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The Expansion of a Wedge of Gas into Vacuum with Small Angle in Two-Dimensional Isothermal Flow*

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Abstract In this paper, the authors consider the expansion problem of a wedge of gas into vacuum for the two-dimensional Euler equations in isothermal flow. By the bootstrapping argument, they prove the global existence of the smooth solution through the direct method in the case $0 < \theta \le \overline{\theta} = \arctan \frac{1}{\sqrt{2+\sqrt{5}}}$, where θ is the half angle of the wedge. Furthermore, they get the uniform $C^{1,1}$ estimates of the solution to the expansion problem.

Keywords Hyperbolic partial differential equation, 2D Riemann problem, Rarefaction wave, Isothermal flow
2000 MR Subject Classification 35L65, 35L80, 35R35, 35L60, 35L50.

1 Introduction

In this paper, we consider the two-dimensional isentropic compressible Euler equations

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1.1}$$

where (u, v) and ρ denote the velocity and the density, respectively, while the pressure p is given by $p(\rho) = \rho$ for the isothermal case. For the Riemann initial data, we may seek the self-similar solutions $(u, v, \rho) = (u, v, \rho)(\xi, \eta)$ $\big(\xi = \frac{x}{t}, \eta = \frac{y}{t}\big)$, for the reason that (1.1) and the Riemann initial data are invariant under the stretch $(x, y, t) \to (kx, ky, kt)$ $(k > 0)$. This kind of initial value problem is called the two-dimensional Riemann problem. The Riemann problem in general is very complicated. A simpler situation is the expansion problem of a wedge of gas into vacuum.

This problem has been an interesting problem for a long time. In [6], Dai and Zhang used the characteristic decomposition method to establish the global smooth solution for the expansion problem of the pressure gradient system. In [14], by the hodograph transformation and the characteristic decompositions of characteristic angles, Li and Zheng obtained various

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priori estimates and constructed classical self-similar solutions to the interaction of two planar rarefaction waves for the two-dimensional polytropic Euler equations. In [12], Li, Yang and Zheng developed the direct approach to recover all the properties of the solutions obtained by the hodograph transformation of Li and Zheng [14]. They obtained the existence of the solution for $\overline{\theta} < \theta < \frac{\pi}{2}$ and $\gamma > 1$. In [18], by the characteristic decompositions and the special symmetric structure of the characteristic forms, Zhao proved the global existence of the solution through the direct approach for $0 < \theta \leq \overline{\theta}$. In [7], Hu, Li and Sheng investigated the two-dimensional isothermal Euler equations and obtained the existence of the global solution for $\overline{\theta} < \theta < \frac{\pi}{2}$. For more related results, readers can see the survey papers (see [1–5, 8–11, 15–17]).

In this paper, we prove the global existence of the solution to the expansion problem of a wedge of gas into vacuum to the two-dimensional isothermal Euler equations by the direct method in the case $0 < \theta \leq \overline{\theta}$, where θ is the half angle of the wedge. Our research relies on the bootstrapping argument and the analysis of α, β instead of c for the reason that $c = 1$ for the isothermal flow. This paper is organized as follows. In Section 2, we give some preliminaries, including the characteristic forms of the isothermal Euler equations and some characteristic decompositions of the inclination angle (α, β). In Section 3, the expansion problem of a wedge of gas into vacuum is considered, and the main results are obtained.

Here is a list of our notations:

$$\delta=\frac{\alpha-\beta}{2},\quad \sigma=\frac{\alpha+\beta}{2},\quad m_1=\alpha-\frac{\pi}{4},\quad m_2=-\beta-\frac{\pi}{4},\quad M=\frac{\pi}{4}-\theta,$$
$$\partial_\pm=\partial_\xi+\Lambda_\pm\partial_\eta,\quad \overline{\partial}^+=\cos\alpha\partial_\xi+\sin\alpha\partial_\eta,\quad \overline{\partial}^-=\cos\beta\partial_\xi+\sin\beta\partial_\eta.$$

2 Preliminaries

Consider the two-dimensional isentropic isothermal compressible Euler equations (1.1). For the smooth self-similar solutions, the system (1.1) can be written as

$$\begin{cases} Uq_\xi + Vq_\eta + q(u_\xi + v_\eta) = 0, \\ Uu_\xi + Vu_\eta + q_\xi = 0, \\ Uv_\xi + Vv_\eta + q_\eta = 0, \end{cases} \tag{2.1}$$

where $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity, and $q = \ln \rho$ for $\rho > 0$. The eigenvalues of (2.1) are

$$\Lambda_0 = \frac{V}{U}, \quad \Lambda_\pm = \frac{UV \pm \sqrt{U^2 + V^2 - 1}}{U^2 - 1}. \tag{2.2}$$

The curves $C_0 : \frac{d\eta}{d\xi} = \Lambda_0$ and $C_\pm : \frac{d\eta}{d\xi} = \Lambda_\pm$ are the (pseudo-)flow characteristics and the (pseudo-) wave characteristics of (2.1), respectively.

We further assume that the flow is ir-rotational, i.e., $u_\eta = v_\xi$. Then (2.1) can be rewritten as

$$\begin{cases} (1 - U^2)u_\xi - UV(u_\eta + v_\xi) + (1 - V^2)v_\eta = 0, \\ u_\eta - v_\xi = 0 \end{cases} \tag{2.3}$$

supplemented by Bernoulli's law (see [13]):

$$\frac{U^2+V^2}{2}+\ln\rho=-\varphi,\quad \varphi_\xi=U,\quad \varphi_\eta=V. \tag{2.4}$$

The characteristic forms of the system are $\partial_\pm u+\Lambda_\mp\partial_\pm v=0$. As in [14] and [12], let α and β be the inclination angle variables of wave characteristics, that is,

$$\tan\alpha=\Lambda_+,\quad \tan\beta=\Lambda_-. \tag{2.5}$$

Then, for the convenience of solving the gas expansion problem, we choose

$$\begin{cases} U=-\dfrac{\cos\sigma}{\sin\delta},\\ V=-\dfrac{\sin\sigma}{\sin\delta}\end{cases}\quad\text{or}\quad\begin{cases} u=\xi-\dfrac{\cos\sigma}{\sin\delta},\\ v=\eta-\dfrac{\sin\sigma}{\sin\delta}.\end{cases} \tag{2.6}$$

Then, using (2.6), we get

$$\begin{cases} \overline{\partial}^+u=\cos\alpha+\dfrac{\cos\beta\overline{\partial}^+\alpha-\cos\alpha\overline{\partial}^+\beta}{2\sin^2\delta},\\ \overline{\partial}^+v=\sin\alpha+\dfrac{\sin\beta\overline{\partial}^+\alpha-\sin\alpha\overline{\partial}^+\beta}{2\sin^2\delta},\\ \overline{\partial}^-u=\cos\beta+\dfrac{\cos\beta\overline{\partial}^-\alpha-\cos\alpha\overline{\partial}^-\beta}{2\sin^2\delta},\\ \overline{\partial}^-v=\sin\beta+\dfrac{\sin\beta\overline{\partial}^-\alpha-\sin\alpha\overline{\partial}^-\beta}{2\sin^2\delta}.\end{cases} \tag{2.7}$$

Then, we have

$$\begin{cases} \overline{\partial}^+\alpha=\cos(2\delta)(-2\sin^2\delta+\overline{\partial}^+\beta),\\ \overline{\partial}^-\beta=\cos(2\delta)(2\sin^2\delta+\overline{\partial}^-\alpha).\end{cases} \tag{2.8}$$

In addition, we cite the following commutator relation of $\overline{\partial}^\pm$ from [12–13] and the characteristic decompositions in [7].

Lemma 2.1 (Commutator Relation of $\overline{\partial}^\pm$) *For any C^2 smooth function $I(\xi,\eta)$, there holds*

$$\overline{\partial}^-\overline{\partial}^+I-\overline{\partial}^+\overline{\partial}^-I=\frac{1}{\sin(2\delta)}\{(\cos(2\delta)\overline{\partial}^+\beta-\overline{\partial}^-\alpha)\overline{\partial}^-I-(\overline{\partial}^+\beta-\cos(2\delta)\overline{\partial}^-\alpha)\overline{\partial}^+I\}. \tag{2.9}$$

Lemma 2.2 *For the inclination angles α and β, we have*

$$\begin{cases} \overline{\partial}^+\overline{\partial}^-\alpha+M_1\overline{\partial}^-\alpha=\dfrac{1}{2}\sin(2\delta)(3\tan^2\delta-1)\overline{\partial}^+\alpha,\\ \overline{\partial}^-\overline{\partial}^+\beta+M_2\overline{\partial}^+\beta=\dfrac{1}{2}\sin(2\delta)(3\tan^2\delta-1)\overline{\partial}^-\beta,\end{cases} \tag{2.10}$$

where

$$\begin{cases} M_1=\dfrac{1}{\sin(2\delta)}\{-8\sin^6\delta-\overline{\partial}^-\alpha+(1-2\sin^2\delta\cos(2\delta))\overline{\partial}^+\beta\},\\ M_2=\dfrac{1}{\sin(2\delta)}\{-8\sin^6\delta+\overline{\partial}^+\beta-(1-2\sin^2\delta\cos(2\delta))\overline{\partial}^-\alpha\}.\end{cases}$$

Lemma 2.3 *For the inclination angle σ of Λ_0-characteristics, we have*

$$\begin{cases} \overline{\partial}^-\overline{\partial}^+\sigma + N_1\overline{\partial}^+\sigma = \tan\delta\cdot a(\delta)\overline{\partial}^-\sigma, \\ \overline{\partial}^+\overline{\partial}^-\sigma + N_2\overline{\partial}^-\sigma = \tan\delta\cdot a(\delta)\overline{\partial}^+\sigma, \end{cases} \tag{2.11}$$

where

$$N_1 = \tan\delta(1-4\sin^2\delta) + \frac{1}{\cos^2\delta}\Big(\frac{1}{2}\tan\delta\cos(2\delta) + \frac{1}{\sin(2\delta)}(\overline{\partial}^+\sigma - \cos(2\delta)\overline{\partial}^-\sigma)\Big),$$
$$N_2 = \tan\delta(1-4\sin^2\delta) + \frac{1}{\cos^2\delta}\Big(\frac{1}{2}\tan\delta\cos(2\delta) - \frac{1}{\sin(2\delta)}(\overline{\partial}^-\sigma - \cos(2\delta)\overline{\partial}^+\sigma)\Big)$$

and

$$a(\delta) = \frac{1}{2}\cos^2\delta\Big(\tan^2\delta - \frac{1}{2+\sqrt{5}}\Big)(\tan^2\delta + 2 + \sqrt{5}).$$

Introduce $\overline{\theta}$ by $\tan^2\overline{\theta} = \frac{1}{2+\sqrt{5}}$. Then, we have $a(\delta) > 0$ if $\delta > \overline{\theta}$.

Lemma 2.4 *For the inclination angles α and β ($\alpha - \beta \neq \frac{\pi}{2}, \pi$), we have*

$$\begin{cases} \overline{\partial}^-\Big(\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}\Big) = \frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}\Big(-(2+\cos(2\delta))\tan\delta + f_1\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} + f_2\frac{\overline{\partial}^-\beta}{\cos(2\delta)}\Big), \\ \overline{\partial}^+\Big(\frac{\overline{\partial}^-\beta}{\cos(2\delta)}\Big) = \frac{\overline{\partial}^-\beta}{\cos(2\delta)}\Big(-(2+\cos(2\delta))\tan\delta + f_1\frac{\overline{\partial}^-\beta}{\cos(2\delta)} + f_2\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}\Big), \end{cases} \tag{2.12}$$

where

$$f_1 = \frac{1}{\sin(2\delta)}, \quad f_2 = \frac{\cos^2(2\delta) + 2\sin^2\delta}{\sin(2\delta)}.$$

3 The Gas Expansion Problem to the Isothermal Euler Equations for the Case That $\theta \in (0, \overline{\theta}]$

In this section, by the characteristic decompositions in the previous section, we discuss the expansion problem of a wedge of gas into vacuum directly in the (ξ, η) plane.

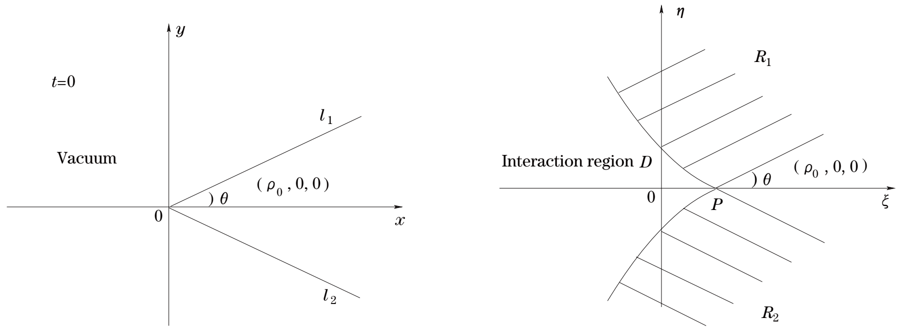
3.1 The expansion problem of a wedge of gas into vacuum

For convenience, we place the wedge of gas symmetrically with respect to the x-axis and the sharp corner at the origin, as in Figure 3.1(a). Let $\theta \in \big(0, \frac{\pi}{2}\big)$ be the wedge half-angle and l_1, l_2 denote the two edges of the wedge. At the time $t = 0$, the wedge is full of the gas, and vacuum is outside. Then the gas would expand into the vacuum. This problem is then formulated mathematically as seeking the solution of (2.3) with the initial data

$$(\rho, u, v)(0, x, y) = \begin{cases} (\rho_0, 0, 0), & -\theta < \omega < \theta, \\ \text{vacuum}, & \text{otherwize}, \end{cases} \tag{3.1}$$

where $\rho_0 > 0$ is a constant, and $\omega = \arctan\frac{y}{x}$ is the polar angle. In fact, this problem can be considered as a two-dimensional Riemann problem of (2.3) with two pieces of initial data (3.1). Through the analysis in the above subsection, the gas away from the wedge expands uniformly to infinity as planar rarefaction waves R_1 and R_2 which satisfy

$$(\rho, u, v)(t, x, y) = \begin{cases} (\rho_1, 0, 0), & \zeta > 1, \\ (\rho, u, v)(\zeta), & -\infty < \zeta \leq 1, \end{cases} \tag{3.2}$$



(a) Initial data (b) Interaction of rarefaction waves

Figure 3.1 The expansion of a wedge of gas

where $\zeta = n_1\xi + n_2\eta$, with $(n_1, n_2) = (\sin\theta, -\cos\theta)$ and $(n_1, n_2) = (\sin\theta, \cos\theta)$, respectively (see Figure 3.1(b)). Then the rarefaction waves R_1 and R_2 emitting from the initial discontinuities l_1 and l_2 begin to interact at $P\big(\frac{1}{\sin\theta}, 0\big)$ in the (ξ, η) plane. The wave interaction region D is formed adjacent to the planar rarefaction waves with boundaries k_1 and k_2 (see [7]).

Gas Expansion Problem Find a solution of (2.3) and (3.1) inside the wave interaction region D, subject to the boundary values on k_1 and k_2, which are determined continuously from the rarefaction waves R_1 and R_2.

3.2 The existence of local solutions

The equations (2.8) can be reduced to a diagonal form

$$\begin{cases} \overline{\partial}^{+}(-\beta + \cot\delta) = \cos(2\delta), \\ \overline{\partial}^{-}(\alpha + \cot\delta) = \cos(2\delta). \end{cases} \tag{3.3}$$

Let $(\xi_{\overline{\delta}}, \xi_{\overline{\delta}}) = \{\delta = \overline{\delta}\} \cap k_1$, and $D_{\overline{\delta}}$ ($\overline{\delta}$ is between θ and $\max\limits_{D}\{\delta\}$) be the region enclosed by the three curves k_1, k_2 and $\xi = \xi_{\overline{\delta}}$ (see Figure 3.2). Then, for the boundary data

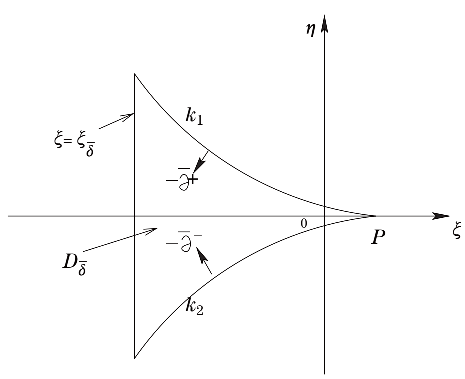


Figure 3.2 The domain $D_{\overline{\delta}}$

$$\alpha|_{k_1} = \theta, \quad \beta|_{k_2} = -\theta, \tag{3.4}$$

we get the following result as Lemma 5.2 in [7].

Theorem 3.1 (Local Existence) *There is a $\delta_0 > 0$ such that the C^1 solution of* (3.3) *and* (3.4) *exists uniquely in the region D_{δ_0}, where δ_0 depends only on the C^0 and C^1 norm of α, β on the boundaries k_1 and k_2.*

3.3 Estimates for the case that $\theta \in (0, \overline{\theta}]$

Lemma 3.1 (Boundary Data Estimates (see [7])) *If $0 < \theta \le \overline{\theta}$, there holds*

$$2\theta \le (\alpha - \beta)|_{k_i} \le \frac{\pi}{2}, \quad i = 1, 2. \tag{3.5}$$

Lemma 3.2 *Assuming that the solution $(\alpha, \beta) \in C^1$, we have that $\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}$ and $\frac{\overline{\partial}^-\beta}{\cos(2\delta)}$ are positive and bounded in $D_{\overline{\delta}}$.*

Here we note that $\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} = 2\sin^2\delta - \overline{\partial}^+\beta$ and $\frac{\overline{\partial}^-\beta}{\cos(2\delta)} = 2\sin^2\delta + \overline{\partial}^-\alpha$, which can be obtained by (2.8).

Proof By (3.4), we get $\overline{\partial}^-\alpha = 0$ and $\overline{\partial}^+\beta = 0$ on k_1 and k_2, respectively. So, we obtain that $\frac{\overline{\partial}^-\beta}{\cos(2\delta)} > 0$ and $\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} > 0$ on k_1 and k_2, respectively. Using the characteristic decompositions (2.12), we get $\frac{\overline{\partial}^-\beta}{\cos(2\delta)} > 0$ and $\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} > 0$ in $D_{\overline{\delta}}$,

$$\begin{aligned} X = \max\Big\{ &\max_{D_{\overline{\delta}}}\Big[(2+\cos(2\delta))\sin(2\delta)\tan\delta, \frac{(2+\cos(2\delta))\sin(2\delta)\tan\delta}{\cos^2(2\delta) + 2\sin^2\delta}\Big], \\ &\max_{k_2}|2\sin^2\delta|, \max_{k_1}|2\sin^2\delta|\Big\} + 2. \end{aligned} \tag{3.6}$$

$$X = \max_{D_{\overline{\delta}}}\Big[(2+\cos(2\delta))\sin(2\delta)\tan\delta, \frac{(2+\cos(2\delta))\sin(2\delta)\tan\delta}{\cos^2(2\delta) + 2\sin^2\delta}\Big] + 2. \tag{3.7}$$

Next, we prove that $0 < \frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} < X$ and $0 < \frac{\overline{\partial}^-\beta}{\cos(2\delta)} < X$. Suppose that $\xi = \xi_{\delta_1}$ is the first time that $\frac{\overline{\partial}^-\beta}{\cos(2\delta)} = X$ or $\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} = X$. Without loss of generality, we assume that $\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} = X$ at the point P_1 on the line $\xi = \xi_{\delta_1}$. From the first equation of (2.12), we have

$$\begin{aligned} \overline{\partial}^-\Big(\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}\Big)\Big|_{P_1} &= \Big\{\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}\Big(-(2+\cos(2\delta))\tan\delta + f_1\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} + f_2\frac{\overline{\partial}^-\beta}{\cos(2\delta)}\Big)\Big\}\Big|_{P_1} \\ &= X\Big\{\Big[-(2+\cos(2\delta))\tan\delta + f_2\frac{\overline{\partial}^-\beta}{\cos(2\delta))}\Big]\Big|_{P_1} + f_1|_{P_1}X\Big\} > 0. \end{aligned}$$

Note that the direction of $\overline{\partial}^-$ is the direction from $D_{\overline{\delta}}$ to the boundary. Thus, we have

$$\overline{\partial}^-\Big(\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}\Big) < 0$$

at point P_1. It leads to a contradiction. Then, we have $0 < \frac{-\overline{\partial}^+\alpha}{\cos(2\delta)} < X$ and $0 < \frac{\overline{\partial}^-\beta}{\cos(2\delta)} < X$.

Introducing new variables $m_1 = \alpha - \frac{\pi}{4}$ and $m_2 = -\beta - \frac{\pi}{4}$, we get

$$\begin{cases} -\overline{\partial}^+ m_1 = -\dfrac{\overline{\partial}^+\alpha}{\cos(2\delta)}\cos(2\delta) = (-m_1 - m_2)F(m_1, m_2), \\ -\overline{\partial}^- m_2 = \dfrac{\overline{\partial}^-\beta}{\cos(2\delta)}\cos(2\delta) = (-m_1 - m_2)G(m_1, m_2), \end{cases} \tag{3.8}$$

where

$$F(m_1, m_2) = \frac{\sin(\frac{\pi}{2} - 2\delta)}{\frac{\pi}{2} - 2\delta}\frac{-\overline{\partial}^+\alpha}{\cos(2\delta)}, \quad G(m_1, m_2) = \frac{\sin(\frac{\pi}{2} - 2\delta)}{\frac{\pi}{2} - 2\delta}\frac{\overline{\partial}^-\beta}{\cos(2\delta)}. \tag{3.9}$$

Then, we have the results as follows.

Lemma 3.3 *If $0 < \theta \leq \overline{\theta}$, $|m_1| \leq M$ and $|m_2| \leq M$, then $F(m_1, m_2)$ and $G(m_1, m_2)$ are positive and bounded in $D_{\overline{\delta}}$.*

Proof Because $|m_1| \leq M$, $|m_2| \leq M$, we have

$$2\theta - \frac{\pi}{2} \leq \frac{\pi}{2} - 2\delta \leq \frac{\pi}{2} - 2\theta. \tag{3.10}$$

Then, we have $\frac{\sin(\frac{\pi}{2}-2\delta)}{\frac{\pi}{2}-2\delta} > 0$. From Lemma 3.2, considering the fact that $\sin x$ and x are equivalent infinitely small, we get our result in this lemma.

Theorem 3.2 *Assuming that there is a C^1 solution in $D_{\overline{\delta}}$, $0 < \theta \leq \overline{\theta}$, then we have*

$$\theta \leq \alpha \leq \frac{\pi}{2} - \theta, \quad -\frac{\pi}{2} + \theta \leq \beta \leq -\theta. \tag{3.11}$$

Moreover, the above inequalities must be strict in the interior of $D_{\overline{\delta}}$, that is,

$$\theta < \alpha < \frac{\pi}{2} - \theta, \quad -\frac{\pi}{2} + \theta < \beta < -\theta. \tag{3.12}$$

Proof Considering $M = \frac{\pi}{4} - \theta$, then, we have that proving the inequalities (3.12) is equivalent to proving $|m_1| < M$ and $|m_2| < M$. In view of the boundary data estimates (3.5), according to the bootstrapping argument, we can get (3.12) if we show $|m_1| < M$, $|m_2| < M$ under the assumption that $|m_1| \leq M$ and $|m_2| \leq M$. Consequently, we only need to prove (3.12) holds provided that $|m_1| \leq M$ and $|m_2| \leq M$.

From (3.4), through the characteristic decompositions, we have that there exists a neighborhood ω_1 of k_1 such that $|m_2| < M$ in ω_1, and there exists a neighborhood ω_2 of k_2 such that $|m_1| < M$ in ω_2. Here, ω_1 and ω_2 are located in the interior of $D_{\overline{\delta}}$. By Lemma 3.1, we get that $-m_1 - m_2 = \frac{\pi}{2} - 2\delta > 0$ on k_1 and k_2. From Lemma 3.3, we get $-\overline{\partial}^+ m_1|_{k_1} > 0$ and $-\overline{\partial}^- m_2|_{k_2} > 0$. According to $-\overline{\partial}^+$ and $-\overline{\partial}^-$ pointing toward the interior of $D_{\overline{\delta}}$ on k_1 and k_2, respectively, and $m_1 = -M$, $m_2 > -M$ on k_1 and $m_1 > -M$, $m_2 = -M$ on k_2 except P, we get the results.

We select a Λ_- characteristic curve $\widetilde{k}_1$ in ω_1 and a Λ_+ characteristic curve $\widetilde{k}_2$ in ω_2, taking them as the new boundaries and the data on them as the initial data in the bootstrapping argument.

Next, we would prove $|m_1| < M$ and $|m_2| < M$ in the interior of $D_{\overline{\delta}}$. Otherwise, there exist some points at which $|m_1| \geq M$ or $|m_2| \geq M$. Suppose that $P_1(\xi_1, \eta_1)$ is the first point at which $|m_1| = M$ or $|m_2| = M$ along the Λ_+ characteristic curve C_1 emitting from the point $P_0(\xi_0, \eta_0)$ on $\widetilde{k}_1$. Noticing (3.8) can be rewritten as

$$-\overline{\partial}^+ m_1 + m_1 F = -m_2 F, \tag{3.13}$$

and solving the equation along C_1 from P_0 to P_1, we get

$$m_1(\xi_1, \eta_1) = \mathrm{e}^{-\int_{C_1} F\mathrm{d}s} m_1(\xi_0, \eta_0) + \mathrm{e}^{-\int_{C_1} F\mathrm{d}s} \int_{C_1} F \mathrm{e}^{\int_{C_1} F\mathrm{d}s}(-m_2)\mathrm{d}s. \tag{3.14}$$

In view of $\mathrm{d}(\mathrm{e}^{\int F\mathrm{d}s}) = F\mathrm{e}^{\int F\mathrm{d}s}\mathrm{d}s$ and Lemma 3.3 and the choice condition of P_1, we have

$$|m_1(\xi_1, \eta_1)| < \mathrm{e}^{-\int_{C_1} F\mathrm{d}s} M + M\mathrm{e}^{-\int_{C_1} F\mathrm{d}s}(\mathrm{e}^{\int_{C_1} F\mathrm{d}s} - 1) < M. \tag{3.15}$$

Similarly, utilizing the second equation of (3.8), we get $|m_2(\xi_1, \eta_1)| < M$. It leads to a contradiction.

From Theorem 3.2, we have $\theta < \delta \leq \frac{\pi}{2} - \theta$. Considering (3.7), we can get that $X = C(\theta, \gamma)$ is a constant independent of $\overline{\delta}$. By (2.7)–(2.8), the gradient estimates can be obtained by Lemma 3.2 directly, similar to [7].

Lemma 3.4 *$\overline{\partial}^\pm u$, $\overline{\partial}^\pm v$, $\overline{\partial}^\pm \alpha$ and $\overline{\partial}^\pm \beta$ are all uniformly bounded for the C^1 solution in $D_{\overline{\delta}}$.*

Lemma 3.5 *Assume that there is a C^1 solution in $D_{\overline{\delta}}$, where the system is hyperbolic $(\alpha - \beta \neq 0, \pi)$. Then the C^1 norm of α, β, u, v have a uniform bound $C = C(\theta, \gamma)$:*

$$\|(\alpha, \beta, u, v)\|_{C^1(D_{\overline{\delta}})} \leq C. \tag{3.16}$$

Theorem 3.3 ($C^{1,1}$ Estimates) *Assume that there exists a smooth solution in the domain $D_{\overline{\delta}}$. Then, there exists a constant $C(\theta, \gamma)$, such that*

$$\|(\alpha, \beta, u, v)\|_{C^{1,1}(D_{\overline{\delta}})} \leq C. \tag{3.17}$$

Proof From Lemma 3.5 and (3.16), it suffices to prove that $|\overline{\nabla}^2 \alpha| < C$, $|\overline{\nabla}^2 \beta| < C$, where $\overline{\nabla}^2 = (\overline{\partial}^+\overline{\partial}^+, \overline{\partial}^-\overline{\partial}^+, \overline{\partial}^+\overline{\partial}^-, \overline{\partial}^-\overline{\partial}^-)$. From the first equation in (2.10) and (3.16), we get $|\overline{\partial}^+\overline{\partial}^- \alpha| < C$ easily. By the commutator relation (2.9), we get

$$|\overline{\partial}^-\overline{\partial}^+ \alpha| < C|\overline{\partial}^- \alpha| + C|\overline{\partial}^+ \alpha| < C. \tag{3.18}$$

Using (2.9) for $I = \overline{\partial}^+ \alpha$, we get

$$\overline{\partial}^-\overline{\partial}^+\overline{\partial}^+ \alpha = \overline{\partial}^+\overline{\partial}^-\overline{\partial}^+ \alpha + W(\alpha, \beta, \overline{\nabla}\alpha, \overline{\nabla}\beta, \overline{\partial}^-\overline{\partial}^+ \alpha, \overline{\partial}^+\overline{\partial}^+ \alpha). \tag{3.19}$$

From (2.8)–(2.10) and (3.18), we get a first-order equation of $\overline{\partial}^+\overline{\partial}^+ \alpha$,

$$\overline{\partial}^-(\overline{\partial}^+\overline{\partial}^+ \alpha) = P(\alpha, \beta, \overline{\nabla}\alpha)\overline{\partial}^+\overline{\partial}^+ \alpha + Q(\alpha, \beta, \overline{\nabla}\alpha), \tag{3.20}$$

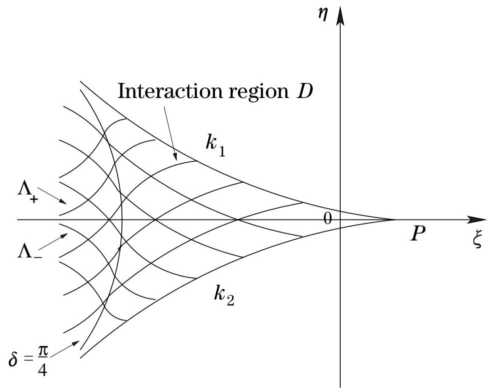


Figure 3.3 Convexity types of characteristics as $0 < \theta \le \overline{\theta}$

where P and Q are algebraic functions of $\alpha, \beta, \overline{\nabla}\alpha$. Integrating (3.20) along the direction $\overline{\partial}^-$ and considering (3.16), we get that $|\overline{\partial}^+\overline{\partial}^+\alpha| < C$. Similarly, we get that $|\overline{\partial}^-\overline{\partial}^-\alpha| < C$. The results of β can be obtained in a similar way.

Through the prior estimates, we could extend the local solution to the global smooth solution.

Theorem 3.4 (Global Existence) *There exists a unique global smooth solution to the interaction of two rarefaction waves with the interaction* (*half*) *angle* $\theta \in (0, \overline{\theta}]$. *As shown in Figure* 3.3, *the* $\Lambda_\pm$ *characteristics are concave and convex, respectively, before they hit the curve* $\delta = \frac{\pi}{4}$, *and the* $\Lambda_\pm$ *characteristics are convex and concave, respectively, after they cross the curve* $\delta = \frac{\pi}{4}$.

Proof The proof follows from the previous results, including Theorems 3.1–3.3 and the fact that the curve $\xi = \xi_{\overline{\delta}}$ is non-characteristic. Here, we omit the details since they are similar to those in [6, 14]. Differentiating (2.5), we get

$$\begin{cases} \overline{\partial}^+\Lambda_+ = \sec^2\alpha\overline{\partial}^+\alpha = -\sec^2\alpha\cos(2\delta)\dfrac{-\overline{\partial}^+\alpha}{\cos(2\delta)}, \\ \overline{\partial}^-\Lambda_- = \sec^2\beta\overline{\partial}^-\beta = \sec^2\beta\cos(2\delta)\dfrac{\overline{\partial}^-\beta}{\cos(2\delta)}. \end{cases} \tag{3.21}$$

By Lemma 3.2, we get the convexity types immediately.

Remark 3.1 The global existence of the solution to the expansion problem of the isothermal Euler equations for $0 < \theta < \frac{\pi}{2}$ is obtained from the results above and Theorem 5.8 in [7].

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