Chin. Ann. Math.
37B(3), 2016, 419–426
DOI: 10.1007/s11401-016-0940-1

Chinese Annals of Mathematics, Series B

Adapted Metrics and Webster Curvature in Finslerian 2-Dimensional Geometry

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Abstract The Webster scalar curvature is computed for the sphere bundle T_1S of a Finsler surface (S, F) subject to the Chern-Hamilton notion of adapted metrics. As an application, it is derived that in this setting $(T_1S, g_{\text{Sasaki}})$ is a Sasakian manifold homothetic with a generalized Berger sphere, and that a natural Cartan structure is arising from the horizontal 1-forms and the author associates a non-Einstein pseudo-Hermitian structure. Also, one studies when the Sasaki type metric of T_1S is generally adapted to the natural co-frame provided by the Finsler structure.

Keywords Webster curvature, Finsler geometry, Sasakian type metric on tangent bundle, Sphere bundle, Adapted metric, Cartan structure, Pseudo-Hermitian structure

2000 MR Subject Classification 53C60, 58B20, 53D10, 53C56

1 Introduction

The present note introduces the Webster scalar curvature discussed by Chern and Hamilton in [5] into the framework of 2-dimensional Finsler geometry. More precisely, we compute the Webster curvature for the sphere bundle T_1S of a Finsler surface $(S, F(x, y))$ by using the structural equations of this bundle. Specifically, the condition of adapted metric of [5] is suitable for only one 1-form (namely ω_3) of the natural co-frame of T_1S endowed with the Sasaki type metric g_{Sasaki} induced by F. This condition, called vertical adapted, reduces the discussion to the Riemannian surfaces by the vanishing of the main scalar I and yields the constant Gaussian curvature $K = 2$. It follows that the Webster curvature is $\frac{1}{2}$ and a natural Cartan structure (in terms of [8, p. 148]) is given by the horizontal 1-forms. Let us remark that an interplay between Cartan structures and the generalized Finsler structures is studied in [13–14].

We apply this computation to prove a structure result, that is, T_1S with g_{Sasaki} is homothetic with a generalized Berger sphere. More precisely, we obtain that under the vertical adapted condition, the vector field e_3, dual of ω_3 with respect to g_{Sasaki}, is a Killing vector field for this metric and then it makes $(g_{\text{Sasaki}}, \omega_3)$ a Sasakian structure on T_1S. Another important result is that in our setting ω_3 is a pseudo-Hermitian form corresponding to a CR structure on T_1S. Although this pseudo-Hermitian structure is non-Einstein, we obtain that its Webster scalar curvature is again $\frac{1}{2}$.

In order to extend the class of metrics, we generalize the concept of adapted metrics; in fact, we modify the original condition of Chern-Hamilton from the scalar 2 to a general $\rho \in \mathbb{R}$

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in order to cover all possibilities; this approach was used in [6]. Also our study is enlarged to all 1-forms providing the natural co-frame of T_1S.

2 Webster Scalar Curvature: The Chern-Hamilton Formalism

Fix (M^3, g) to be a 3-dimensional Riemannian manifold and consider $\{\omega_1, \omega_2, \omega_3\}$ as an orthonormal basis of 1-forms on M; then M is oriented with the volume form $\omega_1 \wedge \omega_2 \wedge \omega_3$. Then there exists a unique skew-symmetric matrix of 1-forms

$$\begin{pmatrix} 0 & \varphi_3 & -\varphi_2 \\ -\varphi_3 & 0 & \varphi_1 \\ \varphi_2 & -\varphi_1 & 0 \end{pmatrix},$$

such that the structural equations

$$\begin{cases} \mathrm{d}\omega_1 = \varphi_2 \wedge \omega_3 - \varphi_3 \wedge \omega_2, \\ \mathrm{d}\omega_2 = \varphi_3 \wedge \omega_1 - \varphi_1 \wedge \omega_3, \\ \mathrm{d}\omega_3 = \varphi_1 \wedge \omega_2 - \varphi_2 \wedge \omega_1 \end{cases} \tag{2.1}$$

hold on M. Making one step further, we derive the existence of the functions $\{K_{ij}; 1 \leq i, j \leq 3\}$ such that $K_{ij} = K_{ji}$ and

$$\begin{cases} \mathrm{d}\varphi_1 = \varphi_2 \wedge \varphi_3 + K_{11}\omega_2 \wedge \omega_3 + K_{12}\omega_3 \wedge \omega_1 + K_{13}\omega_1 \wedge \omega_2, \\ \mathrm{d}\varphi_2 = \varphi_3 \wedge \varphi_1 + K_{21}\omega_2 \wedge \omega_3 + K_{22}\omega_3 \wedge \omega_1 + K_{23}\omega_1 \wedge \omega_2, \\ \mathrm{d}\varphi_3 = \varphi_1 \wedge \varphi_2 + K_{31}\omega_2 \wedge \omega_3 + K_{32}\omega_3 \wedge \omega_1 + K_{33}\omega_1 \wedge \omega_2. \end{cases} \tag{2.2}$$

Recall that the subject of [5] consists in adapted metrics for a contact 1-form ω, i.e., Riemannian metrics satisfying

$$\|\omega\| = 1, \quad \mathrm{d}\omega = 2 * \omega. \tag{2.3}$$

If g is adapted to ω_3, then the Webster scalar curvature W of the triple (M, g, ω_3) is defined as

$$W(M, g, \omega_3) = \frac{1}{8}(K_{11} + K_{22} + 2K_{33} + 4) \tag{2.4}$$

and is computed in [5] for the unit sphere $\mathbb{S}^3$, the unit tangent bundle of a compact orientable surface of genus $g \neq 1$ (for $g = 0$ it results in $W = 1$) and the Heisenberg group Nil_3. In fact, $W(\mathbb{S}^3) = 1$ and $W(Nil_3) = 0$. For another formalism on Webster curvature, see [3, p. 212] and our formula (5.4) below.

A last main notion of this note is that of Cartan structure according to Definition 1.1 of [8, p. 148]: A pair of 1-forms ω_1, ω_2 with

$$\omega_1 \wedge \mathrm{d}\omega_1 = \omega_2 \wedge \mathrm{d}\omega_2 (\neq 0), \quad \omega_1 \wedge \mathrm{d}\omega_2 = 0 = \omega_2 \wedge \mathrm{d}\omega_1. \tag{2.5}$$

3 Finsler 2-Dimensional Geometry and Adapted Metrics

Let S be a 2-dimensional manifold and $\pi : TS \to S$ its tangent bundle. Let $x = (x^i) = (x^1, x^2)$ be the local coordinates on S and $(x, y) = (x^i, y^i) = (x^1, x^2, y^1, y^2)$ the induced local coordinates on TS. Denote by O the null-section of π.

Recall that a Finsler fundamental function on S is a map $F : TS \to \mathbb{R}_+$ with the following properties:

(F1) F is smooth on the slit tangent bundle $TS \setminus O$ and continuous on O;

(F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$;

(F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite.

The tensor field $(g_{ij}(x, y))$ is called the Finsler metric.

Due to the homogeneity condition, all important objects of Finsler geometry actually live on the sphere bundle $p : T_1S = \{(x, y) \in TS; F(x, y) = 1\} \to S$ (see [2, p. 9]). Here T_1S is 3-dimensional and an adapted co-frame consists in three 1-forms denoted by ω_1, ω_2, ω_3. More precisely, after [2, p. 93], we have

$$\begin{cases} \omega_1 = \dfrac{\sqrt{g}}{F}(y^2\mathrm{d}x^1 - y^1\mathrm{d}x^2) := m_1\mathrm{d}x^1 + m_2\mathrm{d}x^2, \\ \omega_2 = F_{y^1}\mathrm{d}x^1 + F_{y^2}\mathrm{d}x^2 := l_1\mathrm{d}x^1 + l_2\mathrm{d}x^2, \\ \omega_3 = \dfrac{\sqrt{g}}{F^2}(y^2\delta y^1 - y^1\delta y^2) = \dfrac{m_1}{F}\delta y^1 + \dfrac{m_2}{F}\delta y^2, \end{cases} \tag{3.1}$$

where $g = \det(g_{ij})$, $F_{y^i} = \frac{\partial F}{\partial y^i}$ and $\delta y^i = \mathrm{d}y^i + N^i_j\mathrm{d}x^j$ with $(N^i_j(x, y))$ being the canonical nonlinear connection of the Finsler geometry (S, F) (see [2, p. 34]). The vector fields $\left(\frac{\partial}{\partial y^i}\right)$ span the vertical distribution while $\left(\frac{\delta}{\delta x^i}\right)$ span the horizontal distribution. The Finsler metric yields the Sasaki type metric on T_1S (see [2, p. 93]):

$$g_{\mathrm{Sasaki}} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 \tag{3.2}$$

making $\{\omega_1, \omega_2, \omega_3\}$ an orthonormal co-frame. If $\{e_1, e_2, e_3\}$ is the dual frame, then e_1 and e_2 are horizontal while e_3 is vertical.

After [2, p. 82], the structural equations of (S, F) are

$$\begin{cases} \mathrm{d}\omega_1 = -I\omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3, \\ \mathrm{d}\omega_2 = -\omega_1 \wedge \omega_3, \\ \mathrm{d}\omega_3 = K\omega_1 \wedge \omega_2 - J\omega_1 \wedge \omega_3, \end{cases} \tag{3.3}$$

where I, J, K are smooth functions defined as follows (see [2, p. 82]):

(i) I is the Cartan (or main) (pseudo-)scalar. Its vanishing characterizes Riemannian surfaces, i.e., $g = g(x)$ which means that $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$ and g_{Sasaki} on TS is exactly the Sasaki lift of the Riemannian metric g. It also follows that $N^i_j(x, y) = \Gamma^i_{jk}(x)y^k$ with $(\Gamma^{\cdot}_{\cdot\cdot})$ being the Christoffel symbols of g.

(ii) J is the Landsberg (pseudo-)scalar. Its vanishing characterizes Landsberg surfaces.

(iii) K is the Gaussian curvature. Its vanishing characterizes flat (in the Finslerian sense) surfaces. Note that ω_3 is a contact form for non-flat Finslerian surfaces since $\omega_3 \wedge \mathrm{d}\omega_3 = \omega_3 \wedge (K\omega_1 \wedge \omega_2 - J\omega_1 \wedge \omega_3) = K\omega_1 \wedge \omega_2 \wedge \omega_3$. Then e_3 can be called the Reeb vector field of (S, F).

Remark that Bianchi equations yield some relations between these functions (see [2, p. 97]):

$$I_3 = J, \quad J_3 = -KI - K_2, \tag{3.4}$$

where the subscript i denotes the derivation in the direction of e_i, i.e., $\mathrm{d}f = f_1\omega_1 + f_2\omega_2 + f_3\omega_3$. It follows that $I = 0$ implies $J = 0$ and also $K_2 = 0$.

In order to enlarge the class of suitable metrics, we consider the following notion which appears (with a factor 2 in RHS) in [11].

Definition 3.1 *Fix a* 1*-form* ω *on a general* (M^3, g) *and the real number* ρ*. The Riemannian metric* g *on* M *is called* ρ*-adapted to* ω *if* $\mathrm{d}\omega = \rho * \omega$.

We conclude from (3.3) the following proposition.

Proposition 3.1 *The metric* $g_{Sasakian}$ *is*
(i) 1*-adapted to the* ω_1 *if and only if* S *is a Riemannian surface;*
(ii) 1*-adapted to* ω_2*;*
(iii) K*-adapted to* ω_3 *in the Landsberg case.*

It follows that the lift of the round metric of S^2 to $T_1S^2 = \mathbb{R}P^3 = SO(3)$ is 1-adapted all ω's.

4 Webster Curvature in Finslerian Geometry of Surfaces

Comparing (2.3) with (3.3), it results that g_{Sasaki} can be an adapted metric only for ω_3, in which case we say that it is vertical adapted due to the character of the Reeb vector field e_3; correspondingly the 1-forms ω_1, ω_2 will be called horizontal. We are ready for the main result of this note.

Theorem 4.1 *The Riemannian metric* g_{Sasaki} *of* T_1S *is vertical adapted if and only if* S *is a Riemannian surface with* $K = 2$*. Then, the horizontal pair* (ω_1, ω_2) *is a Cartan structure and the Webster curvature is*

$$W(T_1S, g_{\mathrm{Sasaki}}, \omega_3) = \frac{1}{2}. \tag{4.1}$$

Proof Since ω_i is a g_{Sasaki}-orthonormal co-frame, we have $*\omega_3 = \omega_1 \wedge \omega_2$, and locking at (3.3_3), we get that g_{Sasaki} is vertical adapted if and only if $J = 0,\ K = 2$. From the second Bianchi relation (3.4), we deduce that $I = 0$, which yields the first part of the conclusion.

Now, the structural equations have the expression

$$\begin{cases} \mathrm{d}\omega_1 = \omega_2 \wedge \omega_3, \\ \mathrm{d}\omega_2 = -\omega_1 \wedge \omega_3, \\ \mathrm{d}\omega_3 = 2\omega_1 \wedge \omega_2, \end{cases} \tag{4.2}$$

and then we get the relations (2.5) with $\omega_1 \wedge \mathrm{d}\omega_1 = \omega_2 \wedge \mathrm{d}\omega_2 = \omega_1 \wedge \omega_2 \wedge \omega_3 =$ being the volume form of the metric g_{Sasaki}. It also follows that

$$\varphi_1 = \omega_1, \quad \varphi_2 = \omega_2, \quad \varphi_3 = 0. \tag{4.3}$$

It results in

$$\begin{cases} \mathrm{d}\varphi_1 = \omega_2 \wedge \omega_3, \\ \mathrm{d}\varphi_2 = -\omega_1 \wedge \omega_3, \\ \mathrm{d}\varphi_3 = 0 \end{cases} \tag{4.4}$$

which gives the matrix of K's:

$$K_{11} = K_{22} = 1, \quad K_{33} = -1, \tag{4.5}$$

all other being zero. Using the definition (2.4), it results in the Webster curvature (4.1).

Remark 4.1 (i) Comparing our result with the second example of [5, p. 285] gives that K_{ii} given by (4.5) coincides with relations (22) of the cited paper for $\epsilon = \frac{1}{2} = W$.

(ii) If S is compact embedded in $\mathbb{R}^3$ (being also oriented), then a classical sphere theorem (from 1897) of Hadamard states that S must be diffeomorphic with a sphere. The following Theorem 4.2 clarifies this claim.

(iii) In [7], the 1-form $\eta = I\omega_3$ is introduced under the name Cartan-type form of (S, F) and it is proved that $\eta \wedge \mathrm{d}\eta$ is the Chern-Simons form of (S, F). In our setting, this Chern-Simons form is zero.

(iv) A Cartan structure is a particular case of taut contact circle according to the Definition 1.1 of [8, p. 148] and then any linear combination $\lambda_1\omega_1 + \lambda_2\omega_2$ with $(\lambda_1, \lambda_2) \in S^1 \subset \mathbb{R}^2$ defines the same volume form, and in our case that is the form of g_{Sasaki}.

As an application of the previous theorem, we have the following structural result.

Theorem 4.2 *If the Riemannian metric g_{Sasaki} of T_1S is vertical adapted, then the manifold $(T_1S, g_{\text{Sasaki}})$ is Sasakian and homothetic with a generalized Berger sphere.*

Proof According to the classification of [9, p. 124], $W = \frac{1}{2}$ implies that if $(T_1S, g_{\text{Sasaki}}, \omega_3)$ is a Sasakian manifold, then it is homothetic with a generalized Berger sphere. Hence we must prove that the vertical adapted condition implies the Sasakian condition for g_{Sasaki}. But from [3, p. 87], we know that in dimension 3 this is equivalent to the cu K-contact condition and then we prove that the vertical adapted condition implies that e_3 is a Killing vector field for g_{Sasaki}.

According to [4, p. 28], we have the general Lie brackets:

$$[e_1, e_2] = -Ke_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = -Ie_1 - e_2 - Je_3 \tag{4.6}$$

which yields the Levi-Civita connection of g_{Sasaki}:

$$\begin{cases} \nabla_{e_1}e_1 = -Ie_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_3}e_3 = Je_1, \\ \nabla_{e_1}e_2 = -\dfrac{K}{2}e_3, \quad \nabla_{e_1}e_3 = Ie_1 + \dfrac{K}{2}e_2, \quad \nabla_{e_2}e_3 = -\dfrac{K}{2}e_1, \\ \nabla_{e_2}e_1 = \dfrac{K}{2}e_3, \quad \nabla_{e_3}e_1 = \Big(\dfrac{K}{2} - 1\Big)e_2 - Je_3, \quad \nabla_{e_3}e_2 = -\Big(\dfrac{K}{2} - 1\Big)e_1. \end{cases} \tag{4.7}$$

Let $X = X^ie_i$ and $Y = Y^ie_i$ be two arbitrary vector fields on T_1S, we get

$$\begin{cases} \nabla_X e_1 = -(IX^1 + JX^3)e_1 + \Big(\dfrac{K}{2} - 1\Big)X^3e_2 + \dfrac{K}{2}X^2e_3, \\ \nabla_X e_2 = -\Big(\dfrac{K}{2} + 1\Big)X^3e_1 - \dfrac{K}{2}X^1e_3, \\ \nabla_X e_3 = \Big[IX^1 - \dfrac{K}{2}X^2 + JX^3\Big]e_1 + \dfrac{K}{2}X^1e_2. \end{cases} \tag{4.8}$$

It follows that the Lie derivatives of the metric are

$$\begin{cases}\mathcal{L}_{e_1} g_{\text{Sasaki}}(X,Y) = -2IX^1J^1 - J(X^1Y^3 + X^3Y^1) + (K-1)(X^2Y^3 + X^3Y^2),\\ \mathcal{L}_{e_2} g_{\text{Sasaki}}(X,Y) = -(K+1)(X^1Y^3 + X^3Y^1),\\ \mathcal{L}_{e_3} g_{\text{Sasaki}}(X,Y) = 2IX^1Y^1 + J(X^1Y^3 + X^3Y^1).\end{cases} \tag{4.9}$$

The vertical adapted condition gives then

$$\begin{cases}\mathcal{L}_{e_1} g_{\text{Sasaki}}(X,Y) = X^2Y^3 + X^3Y^2,\\ \mathcal{L}_{e_2} g_{\text{Sasaki}}(X,Y) = -3(X^1Y^3 + X^3Y^1),\\ \mathcal{L}_{e_3} g_{\text{Sasaki}}(X,Y) = 0\end{cases} \tag{4.10}$$

and we have the final conclusion.

Remark 4.2 (i) The relations in first line of (4.7) yield that under the vertical adapted condition all vector fields e_i are geodesic: $\nabla_{e_i} e_i = 0$. Also, we can determine the generalized Berger sphere structure of $(T_1S^2, g_{\text{Sasaki}})$ according to the computations of [12]. More precisely, we consider $SU(2) = S^3$ with the natural left-invariant and orthonormal frame (X_1, X_2, X_3) of [12, p. 7], and g_{Sasaki} is the metric making orthonormal the frame: $e_1 = \frac{X_2}{\sqrt{2}}$, $e_2 = \frac{X_3}{\sqrt{2}}$, $e_3 = -\frac{X_3}{2}$ as in [12, p. 81].

(ii) Let us remark that our contact structure on T_1S is different from that of [3, p. 175] for which the K-contact condition is characterized via the well-known Tashiro theorem ([3, p. 178]) in terms of constant curvature $+1$ for the base manifold $(S, g(x))$. Let us also note that the Finslerian version of the Tashiro theorem was proved in [1].

(iii) Our Theorem 4.2 is a particular case of Lemma A.1 of Alan Weinstein from the Appendix of [5] that $\varphi_1 = \omega_1$, $\varphi_2 = \omega_2$ implies e_3 is a Killing vector field. Also, from the complex structural equations (39) of [5, p. 290], it follows that $\Omega = \omega_1 + \mathrm{i}\omega_2$ is a closed differential 1-form: $\mathrm{d}\Omega = 0$.

5 An Associated Pseudo-Hermitian Structure on T_1S

From the third equation of (4.8), it results that the vertical adapted condition implies

$$\nabla_X e_3 = -[X^2e_1 - X^1e_2], \tag{5.1}$$

and recall, after [3, p. 87], that the Sasakian condition reads

$$\nabla_X e_3 = -\phi(X) \tag{5.2}$$

in terms of the structural tensor field ϕ of $(1,1)$-type. It gives the expression of ϕ:

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0. \tag{5.3}$$

Let $\mathcal{D} = \ker \omega_3$ be the structural distribution associated to ω_3. A second formula for the Webster scalar formula is [3, p. 213]:

$$W(M, g, \omega_3) = \frac{1}{8}(\tau - \mathrm{Ric}(e_3) + 4), \tag{5.4}$$

where τ is the scalar curvature of the metric g and $\mathrm{Ric}(e_3)$ is the Ricci curvature in the direction of e_3. Note also that in the same way as [3, p. 214], we have

$$\tau = 2K(\mathcal{D}) + 2\mathrm{Ric}(e_3), \tag{5.5}$$

where $K(\mathcal{D})$ is the sectional curvature of the 2-plane $\mathcal{D}$ and from Theorem 7.1 of [3, p. 112] on the 3-dimensional K-contact case it results that $\mathrm{Ric}(e_3) = 2$. Using the Levi-Civita connection (4.7), we obtain $K(\mathcal{D}) = -1$, so then $\tau = 2$ and from (4.14) we arrive again at $W = \frac{1}{2}$.

Remark also that $J = \phi|_{\mathcal{D}}$ is a complex structure satisfying the integrability conditions:

$$[JX, Y] + [X, JY] \in \mathcal{D}, \quad J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] = 0 \tag{5.6}$$

for all $X, Y \in \mathcal{D} = \mathrm{span}\{e_1, e_2\}$. Using the terminology of [10], ω_3 is a pseudo-Hermitian structure on the CR manifold $(T_1S, \mathcal{D}, J)$. Its associated Webster metric:

$$g_{\omega_3}(X, Y) = \mathrm{d}\omega_3(X, JY), \quad g_{\omega_3}(X, e_3) = 0, \quad g_{\omega_3}(e_3, e_3) = 1 \tag{5.7}$$

being

$$g_{\omega_3} = -2\omega_1^2 - 2\omega_2^2 + \omega_3^2 = \mathrm{diag}(-2, -2, 1) \tag{5.8}$$

is not positive definite and hence the pseudo-Hermitian structure is not strictly pseudoconvex. Since the Levi-Civita connection of g_{ω_3} satisfies

$$\nabla^{\omega_3}_{e_1} e_3 = 3e_2, \quad \nabla^{\omega_3}_{e_2} e_3 = -3e_1, \quad \nabla^{\omega_3} \omega_3 = 0, \tag{5.9}$$

it results that

$$\nabla^{\omega_3}_X e_3 = 3(-X^2 e_1 + X^1 e_2), \tag{5.10}$$

and then, as in the previous section, we get that e_3 is a Killing vector field for g_{ω_3}, which means that e_3 is a transversal symmetry (see [10, p. 446]) for the given pseudo-Hermitian structure.

Using the formulae of [10, p. 448] we get a component of the Webster-Ricci tensor of g_{ω_3}:

$$\mathrm{Ric}^W(e_1, e_2) = \frac{2\mathrm{Ric}^{g_{\omega_3}}(e_1, e_1) + g_{\omega_3}(e_1, e_1)}{-2\mathrm{i}} = \frac{0 - 2}{-2\mathrm{i}} = -\mathrm{i} \tag{5.11}$$

and then the Webster scalar curvature of g_{ω_3} is

$$\mathrm{scal}^W = \mathrm{i} g_{\omega_3}(e_1, e_1) \mathrm{Ric}^W(e_1, Je_1) = \mathrm{i} \cdot (-2) \cdot \mathrm{i} = 2 = K = \tau. \tag{5.12}$$

Since we have $\mathrm{Ric}^W \neq -\mathrm{i}\,\mathrm{scal}^W \mathrm{d}\omega_3$, it results that this pseudo-Hermitian structure is not Einstein.

Acknowledgement The author is extremely indebted to an anonymous referee who helped to substantially improve the presentation and the contents of this paper.

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