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Brake Subharmonic Solutions of Subquadratic Hamiltonian Systems*

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Abstract The author mainly uses the Galerkin approximation method and the iteration inequalities of the L-Maslov type index theory to study the properties of brake subharmonic solutions for the Hamiltonian systems $\dot{z}(t) = J\nabla H(t, z(t))$, where $H(t, z) = \frac{1}{2}(\widehat{B}(t)z, z) + \widehat{H}(t, z)$, $\widehat{B}(t)$ is a semipositive symmetric continuous matrix and $\widehat{H}$ is unbounded and not uniformly coercive. It is proved that when the positive integers j and k satisfy the certain conditions, there exists a jT-periodic nonconstant brake solution z_j such that z_j and z_{kj} are distinct.

Keywords Brake subharmonic solution, L-Maslov type index, Hamiltonian systems
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1 Introduction and the Main Results

Consider the Hamiltonian systems

$$\dot{z}(t) = J\nabla H(t, z(t)), \quad \forall z \in \mathbb{R}^{2n},\ \forall t \in \mathbb{R}, \tag{1.1}$$

where $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, I_n is the unit matrix of order n, $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and $\nabla H(t, z)$ is the gradient of $H(t, z)$ with respect to the space variable z. We denote the standard norm and inner product in $\mathbb{R}^{2n}$ by $|\cdot|$ and $(\cdot, \cdot)$, respectively.

Suppose that $H(t, z) = \frac{1}{2}(\widehat{B}(t)z, z) + \widehat{H}(t, z)$ and $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ satisfies the following conditions:

(H1) $\widehat{H}(T + t, z) = \widehat{H}(t, z)$ for all $z \in \mathbb{R}^{2n},\ t \in \mathbb{R}$;

(H2) $\widehat{H}(t, z) = \widehat{H}(-t, Nz)$ for all $z \in \mathbb{R}^{2n},\ t \in \mathbb{R},\ N = \left(\begin{smallmatrix} -I_n & 0 \\ 0 & I_n \end{smallmatrix}\right)$;

(H3) $\widehat{H}''(t, z) > 0$ for all $z \in \mathbb{R}^{2n}\backslash\{0\},\ t \in \mathbb{R}$;

(H4) There exist constants $a_1, a_2 > 0$ and $\alpha \in (0, 1)$ such that

$$|\nabla \widehat{H}(t, z)| \leq a_1|z|^{\alpha} + a_2 \quad \text{for all } z \in \mathbb{R}^{2n},\ t \in [0, T];$$

(H5) $\lim\limits_{|z|\to+\infty} |z|^{-2\alpha} \int_0^T \widehat{H}(t, z)\mathrm{d}t = +\infty$;

(H6) $\widehat{B}(t)$ is a symmetrical continuous matrix, $|\widehat{B}|_{C^0} \leq \beta_0$ for some $\beta_0 > 0$, and $\widehat{B}(t)$ is a semi-positively definite for all $t \in \mathbb{R}$;

(H7) $\widehat{B}(T + t) = \widehat{B}(t) = \widehat{B}(-t)$, $\widehat{B}(t)N = N\widehat{B}(t)$ for all $t \in \mathbb{R}$.

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Recall that a T-periodic solution (z, T) of (1.1) is called a brake solution if $z(t+T) = z(t)$ and $z(t) = Nz(-t)$, and the later is equivalent to $z\big(\frac{T}{2}+t\big) = Nz\big(\frac{T}{2}-t\big)$, in which T is called the brake period of z. Up to the author's knowledge, H. Seifert firstly studied brake orbits in the second-order autonomous Hamiltonian systems in [27] of 1948. Since then, many studies have been carried out for brake orbits of the first-order and second-order Hamiltonian systems. For the minimal periodic problem, multiple existence results about brake orbits for the Hamiltonian systems and more details about brake orbits, one can refer to the papers (see [1, 3–6, 11–13, 20, 22, 25, 29]) and the references therein. S. Bolotin proved first in [5] (also see [6]) of 1978 the existence of brake orbits in the general setting. K. Hayashi in [13], H. Gluck and W. Ziller in [11], and V. Benci in [3] in 1983–1984 proved the existence of brake orbits of second-order Hamiltonian systems under certain conditions. In 1987, P. Rabinowitz in [25] proved the existence of brake orbits of the first-order Hamiltonian systems. In 1989, V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [4]. In 1989, A. Szulkin in [29] proved the existence of brake orbits of the first-order Hamiltonian systems under the $\sqrt{2}$-pinched condition. E. van Groesen in [12] of 1988 and A. Ambrosetti, V. Benci, Y. Long in [1] of 1993 also proved the multiplicity result about brake orbits for the second order Hamiltonian systems under different pinching conditions. Without pinching conditions, in [22] Y. Long, D. Zhang and C. Zhu proved that there exist at least two geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbb{R}^n$ for $n \geq 2$. Recently, C. Liu and D. Zhang in [20] proved that there exist at least $\big[\frac{n}{2}\big]+1$ geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbb{R}^n$ for $n \geq 2$, and there exist at least n geometrically distinct brake orbits on the nondegenerate domain. D. Zhang studied the minimal period problem for brake orbits of nonlinear autonomous reversible Hamiltonian systems in [30].

For the non-autonomous Hamiltonian systems, and the periodic boundary (brake solution) problems, since the Hamiltonian function H is T-periodic in the time variable t, if the system (1.1) has a T-periodic solution (z_1, T), one hopes to find the jT-periodic solution (z_j, jT) for integer $j \geq 1$, for example, (z_1, jT) itself is a jT-periodic solution. The subharmonic solution problem asks when the solutions z_1 and z_j are geometrically distinct. More precisely, in the case of brake solutions, z_1 and z_j are distinct if $\frac{kT}{2} * z_1(\cdot) \equiv z_1(\frac{kT}{2}+\cdot) \neq z_j(\cdot)$ for any integer k. Below we remind that the L_0-indices of the two solutions z_1 and $(kT) * z_1$ for any $k \in \mathbb{Z}$ in the interval $\big[0, \frac{T}{2}\big]$ are the same.

Theorem 1.1 *Suppose that $H \in C^2(\mathbb{R}\times\mathbb{R}^{2n}, \mathbb{R})$ satisfies* (H1)–(H7)*, and then for each integer $1 \leq j < \frac{2\pi}{\beta_0 T}$, there is a jT-periodic nonconstant brake solution z_j of* (1.1) *such that z_j and z_{kj} are distinct for $k \geq 5$ and $kj < \frac{2\pi}{\beta_0 T}$. Furthermore, $\{z_{k^p} \mid p \in \mathbb{N}\}$ is a pairwise distinct brake solution sequence of* (1.1) *for $k \geq 5$ and $1 \leq k^p < \frac{2\pi}{\beta_0 T}$.*

Especially, if $\widehat{B}(t) \equiv 0$, then $\frac{2\pi}{\beta_0 T} = +\infty$. Therefore, one can state the following theorem.

Theorem 1.2 *Suppose that $H \in C^2(\mathbb{R}\times\mathbb{R}^{2n}, \mathbb{R})$ with $\widehat{B}(t) \equiv 0$ satisfies* (H1)–(H5)*, and then for each integer $j \geq 1$, there is a jT-periodic nonconstant brake solution z_j of* (1.1)*. Furthermore, given any integers $j \geq 1$ and $k \geq 5$, z_j and z_{kj} are distinct brake solutions of* (1.1)*, and in particularly, $\{z_{k^p} \mid p \in \mathbb{N}\}$ is a pairwise distinct brake solution sequence of* (1.1).

The first result on subharmonic periodic solutions for the Hamiltonian systems $\dot{z}(t) = J\nabla H(t, z(t))$, where $z \in \mathbb{R}^{2n}$ and $H(t, z)$ is T-periodic in t, was obtained by P. Rabinowitz in his pioneer work [26]. Since then, many new contributions have appeared (see, for example, [8–9, 19, 21, 28] and the references therein). Especially, in [9], I. Ekeland and H. Hofer proved that under a strict convex condition and a superquadratic condition, the Hamiltonian system $\dot{z}(t) = J\nabla H(t, z(t))$ possesses a subharmonic solution z_k for each integer $k \geq 1$ and all of these solutions are pairwise geometrically distinct. In [19], C. Liu obtained a result of subharmonic solutions for the non-convex case by using the Maslov-type index iteration theory. In [14], the author of this paper and C. Liu obtained a result of brake subharmonic solutions for the superquadratic condition by using the L-Maslov type index iteration theory. For the subquadratic Hamiltonian systems, P. Rabinowitz [26] proved the existence of subharmonic solutions for the Hamiltonian system (1.1) under conditions (H4)–(H5) for the special case $\alpha = 0$. In [28], E. A. B. Silva obtained the existence of subharmonic solutions for the Hamiltonian system (1.1) under conditions (H4)–(H5), by establishing a new version of a saddle point theorem for strongly indefinite functionals which satisfy a generalization of the well-known (PS) condition. In this paper, we mainly use the L-Maslov type index iteration theory to study the brake subharmonic solutions under the subquadratic conditions.

The main ingredient in proving Theorems 1.1–1.2 is to transform the brake solution problem into the L_0-boundary problem:

$$\begin{cases} \dot{z}(t) = J\nabla H(t, z(t)), \quad \forall z \in \mathbb{R}^{2n},\ \forall t \in \left[0, \dfrac{T}{2}\right], \\ z(0) \in L_0, \quad z\left(\dfrac{T}{2}\right) \in L_0, \end{cases} \tag{1.2}$$

where $L_0 = \{0\} \oplus \mathbb{R}^n \in \Lambda(n)$. $\Lambda(n)$ is the set of all linear Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0)$, where the standard symplectic form is defined by $\omega_0 = \sum\limits_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$. A Lagrangian subspace L of $\mathbb{R}^{2n}$ is an n dimensional subspace satisfying $\omega_0|_L = 0$. Then we use the Galerkin approximation methods to get a critical point of the action functional which is also a solution of (3.1) with a suitable L_0-index estimate (see Theorem 3.1 below).

The L-Maslov type index theory for any $L \in \Lambda(n)$ was studied in [17] by the algebraic methods. In [22], Y. Long, D. Zhang and C. Zhu established two indices $\mu_1(\gamma)$ and $\mu_2(\gamma)$ for the fundamental solution γ of a linear Hamiltonian system by the methods of functional analysis which are special cases of the L-Maslov type index $i_L(\gamma)$ for Lagrangian subspaces $L_0 = \{0\} \oplus \mathbb{R}^n$ and $L_1 = \mathbb{R}^n \oplus \{0\}$ up to a constant n. In order to prove Theorem 1.1, we need to consider the problem (3.1). The iteration theory of the L_0-Maslov type index theory was developed in [18] and [20], which helps us to distinguish solutions z_j from z_{kj} in Theorems 1.1–1.2.

This paper is divided into 3 sections. In Section 2, we give an introduction to the Maslov-type index theory for symplectic paths with Lagrangian boundary conditions and an iteration theory for the L_0-Maslov type index theory. In Section 3, we give the proofs of Theorems 1.1–1.2.

2 Preliminaries

In this section, we briefly recall the Maslov-type index theory for symplectic paths with Lagrangian boundary conditions and an iteration theory for the L_0-Maslov type index theory. All the details can be found in [16–18, 20].

We denote the $2n$-dimensional symplectic group $Sp(2n)$ by

$$Sp(2n) = \{M \in \mathscr{L}(\mathbb{R}^{2n}) \mid M^T J M = J\},$$

where $\mathscr{L}(\mathbb{R}^{2n})$ is the set of all real $2n \times 2n$ matrices, and M^{T} is the transpose of matrix M. Denote by $\mathscr{L}_s(\mathbb{R}^{2n})$ the subset of $\mathscr{L}(\mathbb{R}^{2n})$ consisting of symmetric matrices. And denote the symplectic path space by

$$\mathcal{P}(2n) = \{\gamma \in C([0,1], Sp(2n)) \mid \gamma(0) = I_{2n}\}.$$

We write a symplectic path $\gamma \in \mathcal{P}(2n)$ in the following form:

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix}, \tag{2.1}$$

where $S(t)$, $T(t)$, $V(t)$ and $U(t)$ are $n \times n$ matrices. The n vectors that come from the column of the matrix $\begin{pmatrix} V(t) \\ U(t) \end{pmatrix}$ are linearly independent and they span a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$. Particularly, at $t = 0$, this Lagrangian subspace is $L_0 = \{0\} \oplus \mathbb{R}^n$.

Definition 2.1 (see [17]) *We define the L_0-nullity of any symplectic path $\gamma \in \mathcal{P}(2n)$ by*

$$\nu_{L_0}(\gamma) \equiv \dim \ker_{L_0}(\gamma(1)) := \dim \ker V(1) = n - \mathrm{rank}\ V(1)$$

with the $n \times n$ matrix function $V(t)$ defined in (2.1).

For $L_0 = \{0\} \oplus \mathbb{R}^n$, We define the following subspaces of $Sp(2n)$ by

$$\begin{aligned} Sp(2n)^*_{L_0} &= \{M \in Sp(2n) \mid \det V_M \neq 0\}, \\ Sp(2n)^0_{L_0} &= \{M \in Sp(2n) \mid \det V_M = 0\}, \\ Sp(2n)^{\pm}_{L_0} &= \{M \in Sp(2n) \mid \pm \det V_M > 0\}, \end{aligned}$$

where $M = \begin{pmatrix} S_M & V_M \\ T_M & U_M \end{pmatrix}$ and $Sp(2n)^*_{L_0} = Sp(2n)^+_{L_0} \cup Sp(2n)^-_{L_0}$. We denote two subsets of $\mathcal{P}(2n)$ by

$$\mathcal{P}(2n)^*_{L_0} = \{\gamma \in \mathcal{P}(2n) \mid \nu_{L_0}(\gamma) = 0\}, \quad \mathcal{P}(2n)^0_{L_0} = \{\gamma \in \mathcal{P}(2n) \mid \nu_{L_0}(\gamma) > 0\}.$$

We note that rank $\begin{pmatrix} V(t) \\ U(t) \end{pmatrix} = n$, so the complex matrix $U(t) \pm \sqrt{-1}V(t)$ is invertible. We define a complex matrix function by

$$Q(t) = (U(t) - \sqrt{-1}V(t))(U(t) + \sqrt{-1}V(t))^{-1}.$$

It is easy to see that the matrix $Q(t)$ is a unitary matrix for any $t \in [0,1]$. We define

$$M_+ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n = \mathrm{diag}\ (-1, 1, \cdots, 1).$$

For a path $\gamma \in \mathcal{P}(2n)^*_{L_0}$, we first adjoin it with a simple symplectic path starting from $J = -M_+$, that is, we define a symplectic path by

$$\widetilde{\gamma}(t) = \begin{cases} I \cos \dfrac{(1-2t)\pi}{2} + J \sin \dfrac{(1-2t)\pi}{2}, & t \in \left[0, \dfrac{1}{2}\right], \\ \gamma(2t-1), & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Then we choose a symplectic path $\beta(t)$ in $Sp(2n)^*_{L_0}$ starting from $\gamma(1)$ and ending at M_+ or M_- according to $\gamma(1) \in Sp(2n)^+_{L_0}$ or $\gamma(1) \in Sp(2n)^-_{L_0}$, respectively. We now define a joint path by

$$\overline{\gamma}(t) = \beta * \widetilde{\gamma} := \begin{cases} \widetilde{\gamma}(2t), & t \in \left[0, \dfrac{1}{2}\right], \\ \beta(2t-1), & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

By the definition, we see that the symplectic path $\overline{\gamma}$ starts from $-M_+$ and ends at either M_+ or M_-. As above, we define

$$\overline{Q}(t) = (\overline{U}(t) - \sqrt{-1}\overline{V}(t))(\overline{U}(t) + \sqrt{-1}\overline{V}(t))^{-1}$$

for $\overline{\gamma}(t) = \left(\begin{smallmatrix} \overline{S}(t) & \overline{V}(t) \\ \overline{T}(t) & \overline{U}(t) \end{smallmatrix}\right)$. We can choose a continuous function $\overline{\Delta}(t)$ in [0,1] such that

$$\det \overline{Q}(t) = \mathrm{e}^{2\sqrt{-1}\overline{\Delta}(t)}.$$

By the above arguments, we see that the number $\frac{1}{\pi}(\overline{\Delta}(1) - \overline{\Delta}(0)) \in \mathbb{Z}$ and it does not depend on the choice of the function $\overline{\Delta}(t)$.

Definition 2.2 (see [17]) *For a symplectic path $\gamma \in \mathcal{P}(2n)^*_{L_0}$, we define the L_0-index of γ by $i_{L_0}(\gamma) = \frac{1}{\pi}(\overline{\Delta}(1) - \overline{\Delta}(0))$.*

Definition 2.3 (see [17]) *For a symplectic path $\gamma \in \mathcal{P}(2n)^0_{L_0}$, we define the L_0-index of γ by $i_{L_0}(\gamma) = \inf\{i_{L_0}(\widetilde{\gamma}) \mid \widetilde{\gamma} \in \mathcal{P}(2n)^*_{L_0}$, and $\widetilde{\gamma}$ is sufficiently close to $\gamma\}$.*

We know that $\Lambda(n) = \frac{U(n)}{O(n)}$, which means that for any linear subspace $L \in \Lambda(n)$, there is an orthogonal symplectic matrix $P = \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$ with $A \pm \sqrt{-1}B \in U(n)$, the unitary matrix, such that $PL_0 = L$. P is uniquely determined by L up to an orthogonal matrix $C \in O(n)$. It means that for any other choice P' satisfying the above conditions, there exists a matrix $C \in O(n)$ such that $P' = P\left(\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right)$ (see [23]). We define the conjugated symplectic path $\gamma_c \in \mathcal{P}(2n)$ of γ by $\gamma_c(t) = P^{-1}\gamma(t)P$.

Definition 2.4 (see [17]) *We define the L-nullity of any symplectic path $\gamma \in \mathcal{P}(2n)$ by*

$$\nu_L(\gamma) \equiv \dim \ker_L(\gamma(1)) := \dim \ker V_c(1) = n - \mathrm{rank}\ V_c(1),$$

where the $n \times n$ matrix function $V_c(t)$ is defined in (2.1) *with the symplectic path γ replaced by γ_c, i.e., $\gamma_c(t) = \left(\begin{smallmatrix} S_c(t) & V_c(t) \\ T_c(t) & U_c(t) \end{smallmatrix}\right)$.*

Definition 2.5 (see [17]) *For a symplectic path $\gamma \in \mathcal{P}(2n)$, we define the L-index of γ by $i_L(\gamma) = i_{L_0}(\gamma_c)$.*

In the case of linear Hamiltonian systems,

$$\dot{y} = JB(t)y, \quad \forall y \in \mathbb{R}^{2n}, \tag{2.2}$$

where $B \in C(\mathbb{R}, \mathscr{L}_s(\mathbb{R}^{2n}))$. Its fundamental solution $\gamma = \gamma_B$ is a symplectic path starting from the identity matrix I_{2n}, i.e., $\gamma = \gamma_B \in \mathcal{P}(2n)$. We denote

$$i_L(B) = i_L(\gamma_B), \quad \nu_L(B) = \nu_L(\gamma_B).$$

Theorem 2.1 (see [17]) *Suppose that $\gamma \in \mathcal{P}(2n)$ is a fundamental solution of* (2.2) *with* $B(t) > 0$. *There holds* $i_L(\gamma) \geq 0$.

Suppose that the continuous symplectic path $\gamma : [0,2] \to Sp(2n)$ is the fundamental solution of (2.2) with $B(t)$ satisfying $B(t+2) = B(t)$ and $B(1+t)N = NB(1-t)$. This implies that $B(t)N = NB(-t)$. By the unique existence theorem of the differential equations, we get

$$\gamma(1+t) = N\gamma(1-t)\gamma(1)^{-1}N\gamma(1), \quad \gamma(2+t) = \gamma(t)\gamma(2).$$

We define the iteration path of $\gamma|_{[0,1]}$ by

$$\begin{aligned}
\gamma^1(t) &= \gamma(t), \quad t \in [0,1],\\
\gamma^2(t) &= \begin{cases} \gamma(t), \quad t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), \quad t \in [1,2], \end{cases}\\
\gamma^3(t) &= \begin{cases} \gamma(t), \quad t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), \quad t \in [1,2], \\ \gamma(t-2)\gamma(2), \quad t \in [2,3], \end{cases}\\
\gamma^4(t) &= \begin{cases} \gamma(t), \quad t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), \quad t \in [1,2], \\ \gamma(t-2)\gamma(2), \quad t \in [2,3], \\ N\gamma(4-t)\gamma(1)^{-1}N\gamma(1)\gamma(2), \quad t \in [3,4], \end{cases}
\end{aligned}$$

and in general, for $k \in \mathbb{N}$, we define

$$\gamma^{2k-1}(t) = \begin{cases} \gamma(t), \quad t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), \quad t \in [1,2], \\ \cdots\cdots \\ N\gamma(2k-2-t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2k-5}, \quad t \in [2k-3, 2k-2], \\ \gamma(t-2k+2)\gamma(2)^{2k-4}, \quad t \in [2k-2, 2k-1], \end{cases}$$

$$\gamma^{2k}(t) = \begin{cases} \gamma(t), \quad t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), \quad t \in [1,2], \\ \cdots\cdots \\ \gamma(t-2k+2)\gamma(2)^{2k-4}, \quad t \in [2k-2, 2k-1], \\ N\gamma(2k-t)\gamma(1)^{-1}N\gamma(1)\gamma(2)^{2k-3}, \quad t \in [2k-1, 2k]. \end{cases}$$

Recall that $(i_\omega(\gamma), \nu_\omega(\gamma))$ is the ω-index pair of the symplectic path γ introduced in [21], and $(i_\omega^{L_0}(\gamma), \nu_\omega^{L_0}(\gamma))$ is defined in [20].

Theorem 2.2 (see [20]) *Suppose that $\omega_k = \mathrm{e}^{\pi\frac{\sqrt{-1}}{k}}$. For odd k we have*

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} i_{\omega_k^{2i}}(\gamma^2), \quad \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} \nu_{\omega_k^{2i}}(\gamma^2),$$

and for even k, we have

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i^{L_0}_{\omega_k^{\frac{k}{2}}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} i_{\omega_k^{2i}}(\gamma^2),$$

$$\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \nu^{L_0}_{\omega_k^{\frac{k}{2}}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} \nu_{\omega_k^{2i}}(\gamma^2),$$

where $\omega_k^{\frac{k}{2}} = \sqrt{-1}$.

Theorem 2.3 (see [20]) *There hold*

$$i_1(\gamma^2) = i_{L_0}(\gamma^1) + i_{L_1}(\gamma^1) + n, \quad \nu_1(\gamma^2) = \nu_{L_0}(\gamma^1) + \nu_{L_1}(\gamma^1),$$

where $L_1 = \mathbb{R}^n \oplus \{0\} \in \Lambda(n)$.

In the following section, we need the following two iteration inequalities.

Theorem 2.4 (see [18]) *For any* $\gamma \in \mathcal{P}(2n)$ *and* $k \in \mathbb{N}$, *there hold*

$$\begin{aligned} & i_{L_0}(\gamma^1) + \frac{k-1}{2}(i_1(\gamma^2) + \nu_1(\gamma^2) - n) \le i_{L_0}(\gamma^k) \\ \le\ & i_{L_0}(\gamma^1) + \frac{k-1}{2}(i_1(\gamma^2) + n) - \frac{1}{2}\nu_1(\gamma^{2k}) + \frac{1}{2}\nu_1(\gamma^2), \quad \text{if } k \in 2\mathbb{N}-1, \\ & i_{L_0}(\gamma^1) + i^{L_0}_{\sqrt{-1}}(\gamma^1) + \Big(\frac{k}{2}-1\Big)(i_1(\gamma^2) + \nu_1(\gamma^2) - n) \le i_{L_0}(\gamma^k) \\ \le\ & i_{L_0}(\gamma^1) + i^{L_0}_{\sqrt{-1}}(\gamma^1) + \Big(\frac{k}{2}-1\Big)(i_1(\gamma^2) + n) \\ & - \frac{1}{2}\nu_1(\gamma^{2k}) + \frac{1}{2}\nu_1(\gamma^2) + \frac{1}{2}\nu_{-1}(\gamma^2), \quad \text{if } k \in 2\mathbb{N}. \end{aligned}$$

Remark 2.1 From (3.21) of [20] and Proposition B of [22], we have that

$$i_{L_0}(B) \le i^{L_0}_{\omega}(B) \le i_{L_0}(B) + n, \quad |i_{L_0}(B) - i_{L_1}(B)| \le n,$$

where $L_1 = \mathbb{R}^n \oplus \{0\} \in \Lambda(n)$.

3 Proof of Theorems 1.1–1.2

In reference [14], we have proved the following Lemma 3.1.

Lemma 3.1 *Suppose that the Hamiltonian function* H *satisfies* (H1)–(H2) *and* (H7). *If* $\big(z, \frac{T}{2}\big)$ *is a solution of the problem* (1.2), *then* $(\widetilde{z}, T)$ *is a* T*-periodic solution of the Hamiltonian system* (1.1) *satisfying the brake condition* $\widetilde{z}\big(\frac{T}{2}+t\big) = N\widetilde{z}\big(\frac{T}{2}-t\big)$, *where* $\widetilde{z}$ *is defined by*

$$\widetilde{z}(t) = \begin{cases} z(t), & t \in \Big[0, \dfrac{T}{2}\Big], \\ Nz(T-t), & t \in \Big(\dfrac{T}{2}, T\Big]. \end{cases}$$

By this observation, we consider the following Hamiltonian system:

$$\begin{cases} \dot{z}(t) = J\nabla H(t, z(t)), \quad \forall z \in \mathbb{R}^{2n},\ \forall t \in \Big[0, \dfrac{jT}{2}\Big], \\ z(0) \in L_0, \quad z\Big(\dfrac{jT}{2}\Big) \in L_0, \end{cases} \tag{3.1}$$

where $j \in \mathbb{N}$. The following result is the first part of Theorem 1.1.

Theorem 3.1 *Suppose that* $H(t,z) \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ *satisfies* (H4)–(H6), *and then* (3.1) *possesses at least one nontrivial solution* z_j *whose* L_0*-index pair* $(i_{L_0}(z_j), \nu_{L_0}(z_j))$ *satisfies*

$$i_{L_0}(z_j) \le 1 \le i_{L_0}(z_j) + \nu_{L_0}(z_j).$$

So we get a nonconstant brake solution $(\widetilde{z}_j, jT)$ *with a brake period* jT *of the Hamiltonian system* (1.1) *by Lemma* 3.1.

In order to prove Theorem 3.1, we need the following arguments. For simplicity, we suppose $T=2$. Let $X := \left\{z \in W^{\frac{1}{2},2}([0,j],\mathbb{R}^{2n}) \mid z = \sum\limits_{l\in\mathbb{Z}} \mathrm{e}^{\frac{l\pi}{j}Jt} z_l,\ z_l \in L_0,\ \|z\|_X < +\infty\right\}$ be the Hilbert space with the inner product

$$(u,v)_X = j(u_0,v_0) + j\sum_{l\in\mathbb{Z}} |l|(u_l,v_l), \quad \forall u,v \in X.$$

In the following, we use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to denote the inner product and the norm in X, respectively. It is well known that for any $z \in X$, one has $z \in L^r([0,j],\mathbb{R}^{2n})$ for any $r \in [1,+\infty)$, and there exists a constant $c_r > 0$ such that $\|z\|_{L^r} \le c_r\|z\|$.

We define the linear operators A and $\widehat{B}$ on X by extending the bilinear form

$$\langle Au,v\rangle = \int_0^j (-J\dot{u},v)\mathrm{d}t, \quad \langle \widehat{B}u,v\rangle = \int_0^j (\widehat{B}(t)u,v)\mathrm{d}t.$$

Then $\widehat{B}$ is a compact self-adjoint operator (see [21]) and A is a self-adjoint operator, i.e., $\langle Au,v\rangle = \langle u,A^*v\rangle = \langle u,Av\rangle$.

We take the spaces

$$\begin{aligned}
X_m &= \Big\{z \in X \mid z = \sum_{l=-m}^{m} \mathrm{e}^{\frac{l\pi}{j}Jt} z_l,\ z_l \in L_0\Big\},\\
X^+ &= \Big\{z \in X \mid z = \sum_{l>0} \mathrm{e}^{\frac{l\pi}{j}Jt} z_l,\ z_l \in L_0\Big\},\\
X^- &= \Big\{z \in X \mid z = \sum_{l<0} \mathrm{e}^{\frac{l\pi}{j}Jt} z_l,\ z_l \in L_0\Big\},\\
X^0 &= L_0,
\end{aligned}$$

and $X_m^+ = X_m \cap X^+$, $X_m^- = X_m \cap X^-$. We have $X_m = X_m^+ \oplus X^0 \oplus X_m^-$. We also know that

$$\langle Az,z\rangle = \frac{\pi}{j}\|z\|^2, \quad \forall z \in X_m^+, \tag{3.2}$$

$$\langle Az,z\rangle = -\frac{\pi}{j}\|z\|^2, \quad \forall z \in X_m^-. \tag{3.3}$$

Equalities (3.2) and (3.3) can be proved by definition and direct computation. Let $P^m : X \to X_m$ be the corresponding orthogonal projection for $m \in \mathbb{N}$. Then $\Gamma = \{P^m;\ m \in \mathbb{N}\}$ is a Galerkin approximation scheme with respect to A (see [16]).

For any Lagrangian subspace $L \in \Lambda(n)$, suppose $P \in Sp(2n) \cap O(2n)$ such that $L = PL_0$. Then we define $X_L = PX$ and $X_L^m = PX_m$. Let $P^m : X_L \to X_L^m$. Then as above, $\overline{\Gamma} = \{P^m;\ m \in \mathbb{N}\}$ is a Galerkin approximation scheme with respect to A. For $d > 0$, we denote by $M_d^*(Q)$, $* = +,0,-$, the eigenspaces corresponding to the eigenvalues λ of the linear operator $Q : X_L \to X_L$ belonging to $[d,+\infty)$, $(-d,d)$ and $(-\infty,-d]$, respectively. And denote by $M^*(Q)$, $* = +,0,-$, the eigenspaces corresponding to the eigenvalues λ of Q belonging to $(0,+\infty)$, $\{0\}$ and $(-\infty,0)$, respectively. For any adjoint operator Q, we denote $Q^\sharp = (Q|_{ImQ})^{-1}$, and we also denote $P^mQP^m = (P^mQP^m)|_{X_L^m}$. The following result is the well-known Galerkin approximation formula, which is proved in [16].

Theorem 3.2 *For any $B(t) \in C([0,1],\mathscr{L}_s(\mathbb{R}^{2n}))$ with its L-index pair $(i_L(B),\nu_L(B))$ and any constant $0 < d \le \frac{1}{4}\|(A-B)^\sharp\|^{-1}$, there exists $m_0 > 0$ such that for $m \ge m_0$, we have*

$$\dim M_d^+(P^m(A-B)P^m) = mn - i_L(B) - \nu_L(B),$$

$$\dim M_d^-(P^m(A-B)P^m) = mn + i_L(B) + n,$$
$$\dim M_d^0(P^m(A-B)P^m) = \nu_L(B).$$

Define a function φ on X by

$$\varphi(z) = \frac{1}{2}\langle Az, z\rangle - \frac{1}{2}\langle \widehat{B}z, z\rangle - \int_0^j \widehat{H}(t, z(t))\mathrm{d}t.$$

Suppose that W is a real Banach space, $g \in C^1(W, \mathbb{R})$. g is said to satisfy the (PS) condition, if for any sequence $\{x_q\} \subset W$ satisfying that $g(x_q)$ is bounded and $g'(x_q) \to 0$ as $q \to \infty$, there exists a convergent subsequence $\{x_{q_j}\}$ of $\{x_q\}$ (see [24]). Let $\varphi_m = \varphi|_{X_m}$ be the restriction of φ on X_m. When H satisfies (H4) and (H5), by Proposition A in [2], we have the following two lemmas.

Lemma 3.2 *For all $m \in \mathbb{N}$, φ_m satisfies the* (PS) *condition on X_m.*

Lemma 3.3 *φ satisfies the* (PS)* *condition on X with respect to $\{z_m\}$, i.e., for any sequence $\{z_m\} \subset X$ satisfying that $z_m \in X_m$, $\varphi_m(z_m)$ is bounded and that $\|\varphi_m'(z_m)\|_{(X_m)'} \to 0$ in $(X_m)'$ as $m \to +\infty$, where $(X_m)'$ is the dual space of X_m, there exists a convergent subsequence $\{z_{m_j}\}$ of $\{z_m\}$ in X.*

In order to prove Theorem 3.1, we need the following definition and the saddle-point theorem.

Definition 3.1 (see [10]) *Let E be a C^2-Riemannian manifold and D be a closed subset of E. A family $\phi(\alpha)$ of subsets of E is said to be a homological family of dimensional q with boundary D if for some nontrivial class, $\alpha \in H_q(E, D)$. The family $\phi(\alpha)$ is defined by*

$$\phi(\alpha) = \{G \subset E : \alpha \text{ is in the image of } i_* : H_q(G, D) \to H_q(E, D)\},$$

where i_* is the homomorphism induced by the immersion $i : G \to E$.

Theorem 3.3 (see [10]) *For the above E, D and α, let $\phi(\alpha)$ be a homological family of dimension q with boundary D. Suppose that $f \in C^2(E, \mathbb{R})$ satisfies the* (PS) *condition. Define*

$$c = \inf_{G \in \phi(\alpha)} \sup_{x \in G} f(x).$$

Suppose that $\sup_{x \in D} f(x) < c$ and f' is Fredholm on

$$\mathscr{K}_c(f) \equiv \{x \in E : f'(x) = 0, f(x) = c\}.$$

Then there exists an $x \in \mathscr{K}_c(f)$ such that the Morse index $m^-(x)$ and the nullity $m^0(x)$ of the functional f at x satisfy

$$m^-(x) \le q \le m^-(x) + m^0(x).$$

It is clear that a critical point of φ is a solution of (3.1). For a critical point $z = z(t)$, let $B(t) = H''(t, z(t))$, and define the linearized systems at $z(t)$ by

$$\begin{cases} \dot{y}(t) = JH''(t, z(t))y(t), & \forall y \in \mathbb{R}^{2n},\ \forall t \in [0, j], \\ y(0) \in L_0, \quad y(j) \in L_0. \end{cases}$$

Then the L_0-index pair of z is defined by $(i_{L_0}(z), \nu_{L_0}(z)) = (i_{L_0}(B), \nu_{L_0}(B))$.

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1 We carry out the proof in 2 steps.

Step 1 The critical points of φ_m.

Set $S_m = X_m^- \oplus X^0$. Then $\dim S_m = mn + \dim X^0 = mn + \dim \ker A = mn + n$, $\dim X_m^+ = mn$.

In the following, we prove that $\varphi_m(z)$ satisfies:

(I) $\varphi_m(z) \geq \beta > 0$, $\forall z \in Y_m = X_m^+ \cap \partial B_\rho(0)$,

(II) $\varphi_m(z) \leq 0 < \beta$, $\forall z \in \partial Q_m$, where $Q_m = \{re \mid r \in [0, r_1]\} \oplus (B_{r_2}(0) \cap S_m)$, $e \in X_m^+ \cap \partial B_1(0)$, $r_1 > \rho$, $r_2 > 0$.

First we prove (I). By (H4), we have $\widehat{H}(t,z) \leq d_1|z|^{1+\alpha} + d_2|z| + d_3$, where d_1, d_2, $d_3 > 0$. Take $z \in Y_m$, and then

$$\begin{aligned}
\Big|\int_0^j \widehat{H}(t,z)\mathrm{d}t\Big| &\leq \int_0^j |\widehat{H}(t,z)|\mathrm{d}t \\
&\leq d_1\|z\|_{L^{\alpha+1}}^{\alpha+1} + d_2\|z\|_{L^1} + d_3 j \\
&\leq d_1 c_{\alpha+1}^{\alpha+1}\|z\|^{\alpha+1} + d_2 c_1\|z\| + d_3 j \\
&= \overline{d_1}\|z\|^{\alpha+1} + \overline{d_2}\|z\| + d_3 j.
\end{aligned} \tag{3.4}$$

Hence by (3.2) and (3.4),

$$\begin{aligned}
\varphi_m(z) &= \frac{1}{2}\langle Az, z\rangle - \frac{1}{2}\langle \widehat{B}z, z\rangle - \int_0^j \widehat{H}(t, z(t))\mathrm{d}t \\
&\geq \frac{\pi}{2j}\|z\|^2 - \frac{\beta_0}{2}\|z\|^2 - \int_0^j \widehat{H}(t, z(t))\mathrm{d}t \\
&\geq \frac{\pi}{2j}\|z\|^2 - \frac{\beta_0}{2}\|z\|^2 - \overline{d_1}\|z\|^{\alpha+1} - \overline{d_2}\|z\| - d_3 j \\
&= \Big(\frac{\pi}{2j} - \frac{\beta_0}{2}\Big)\rho^2 - \overline{d_1}\rho^{\alpha+1} - \overline{d_2}\rho - d_3 j.
\end{aligned}$$

Since $1 \leq j < \frac{\pi}{\beta_0}$, choose a large enough $\rho > 0$ independent of m such that for $z \in Y_m$, $\varphi_m(z) \geq \beta > 0$. Hence (I) holds.

Now we prove (II). Let $e \in X_m^+ \cap \partial B_1$ and $z = z^- + z^0 \in S_m$. By (3.2) and (3.3),

$$\begin{aligned}
\varphi_m(z + re) &= \frac{1}{2}\langle Az^-, z^-\rangle + \frac{1}{2}r^2\langle Ae, e\rangle - \frac{1}{2}\langle \widehat{B}(z+re), z+re\rangle - \int_0^j \widehat{H}(t, z+re)\mathrm{d}t \\
&\leq -\frac{\pi}{2j}\|z^-\|^2 + \frac{\pi}{2j}r^2 - \int_0^j \widehat{H}(t, z+re)\mathrm{d}t \\
&= -\frac{\pi}{2j}\|z^-\|^2 + \frac{\pi}{2j}r^2 - \int_0^j \widehat{H}(t, z^0)\mathrm{d}t - \int_0^j [\widehat{H}(t, z+re) - \widehat{H}(t, z^0)]\mathrm{d}t,
\end{aligned}$$

we have

$$\begin{aligned}
&\Big|\int_0^j \widehat{H}(t, z+re)\mathrm{d}t - \int_0^j \widehat{H}(t, z^0)\mathrm{d}t\Big| \\
&\leq \int_0^j \int_0^1 |\nabla \widehat{H}(t, z^0 + sw)| \cdot |w|\mathrm{d}s\mathrm{d}t \\
&\leq \int_0^j a_1|z^0 + w|^\alpha \cdot |w|\mathrm{d}t + \int_0^j a_2|w|\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^j a_1(|z^0|^\alpha + |w|^\alpha)|w|\mathrm{d}t + \int_0^j a_2|w|\mathrm{d}t \\
&\le a_1|z^0|^\alpha\|w\|_{L^1} + a_1\|w\|_{L^{\alpha+1}}^{\alpha+1} + a_2\|w\|_{L^1} \\
&\le \widetilde{a_1}|z^0|^\alpha\|w\| + \overline{a_1}\|w\|^{\alpha+1} + \widetilde{a_2}\|w\| \\
&= (\widetilde{a_1}|z^0|^\alpha + \widetilde{a_2})(\|z^-\| + r) + \overline{a_1}(\|z^-\| + r)^{\alpha+1},
\end{aligned}$$

where $w = z^- + re$ and $\|w\| = \|z^-\| + r$, $\widetilde{a_1} = a_1c_1$, $\overline{a_1} = a_1c_{\alpha+1}^{\alpha+1}$ and $\widetilde{a_2} = a_2c_1$. Then we can obtain

$$\begin{aligned}
\varphi_m(z + re) &\le -\frac{\pi}{2j}\|z^-\|^2 + \frac{\pi}{2j}r^2 - \int_0^j \widehat{H}(t, z^0)\mathrm{d}t - \int_0^j [\widehat{H}(t, z + re) - \widehat{H}(t, z^0)]\mathrm{d}t \\
&\le -\frac{\pi}{2j}\|z^-\|^2 + \frac{\pi}{2j}r^2 - \int_0^j \widehat{H}(t, z^0)\mathrm{d}t + (\widetilde{a_1}|z^0|^\alpha + \widetilde{a_2})(\|z^-\| + r) \\
&\quad + \overline{a_1}(\|z^-\| + r)^{\alpha+1}.
\end{aligned}$$

It follows from (H5) that $\int_0^j H(t, v)\mathrm{d}t$ is bounded from below on $\mathbb{R}^{2n}$, so then $-\int_0^j H(t, z^0)\mathrm{d}t$ has an upper bound. Choose r_1 and r_2 independent of m such that $\varphi_m(z + re) \le 0 < \beta$ on ∂Q_m. Hence (II) holds.

Because Q_m is the deformation retract of X_m, then $H_q(Q_m, \partial Q_m) \cong H_q(X_m, \partial Q_m)$, where $q = \dim S_m + 1 = mn + n + 1 = \dim Q_m$, and ∂Q_m is the boundary of Q_m in $S_m \oplus \{\mathbb{R}e\}$. But $H_q(Q_m, \partial Q_m) \cong H_{q-1}(S^{q-1}) \cong \mathbb{R}$. Denote by $i : Q_m \to X_m$ the inclusion map. Let $\alpha = [Q_m] \in H_q(Q_m, D)$ be a generator. Then $i_*\alpha$ is nontrivial in $H_q(X_m, \partial Q_m)$, and $\phi(i_*\alpha)$ defined by Definition 3.1 is a homological family of dimension q with boundary $D := \partial Q_m$ and $Q_m \in \phi(i_*\alpha)$. ∂Q_m and Y_m are homologically linked (see [7]). By Lemma 3.2, φ_m satisfies the (PS) condition. Define $c_m = \inf\limits_{G \in \phi(i_*\alpha)} \sup\limits_{z \in G} \varphi_m(z)$. We have

$$\sup_{z \in \partial Q_m} \varphi_m(z) \le 0 < \beta \le c_m \le \sup_{z \in Q_m} \varphi_m(z) \le \frac{\pi}{2j}r_1^2. \tag{3.5}$$

Since X_m is finite dimensional, φ_m' is Fredholm. By Theorem 3.3, φ_m has a critical point z_j^m with critical value c_m, and the Morse index $m^-(z_j^m)$ and nullity $m^0(z_j^m)$ of z_j^m satisfy

$$m^-(z_j^m) \le mn + n + 1 \le m^-(z_j^m) + m^0(z_j^m). \tag{3.6}$$

Since $\{c_m\}$ is bounded, passing to a subsequence, suppose $c_m \to c \in [\beta, \frac{\pi}{2j}r_1^2]$. By the (PS)* condition of Lemma 3.3, passing to a subsequence, there exists a $z_j \in X$ such that

$$z_j^m \to z_j, \quad \varphi(z_j) = c, \quad \varphi'(z_j) = 0.$$

Step 2 Let $B(t) = H''(t, z_j(t))$, $d = \frac{1}{4}\|(A - B)^\sharp\|^{-1}$. Since

$$\|\varphi''(x) - (A - B)\| \to 0 \quad \text{as } \|x - z_j\| \to 0,$$

there exists an $r_3 > 0$ such that

$$\|\varphi''(x) - (A - B)\| < \frac{1}{4}d, \quad \forall x \in V_{r_3}(z_j) = \{x \in X \mid \|x - z_j\| \le r_3\}.$$

Then for m large enough, there holds

$$\|\varphi_m''(x) - P^m(A - B)P^m\| < \frac{1}{2}d, \quad \forall x \in V_{r_3}(z_j) \cap X_m. \tag{3.7}$$

For $x \in V_{r_3}(z_j) \cap X_m,\ \forall u \in M_d^-(P^m(A-B)P^m) \setminus \{0\}$, from (3.7) we have

$$\begin{aligned}\langle \varphi_m''(x)u, u\rangle &\le \langle P^m(A-B)P^m u, u\rangle + \|\varphi_m''(x) - P^m(A-B)P^m\| \cdot \|u\|^2 \\ &\le -\frac{1}{2}d\|u\|^2 < 0. \end{aligned}\tag{3.8}$$

Thus by (3.8),

$$\dim M^-(\varphi_m''(x)) \ge \dim M_d^-(P^m(A-B)P^m), \quad \forall x \in V_{r_3}(z_j) \cap X_m. \tag{3.9}$$

Similarly, we have

$$\dim M^+(\varphi_m''(x)) \ge \dim M_d^+(P^m(A-B)P^m), \quad \forall x \in V_{r_3}(z_j) \cap X_m. \tag{3.10}$$

By Theorem 3.2 and (3.6), (3.9)–(3.10), for large m we have

$$\begin{aligned} mn+n+1 &\ge m^-(z_j^m) \\ &\ge \dim M_d^-(P^m(A-B)P^m) \\ &= mn + i_{L_0}(B) + n. \end{aligned}\tag{3.11}$$

We also have

$$\begin{aligned} mn+n+1 &\le m^-(z_j^m) + m^0(z_j^m) \\ &\le \dim M_d^-(P^m(A-B)P^m) \oplus \dim M_d^0(P^m(A-B)P^m) \\ &= mn + i_{L_0}(B) + n + \nu_{L_0}(B). \end{aligned}\tag{3.12}$$

Combining (3.11) and (3.12), we have

$$i_{L_0}(z_j) \le 1 \le i_{L_0}(z_j) + \nu_{L_0}(z_j).$$

The proof of Theorem 3.1 is complete.

It is the time to give the proof of Theorem 1.1.

Proof of Theorem 1.1 For $1 \le k < \frac{\pi}{\beta_0}$, by Theorem 3.1, we obtain that there is a nontrivial solution (z_k, k) of the Hamiltonian systems (3.1) and its L_0-index pair satisfies

$$i_{L_0}(z_k,k) \le 1 \le i_{L_0}(z_k,k) + \nu_{L_0}(z_k,k). \tag{3.13}$$

Then by Lemma 3.1, $(\widetilde{z}_k, 2k)$ is a nonconstant brake solution of (1.1).

For $k \in 2\mathbb{N}-1$, we suppose that $(\widetilde{z}_1, 2)$ and $(\widetilde{z}_k, 2k)$ are not distinct. By (3.13), Theorems 2.3–2.4, we have

$$\begin{aligned} 1 \ge i_{L_0}(z_k,k) &\ge i_{L_0}(z_1,1) + \frac{k-1}{2}(i_1(\widetilde{z}_1,2) + \nu_1(\widetilde{z}_1,2) - n) \\ &\ge i_{L_0}(z_1,1) + \frac{k-1}{2}(i_{L_0}(z_1,1) + i_{L_1}(z_1,1) + n + \nu_{L_0}(z_1,1) + \nu_{L_1}(z_1,1) - n) \\ &= i_{L_0}(z_1,1) + \frac{k-1}{2}(i_{L_0}(z_1,1) + i_{L_1}(z_1,1) + \nu_{L_0}(z_1,1) + \nu_{L_1}(z_1,1)), \end{aligned}\tag{3.14}$$

where $L_1 = \mathbb{R}^n \oplus \{0\} \in \Lambda(n)$. By (H3), (H6) and Theorem 2.1, we have $i_{L_1}(z_1,1) \ge 0$. We also know that $\nu_{L_1}(z_1,1) \ge 0$ and $i_{L_0}(z_1,1) + \nu_{L_0}(z_1,1) \ge 1$. Then (3.14) is

$$1 \ge i_{L_0}(z_1,1) + \frac{k-1}{2}. \tag{3.15}$$

By $0 \le i_{L_0}(z_1, 1) \le 1$, from (3.15) we have $\frac{k-1}{2} \le 1$, i.e., $k \le 3$. It is contradictory to $k \ge 5$. Similarly, we have that for each $k \in 2\mathbb{N} - 1$, $k \ge 5$ and $kj < \frac{\pi}{\beta_0}$, $1 \le j < \frac{\pi}{\beta_0}$, $(\widetilde{z}_j, 2j)$ and $(\widetilde{z}_{kj}, 2kj)$ are distinct brake solutions of (1.1). Furthermore, $(\widetilde{z}_1, 2)$, $(\widetilde{z}_k, 2k)$, $(\widetilde{z}_{k^2}, 2k^2)$, $(\widetilde{z}_{k^3}, 2k^3)$, $\cdots$, $(\widetilde{z}_{k^p}, 2k^p)$ are pairwise distinct brake solutions of (1.1), where $k \in 2\mathbb{N} - 1$, $k \ge 5$ and $1 \le k^p < \frac{\pi}{\beta_0}$ with $p \in \mathbb{N}$.

For $k \in 2\mathbb{N}$, as above, we suppose that $(\widetilde{z}_1, 2)$ and $(\widetilde{z}_k, 2k)$ are not distinct. By (3.13), Theorems 2.3–2.4, we have

$$\begin{aligned}
1 &\ge i_{L_0}(z_k, k) \\
&\ge i_{L_0}(z_1, 1) + i^{L_0}_{\sqrt{-1}}(z_1, 1) + \Big(\frac{k}{2} - 1\Big)(i_1(\widetilde{z}_1, 2) + \nu_1(\widetilde{z}_1, 2) - n) \\
&\ge i_{L_0}(z_1, 1) + i^{L_0}_{\sqrt{-1}}(z_1, 1) + \Big(\frac{k}{2} - 1\Big)(i_{L_0}(z_1, 1) + i_{L_1}(z_1, 1) + n \\
&\quad + \nu_{L_0}(z_1, 1) + \nu_{L_1}(z_1, 1) - n) \\
&= i_{L_0}(z_1, 1) + i^{L_0}_{\sqrt{-1}}(z_1, 1) + \Big(\frac{k}{2} - 1\Big)(i_{L_0}(z_1, 1) + i_{L_1}(z_1, 1) + \nu_{L_0}(z_1, 1) + \nu_{L_1}(z_1, 1)). \qquad (3.16)
\end{aligned}$$

Similarly, we also know that $i_{L_1}(z_1, 1) \ge 0$, $\nu_{L_1}(z_1, 1) \ge 0$, $i_{L_0}(z_1, 1) + \nu_{L_0}(z_1, 1) \ge 1$. By Remark 2.1, we have $i^{L_0}_{\sqrt{-1}}(z_1, 1) \ge i_{L_0}(z_1, 1) \ge 0$. Then (3.16) is

$$1 \ge i_{L_0}(z_1, 1) + \Big(\frac{k}{2} - 1\Big). \qquad (3.17)$$

By $0 \le i_{L_0}(z_1, 1) \le 1$, from (3.17) we have $\frac{k}{2} - 1 \le 1$, i.e., $k \le 4$. It contradicts $k \ge 5$. Similarly we have that for each $k \in 2\mathbb{N}$, $k \ge 6$ and $kj < \frac{\pi}{\beta_0}$, $1 \le j < \frac{\pi}{\beta_0}$, $(\widetilde{z}_j, 2j)$ and $(\widetilde{z}_{kj}, 2kj)$ are distinct brake solutions of (1.1). Furthermore, $(\widetilde{z}_1, 2)$, $(\widetilde{z}_k, 2k)$, $(\widetilde{z}_{k^2}, 2k^2)$, $(\widetilde{z}_{k^3}, 2k^3)$, $\cdots$, $(\widetilde{z}_{k^p}, 2k^p)$ are pairwise distinct brake solutions of (1.1), where $k \in 2\mathbb{N}$, $k \ge 6$ and $1 \le k^p < \frac{\pi}{\beta_0}$ with $p \in \mathbb{N}$.

In all, for any integer $1 \le j < \frac{\pi}{\beta_0}$, $\widetilde{z}_j$ and $\widetilde{z}_{kj}$ are distinct brake solutions of (1.1) for $k \ge 5$ and $kj < \frac{\pi}{\beta_0}$. Furthermore, $\{\widetilde{z}_{k^p} \mid p \in \mathbb{N}\}$ is a pairwise distinct brake solution sequence of (1.1) for $k \ge 5$ and $1 \le k^p < \frac{\pi}{\beta_0}$. The proof of Theorem 1.1 is complete.

We note that Theorem 1.2 is a direct consequence of Theorem 1.1.

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