β. That is, β^{-1} is obtained from β by revolving 180^0. Replacing each $i \in V_1$ by $i' \in V_2$ in β^{-1} gives an ordering of V_2, which is denoted by β'. So we have the following fact.

Fact 2.1 For the generalized Petersen graph $P(n,k)$, if an ordering of V_1 is β, then $V(P(n,k))$ has an ordering $\beta\beta'$.

Using Fact 2.1, we can draw the next lemma.

Lemma 2.1 *If $P(n,k)[V_1]$ and $P(n,k)[V_2]$ can be embedded in p_1 and p_2 pages with the vertex orderings β and β' respectively, then $pn(P(n,k)) \leq \max(p_1,p_2)+1$.*

Proof Since $P(n,k)[V_1] \cap P(n,k)[V_2] = \emptyset$, $pn(P(n,k)[V_1] \cup P(n,k)[V_2]) = \max(p_1,p_2)$. By the definition of $P(n,k)$, there is a perfect matching M, where each edge $e = uv \in M$ satisfies $u \in V_1$, $v \in V_2$, and $P(n,k) = P(n,k)[V_1] \cup P(n,k)[V_2] \cup M$. Since β is the vertex ordering of $P(n,k)[V_1]$ with $pn(P(n,k)[V_1]) = p_1$, by Fact 2.1, we have an ordering of $V(P(n,k))$, denoted by $\beta\beta'$. By the construction of $\beta\beta'$, M needs one page to be embedded, and the edges in M cross the edges in $P(n,k)[V_1]$ and $P(n,k)[V_2]$. So $pn(P(n,k)) \leq \max(p_1,p_2)+1$.

Proof of Theorem 1.1 Let C_i $(i \in [d-1] \cup 0)$ be an ordered $\frac{n}{d}$-element array and

$$
\begin{aligned}
&C_0 = \cdots \Big(0, k, 2k, \cdots, \Big(\frac{n}{d}-2\Big)k, \Big(\frac{n}{d}-1\Big)k \cdots\Big),\\
&C_1 = \cdots \Big(\Big(\frac{n}{d}-1\Big)k+1, \Big(\frac{n}{d}-2\Big)k+2, \cdots, 2k+1, k+1, 0+1 \cdots\Big),\\
&\qquad\qquad\qquad\qquad \cdots\\
&C_i = \cdots \Big(0+i, k+i, 2k+i, \cdots, \Big(\frac{n}{d}-2\Big)k+i, \Big(\frac{n}{d}-1\Big)k+i \cdots\Big), \quad \text{if } i \text{ is even},\\
&C_i = \cdots \Big(\Big(\frac{n}{d}-1\Big)k+i, \Big(\frac{n}{d}-2\Big)k+i, \cdots, 2k+i, k+i, 0+i \cdots\Big), \quad \text{if } i \text{ is odd},\\
&\qquad\qquad\qquad\qquad \cdots\\
&C_{d-1} = \cdots \Big(\Big(\frac{n}{d}-1\Big)k+(d-1), \Big(\frac{n}{d}-2\Big)k+(d-1), \cdots, k+(d-1), 0+(d-1) \cdots\Big).
\end{aligned}
$$

Thus $\bigcup_{i=0}^{d-1} C_i = V_1$ because $|C_i| = \frac{N}{d}$, $C_i \cap C_j = \emptyset$ and $0 \le v \le N-1$ for $v \in C_i$ and $i = 0, 1, \cdots, d-1$.

Put C_i in the line with the ordering of $C_0, C_1, \cdots, C_{d-1}$, and then all vertices of V_1 are assigned. Let the ordering of V_1 be β. By Fact 2.1, we have an ordering $\beta\beta'$ of $V(P(n,k))$. We denote $\beta' = C_{(d-1)'}, C_{(d-2)'}, \cdots, C_{0'}$, where $C_{i'} = \cdots \big(\big((\frac{n}{d}-1)k+i\big)', \big((\frac{n}{d}-2)k+i\big)', \cdots, (2k+i)', (k+i)', (0+i)' \cdots\big)$, if i' is even, or $C_{i'} = \cdots \big((0+i)', (k+i)', (2k+i)', \cdots, \big((\frac{n}{d}-2)k+i\big)', \big((\frac{n}{d}-1)k+i\big)' \cdots\big)$, if i' is odd. There are some properties as follows.

Property 1 The ordering of $V(P(n,k))$ is $C_0 \to C_1 \to \cdots \to C_{d-1} \to C_{(d-1)'} \to C_{(d-2)'} \to C_{0'}$ (see Figure 1).

Property 2 The edge sets $\{E(C_i) \mid i = 0, 1, \cdots, d-1\}$ and $\{E[C_i, C_{i+1}] \mid i = 0, 1, \cdots, (d-2)\} \cup \{EC_{d-1}, C_0\}$ are contained in $E(P(n,k)[V_1])$ which does not have a 1-edge, and the edge set $\{E[C_{i'}, C_{(i+1)'}] \mid i' = 0', 1', \cdots, (d-2)'\} \cup E[C_{(d-1)'}, C_{0'}]$ is contained in $E(P(n,k)[V_2])$ which does not have a k-edge.

Property 3 All k-edges can be embedded in one page without crossing (see Figure 2).

Property 4 Edges in $E(V_1, V_2)$ do not cross each other (see Figure 4).

By Properties 1–4, we have k-edges and $E[V_1, V_2]$ are embedded in two pages. Thus we only need to embed 1-edges in pages.

Claim 1 1-edges can be embedded in three pages without crossing.

Proof In the ordering of $V(P(n,k))$, if i' is even, $E[C_{i'}, C_{(i+1)'}]$ contains $\frac{n}{d}$ edges and they are $\{(jk+i)', (jk+i+1)'\}$, where j' goes from $0'$ to $\big(\frac{n}{d}-1\big)'$ and $i' \in [(d-2)'] \cup \{0'\}$ (mod n is omitted), which can be embedded in one page without crossing. If i is odd, the edges of $E[C_{i'}, C_{(i+1)'}]$ are $\{(jk+i)', (jk+i+1)'\}$, where j' goes from $\big(\frac{n}{d}-1\big)'$ to $0'$ and $i' \in [(d-2)'] \cup \{0'\}$, which can be embedded in another page without crossing. Next we can embed $E[C_{(d-1)'}, C_{0'}]$ into two pages. The edge set $\{(d-1+ik)', (d+ik)'\}$ with $i' \in \big[\big(\lfloor\frac{n-d}{k}\rfloor - 1\big)'\big] \cup \{0'\}$ (denoted by I-edges) can be assigned in one page and the other edges of $E[C_{(d-1)'}, C_{0'}]$ which are $\{(n-1+ik)', (ik)'\}$ with $i' \in \big[\big(\frac{n}{d} - \lfloor\frac{n-d}{k}\rfloor - 1\big)'\big]$ (denoted by II-edges) can be assigned in the other page.

By Property 3, Claim 1 and Lemma 2.1, we know that $P(n,k)$ can be embedded in a 4-page book.

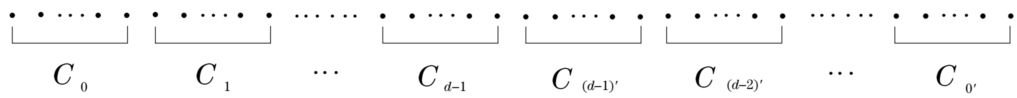


Figure 1

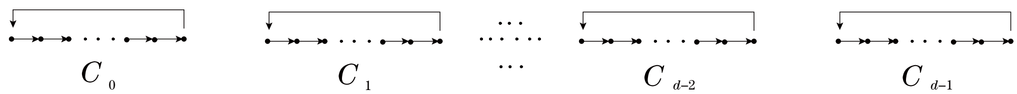
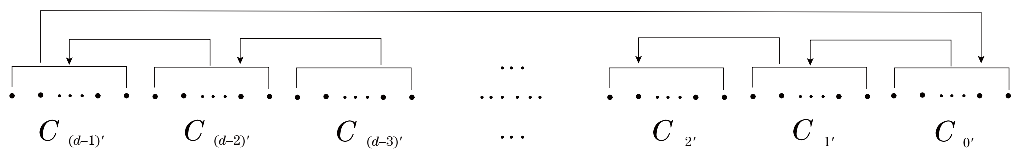


Figure 2



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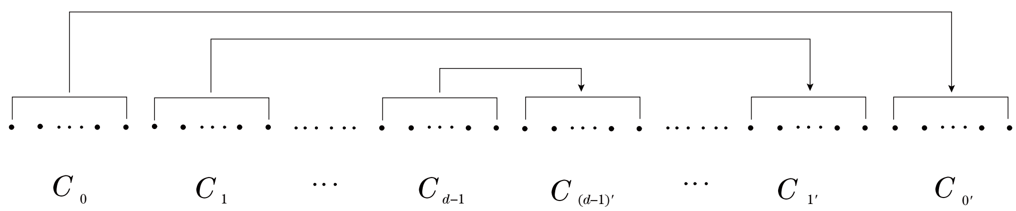

Figure 4

Page 1: II-edges.

Page 2: k-edges and $E[C_i', C_{(i+1)'}]$, i' is even.

Page 3: $E[C_i', C_{(i+1)'}]$, i' is odd and I-edges.

Page 4: $E[V_1, V_2]$.

Proof of Corollary 1.1 If $k = 2$, since V_2 can be partitioned into two parts $C_{0'}$ and $C_{1'}$, where $C_{0'} = (0', 2', 4', \cdots, n')$ and $C_{0'} = ((n-1)', (n-2)', \cdots, 5', 3', 1')$, all 1-edges $\{(0', 1'), (1', 2'), (2', 3') \cdots ((n-2)', (n-1)'), ((n-1)', 0')\}$ can be embedded in two pages. So $P(n, k)$ can be embedded in a 3-page book.

Page 1: 1-edges except the edge $(n-1, 0)$.

Page 2: k-edges and $(n-1, 0)$.

Page 3: $E[V_1, V_2]$.

Proof of Theorem 1.2 Let $C_0 = (0, 2, \cdots, n-4, n-2)$ and $C_1 = (n-2+k, n-4+k, \cdots, 2+k, k)$ (mod n is omitted) be the ordered vertex set of V_1, $C_0 \cap C_1 = \emptyset$, and $C_0 \cup C_1 = V_1$. We put C_i in the spine with the ordering of C_0, C_1. Then all vertices of V_1 are assigned.

Denote the vertex ordering of V_1 by β, and by Fact 2.1, we have an ordering $\beta\beta'$ of $V(P(n, k))$. Thus all vertices of $P(n, k)$ are assigned in the spine. In the ordering, 1-edges can be embedded in four pages: $\{(2i)', (2i+1)'\}$ with $i \in \left[\left\lfloor \frac{k}{2} - 1\right\rfloor\right] \cup \{0\}$ (denoted by E_1), $\{(2i)', (2i+1)'\}$ with $i \in \left[\frac{n}{2}\right] \backslash \left\lfloor \frac{k}{2} - 1\right\rfloor$ (denoted by E_2), $\{(k+2i)', (k+2i+1)'\}$ with $i \in \left[\frac{n-1-k}{2} - 1\right] \cup \{0\}$ (denoted by E_3) and $\{(n-1+2i)', i'\}$ with $i \in \left[\frac{k+1}{2} - 1\right] \cup \{0\}$ (denoted by E_4). k-edges can be embedded in three pages as follows.

k-edges can be embedded in three pages which are $\{2i, 2i+k\}$ with $i \in \left[\frac{n}{2} - 1\right] \cup \{0\}$, $\{k+2i, 2k+2i\}$ with $i \in \left[\frac{n}{2} - k - 1\right] \cup \{0\}$, and $\{n-k+2i, 2i\}$ with $i \in [k-1] \cup \{0\}$.

By Lemma 2.1, we know that $P(n,k)$ can be embedded in a 5-page book.

Next, we will embed $P(n,k)$ when n is odd. It is more complicated.

Proof of Theorem 1.3

Case 1 When n is odd, k is even, and $d = 1$. Let C_i be an ordered array and

$$
\begin{aligned}
&C_0 = \cdots (0, k, \cdots, (c-1)k, ck \cdots), \\
&C_1 = \cdots (ck+1, (c-1)k+1 \cdots, k+1, 0+1 \cdots), \\
&\qquad\qquad \cdots \\
&C_i = \cdots (i, k+i, \cdots, (c-1)k+i, ck+i) \cdots), \quad i \text{ is even}, \\
&C_i = \cdots (ck+i, (c-1)k+i, \cdots, k+i, 0+i \cdots), \quad i \text{ is odd}, \\
&\qquad\qquad \cdots \\
&C_{k-1} = \cdots (ck+(k-1), (c-1)k+(k-1), \cdots, k+(k-1), 0+(k-1) \cdots).
\end{aligned}
$$

Thus $|C_i| = c+1$, $c = q$ if $ck+i < n$; otherwise, $c = q-1$, and $C_i \cap C_j = \emptyset$. Since $n = qk+r$, there are r parts having $q+1$ vertices and other parts having q vertices, and $\bigcup_{i=0}^{k-1} C_i = V_1$.

Claim 1 $|C_i| = q+1$ with $0 \le i \le r-1$ and $|C_i| = q$ with $r-1 < i \le k-1$.

Proof Let $C_i = \{v_1, v_2, \cdots, v_c\}$ and $C_i + 1 = \{v_c+1, v_{c-1}+1, \cdots, v_1+1\}$. From the structure of C_i, we find that the vertex in C_i+1 is equal to C_{i+1}, $0 \le i \le r-2$ and $q+1 = |C_0| = |C_1| = \cdots = |C_{r-1}|$. Assume $C_{r-1} = \{r-1, k+r-1, \cdots, (q-1)k+r-1,\}$, and then $C_{r-1}+1 = \{qk+r, (q-1)k+r, \cdots, k+r, r\}$. $|C_r| = q$ because $qk+r \equiv 0 \pmod{n}$ and $0 \in C_0$. So $q = |C_r| = |C_{r+1}| = \cdots = |C_{k-1}|$.

Put C_i in the spine with the ordering of $C_0, C_1, \cdots, C_{k-1}$, and then all vertices of V_1 are assigned. Let the ordering of V_1 be β. By Fact 2.1, we have an ordering $\beta\beta'$ of $V(P(n,k))$.

Therefore, each vertex of $p(n,k)$ is assigned in the spine by the vertex set ordering $\beta\beta'$. We have the following properties.

Property 1 The order of $V(P(n,k))$ is $C_0 \to C_1 \to \cdots \to C_{k-1} \to C_{(k-1)'} \to C_{(k-2)'} \to \cdots \to C_{0'}$ (see Figure 5).

Property 2 The edge sets $\{E(C_i) \mid i = 0, 1, \cdots, k-1\}$ and $\{E(C_{i'}) \mid i' = 0', 1', \cdots, (k-1)'\}$ are contained in $E(P(n,k)[V_1])$, and they contain no 1-edge. The edge set $E[C_{i'}, C_{j'}] = E[C_{i'}, C_{(i+1)'}] \cup E[C_{(k-1)'}, C_{0'}] \cup ((n-k)', 0')$ $(i', j' = 0', 1', \cdots, (k-1)')$ is contained in $E(P(n,k)[V_2])$, and they contain no k-edge.

Property 3 Edges do not cross each other in $E[V_1, V_2]$ (see Figure 8).

Next, we embed 1-edges and k-edges of $P(n,k)$ in $2s$ pages without crossing.

Claim 2 The k-edges can be embedded in $2s$ pages without crossing, where $s = k - r$.

Proof Let $E_k(C_i)$ denote the k-edges in $E(C_i)$. Similarly, $E_k[C_i, C_j]$ denotes the k-edges in $E[C_i, C_j]$. From the partition and the ordering of V_1, we know that $E_k[C_0, C_s], E_k[C_s, C_{2s}], \cdots,$

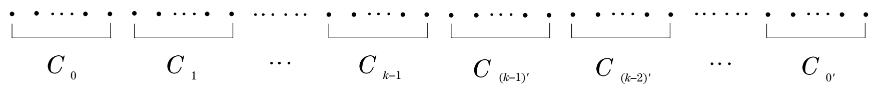

Figure 5

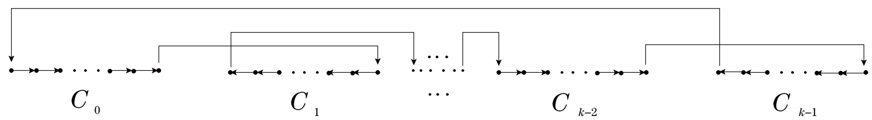

Figure 6

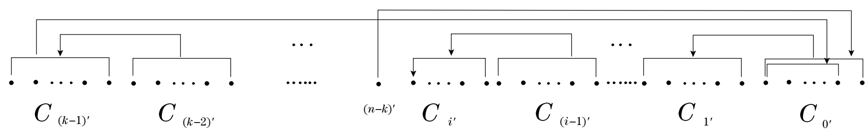

Figure 7

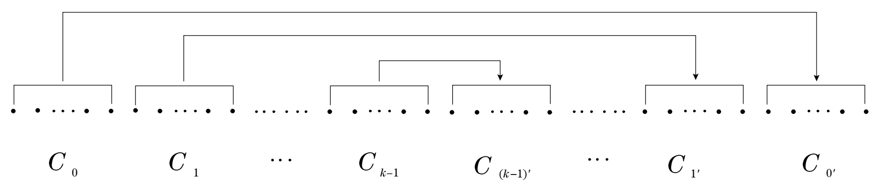

Figure 8

$E_k[C_{(\lfloor\frac{k}{s}\rfloor-1)s}, C_{\lfloor\frac{k}{s}\rfloor s}]$ and $E_k[C_{\lfloor\frac{k}{s}\rfloor s}, C_{k-\lfloor\frac{k}{s}\rfloor s}]$ $\big((k-\lfloor\frac{k}{s}\rfloor s)=1\big)$ can be embedded in two pages. Obviously, $E_k(C_0), E_k(C_s), \cdots, E_k(C_{\lfloor\frac{k}{s}\rfloor})$ can be embedded in the same page. Similarly, under the vertex ordering, for $i<s$, every $E_k[C_i, C_{i+s}], E_k[C_{i+s}, C_{i+2s}], \cdots, E_k[C_{i+(\lfloor\frac{k}{s}\rfloor-1)s}, C_{i+\lfloor\frac{k}{s}\rfloor s}]$ and $E_k[C_{i+\lfloor\frac{k}{s}\rfloor s}, C_{i+k-\lfloor\frac{k}{s}\rfloor s}]$ can be embedded in two pages too. Similarly, $E_k(C_i), E_k(C_{i+s}), \cdots,$ $E_k(C_{i+\lfloor\frac{k}{s}\rfloor s})$ can be embedded in the same page. So we embed all k-edges in $2s$ pages. Specially, if $s=1$, $E_k[C_i, C_{i+1}],\ i\in\{0,1,\cdots,k-1\}$ is embedded in the spine with a natural order, and all k-edges can be embedded in two pages (see Figure 6).

Claim 3 Embedding of 1-edges needs three pages.

Proof Because of the vertex ordering and the structure of $C_{i'}$, 1-edges are $E[C_{i'}, C_{(i+1)'}]\cup E(C_{(k-1)'}, C_0)\cup((n-1)', 0')$ $(i'\neq 1)$. Obviously, the edge set $E[C_{i'}, C_{(i+1)'}]$ (i' is even) can be embedded in one page. Similarly, when i' is odd, the embedding of $E[C_{i'}, C_{(i+1)'}]$ and $E[C_{(k-1)'}, E(C_{0'})]$ also needs one page. $((n-1)', 0')$ needs another page (see Figure 7).

Combining the above properties and claims, $P(n,k)$ can be embedded in a $(2s+1)$-page book if $d=1$.

Case 2 When n is odd, k is even, and $d\neq 1$. Let C_i be an ordered array and

$$C_0=\cdots(0,k,\cdots,(c-1)k, ck\cdots),$$
$$C_1=\cdots(ck+1,(c-1)k+1,\cdots,k+1,0+1\cdots),$$
$$\cdots$$
$$C_i=\cdots(0+i,k+i,\cdots,(c-1)k+i), ck+i\cdots),\ i \text{ is even},$$

$$C_i = \cdots (ck+i, (c-1)k+i, \cdots, k+i, 0+i\cdots),\ i \text{ is odd},$$
$$\cdots$$
$$C_{k-1} = \cdots (ck+k-1, (c-1)k+k-1, \cdots, k+k-1, 0+k-1\cdots).$$

$|C_i| = c+1$, $c=q$ if $ck+i<n$; otherwise, $c=q-1$, and $C_i \cap C_j = \emptyset$. Since $n = qk+r$, there are r parts having $q+1$ vertices and other parts having q vertices. Thus $\bigcup\limits_{i=0}^{k-1} C_i = V_1$.

The graph $P(n,k)[V_1]$ can be divided into d copies because $(n,k) = d \neq 1$. The copy containing 0 is denoted by G_0, and the other copies are denoted by $G_1, G_2, \cdots, G_{d-1}$. We know $|V(G_i)| = \frac{n}{d}$ with $i = 0, 1, \cdots, d-1$ and $V(G_i) = V(G_0)+i$, where $V(G_0)+i = \{v_0+i = v_i \mid v_i \in V(G_0),\ v_0 \in V(G_0)\}$. If we only embed k-edges, each copy has the same k-edges embedding.

The vertex set V_1 can be assigned in the spine in this ordering $C_0, C_1, \cdots, C_{k-1}$. Clearly, all vertices of V_1 are assigned. We use β to denote this ordering. By Fact 2.1, we have an ordering of $V(P(n,k))$, denoted by $\beta\beta'$. Therefore, each vertex of $P(n,k)$ is assigned in the spine and has a position by the vertex set ordering $\beta\beta'$. We have the following properties.

Property 4 The ordering of $V(P(n,k))$ is $C_0 \to C_1 \to \cdots \to C_{k-1} \to C_{(k-1)'} \to \cdots \to C_{(k-2)'} \cdots \to C_{1'} \to C_{0'}$ (see Figure 9).

Property 5 $\{E(C_i) \mid i = 0,1,\cdots,k-1\}$ and $\{E[C_i, C_{i+1}] \mid i = 0,1,\cdots,(k-2)\} \cup \{E[C_{d-1}, C_0]\}$ are contained in $E(P(n,k)[V_1])$, and then they contain no 1-edges. The edge set $\{E[C_{i'}, C_{(i+1)'}] \mid i' = 0', 1', \cdots, (k-2)'\} \cup \{E[C_{(k-1)'}, C_{0'}]\}$ is contained in $E(P(n,k)[V_2])$, and then they contain no k-edges.

Property 6 Edges in $E[V_1, V_2]$ do not cross each other (see Figure 12).

Now we embed the edges of $P(n,k)$ in $2s+1$ pages without crossing.

Claim 4 All k-edges can be embedded in $2s$ pages without crossing.

Proof Since $ck+is+k \equiv (i+1)s \pmod{n}$, where $ck+is+k \in C_{is}$ and $(i+1)s \in C_{(i+1)s}$, we have $\bigcup\limits_{i=0}^{\frac{k}{d}-1} C_{is} = V(G_0)$. That is $\bigcup\limits_{j=0}^{\frac{k}{d}-1} C_{jd} = V(G_0)$ because s is a multiple of d, and ordered array $C_{id} = \{ck+id, (c-1)k+id, \cdots, k+d, d\}$ where i is even, and $C_{id} = \{d, k+d, \cdots, (c-1)k, ck+id\}$ where i is odd. k-edges of G_0 can be embedded in $2\frac{s}{d}$ pages in $\beta\beta'$ because $ck+id+k = s+id \equiv (i+\frac{s}{d})d \pmod{n}$, the number of C_i $(i = 0,1,\cdots,s-1)$ in $V[G_0]$ is $\frac{s}{d}$, $ck+id$ is the first (when i is odd) or the last (when i is even) vertex of C_{id}, and $(i+\frac{s}{d})d$ is the last (when i is odd) or the first (when i is even) vertex of $C_{(i+\frac{s}{d})d}$. Since G_i with $i = 0,1,\cdots,d-1$ has the same k-edges embedding, all k-edges can be embedded in a $(2s)$-page book. Specially, if $s = d$, k-edges can be embedded in $2d$ pages. Then $ck+id+k = s+id \equiv (i+1)d \pmod{n}$ and $id \in C_{id}$ (see Figure 6).

Claim 5 All 1-edges can be embedded in three pages.

Proof The edge set $E[C_{i'}, C_{(i+1)'}]$, $i' = 0', 1', \cdots, (k-2)$, and $E[C_{(k-1)'}, C_{0'}]$ can be embedded in three pages without crossing. When i' is even or odd, $E[C_{i'}, C_{(i+1)'}]$ (i' from $0'$ to $(k-2)', i'$ is even or odd) can be embedded in one page. The edge $(n-1, 0)$ needs another page (see Figure 11).

Combining the above properties and claims, $P(n,k)$ can be embedded in $2s+1$ pages if $d \neq 1$.

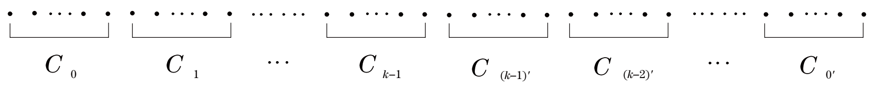

Figure 9

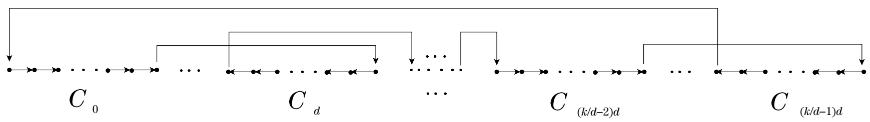

Figure 10

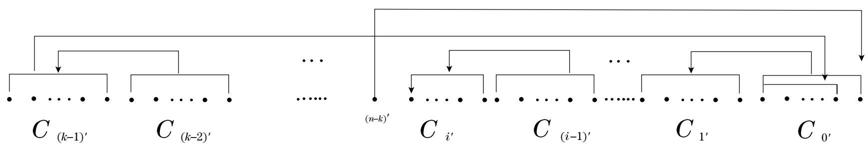

Figure 11

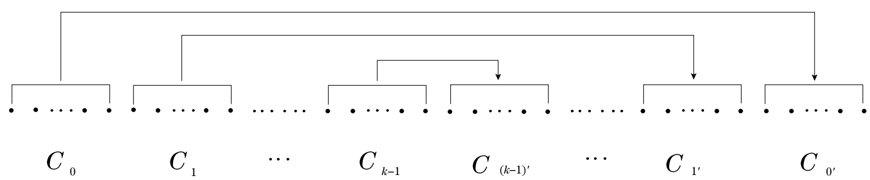

Figure 12

Proof of Corollary 1.2 When $k = 2$ and $s = 1$, all k-edges can be embedded in two pages and 1-edges can be embedded in one of pages of the k-edges. By Lemma 2.1, the embedding of $P(n, 2)$ needs 3 pages.

Proof of Theorem 1.4 Let $C_0 = \{0, 2, \cdots, n-1\}$ and $C_1 = \{n-2, n-4, \cdots, 1\}$ be the ordered vertex set of V_1, $C_0 \cap C_1 = \emptyset$, and $C_0 \cup C_1 = V_1$. We put C_i in the line with the ordering of C_0, C_1. So all vertices of V_1 are assigned.

Denote the vertex ordering of V_1 by β. By Fact 2.1, we have an ordering $\beta\beta'$ of $V(P(n, k))$. Therefore, each vertex of $P(n, k)$ is assigned in the spine.

Note that 1-edges can be embedded in two pages. Next we embed k-edges.

The edge set $\{(2i, 2i+k)\}$ with $i \in \left[\frac{n-k}{2} - 1\right] \cup \{0\}$ can be embedded in one page, and the edge set $\{1+2i, 1+2i+k\}$ with $i \in \left[\frac{n-k}{2} - 1\right] \cup \{0\}$ can also be embedded in one page. Edges in the edge set $\{(n-k+2i, 2i)\}$ with $i \in \left[\frac{k+1}{2} - 1\right] \cup \{0\}$ (denoted by A_1) cross each other. Edges in the edge set $\{(n-2i, n-2i+k)\}$ with $i \in \left[\frac{k-1}{2}\right]$ (denoted by A_2) also cross each other, while A_1, A_2 do not cross each other. Since $\max\{|A_1|, |A_2|\} = \frac{k+1}{2}$, k-edges at most need $\frac{k+1}{2} + 2$ pages to be embedded.

By Lemma 2.1, $P(n, k)$ can be embedded in $\frac{k+1}{2} + 3$ pages.

3 Conclusion

The connected graph G can be embedded in one page if and only if G is outplanar, and in two pages if and only if G is a subgraph of a Hamiltonian planar graph. So for a connected graph which is not an outplanar or a subhamiltonian planar graph, we have $pn(G) \geq 3$. A graph is planar if and only if it does not contain subdivision of either $K_{3,3}$ or K_5, or it has no Kuratowski minor (a minor which is isomorphic to $K_{3,3}$ or K_5). The general Peterson graph is not a planar graph for $k \neq 1$. For example, K_5 is a minor of $P(5, 2)$, so $pn(P(n, k)) \geq 3$. In this paper, we completely describe the upper bounds of $pn(P(n, k))$, and in some situation, $pn(P(n, k)) = 3$, which is the best possible.

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