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Gröbner-Shirshov Bases of Irreducible Modules of the Quantum Group of Type G_2 *

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Abstract First, the authors give a Gröbner-Shirshov basis of the finite-dimensional irreducible module $V_q(\lambda)$ of the Drinfeld-Jimbo quantum group $U_q(G_2)$ by using the double free module method and the known Gröbner-Shirshov basis of $U_q(G_2)$. Then, by specializing a suitable version of $U_q(G_2)$ at $q = 1$, they get a Gröbner-Shirshov basis of the universal enveloping algebra $U(G_2)$ of the simple Lie algebra of type G_2 and the finite-dimensional irreducible $U(G_2)$-module $V(\lambda)$.

Keywords Quantum group, Gröbner-Shirshov basis, Double free module, Indecomposable module, Highest weight module
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1 Introduction

Reduction is a fundamental problem in studying the structures of algebras. Precisely, let A be an algebra given by a group of generators and a set of relations between them. We denote by S and $\langle S\rangle$ the set of these relations and the ideal generated by them, respectively. For any element a in S, we often need to decide whether a belongs to $\langle S\rangle$ or not. This is the so-called "membership problem" in algebra and it is often very difficult but important.

In his thesis [6], Buchberger provided a method to solve this problem in commutative algebra and called his theory the Gröbner bases theory. Later, Bergman [1] generalized Buchberg's theory to associative algebra. On the other hand, Shirshov [18] developed the same theory for Lie algebras. In [2], Bokut proved that Shirshov's method is also valid for associative algebras, so the theory of Shirshov for Lie algebras and the universal enveloping algebras is called the Gröbner-Shirshov bases theory.

In [5], Bokut and Malcolmson developed the Gröbner-Shirshov bases theory for the Drinfeld-Jimbo quantum groups and as an application, constructed a Gröbner-Shirshov basis for the quantum group of type A_n. Recently, in [14–16, 19] the authors, by using the representation theory of algebras, constructed a Gröbner-Shirshov basis for the quantum groups of types G_2, D_4, E_6, and F_4.

In [12], Kang and Lee developed the Gröbner-Shirshov bases theory for the modules over associative algebras and in [13], by using their theory, constructed a Gröbner-Shirshov basis for the irreducible modules of simple Lie algebras of type A_n. Several years later, in [8] Chibrikov used another approach to deal with Gröbner-Shirshov bases for the modules, and the key idea

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in his approach is that a module of an algebra is viewed as a free module over a free algebra. Later, in [7], authors gave a Gröbner-Shirshov basis for the modules over associative algebras by using the idea of Chibrikov.

In this paper, based on the Gröbner-Shirshov basis for the Drinfeld-Jimbo quantum group $U_q(G_2)$ given in [16], we construct a Gröbner-Shirshov basis for the irreducible module $V_q(\lambda)$ of $U_q(G_2)$ by using the method in [7] and by specializing a suitable version of $U_q(G_2)$ at $q = 1$, we get a Gröbner-Shirshov basis for the universal enveloping algebra $U(G_2)$ of the simple Lie algebra of type G_2 and the finite-dimensional irreducible module $V(\lambda)$ over it. And by comparing this new Gröbner-Shirshov basis for $U(G_2)$ with the one obtained in [3] we found that the new one contains the minimal basis in [3].

2 Some Preliminaries

For the convenience of the reader, in this section we recall some notions and results about the Gröbner-Shirshov bases of double-free modules and the quantum groups from [7], [9] and [11], respectively.

Let k be a field, X a non-empty set of letters with integer index, and X^* a free monoid of monomials in the letters in X. Let $k\langle X\rangle$ be the free associative k-algebra generated by X. In order to determine the leading term of an element $f \in k\langle X\rangle$, we choose a well ordering " $<$ " on X^*, and then this ordering naturally induces an ordering in the free associative algebra $k\langle X\rangle$. For any element $f \in k\langle X\rangle$, we denote by $\overline{f}$ the leading term of f. If the coefficient of the leading term of f is 1, then we say f is monic. If f and g are two monic elements in $k\langle X\rangle$, and their leading terms are $\overline{f}$ and $\overline{g}$, then the composition of f and g are defined as follows.

(a) If there are $a, b \in X^*$ such that $\overline{f}a = b\overline{g} = \omega$ and the length of $\overline{f}$, the number of the letters in $\overline{f}$, is bigger than the length of b, then the composition of intersection is defined to be $(f, g)_\omega = fa - bg$.

(b) If there are $a, b \in X^*$ such that $\overline{f} = a\overline{g}b = \omega$, then the composition of inclusion is defined to be $(f, g)_\omega = f - agb$.

Note that in both cases above, we have $\overline{(f, g)_\omega} < \omega$.

Let S be a non-empty subset of $k\langle X\rangle$ generated by some monic elements. We define a congruence relation with respect to S on $k\langle X\rangle$ as follows: For any $f, g \in k\langle X\rangle$ and $\omega \in X^*$,

$$f \equiv g \text{ mod } (S;\omega) \Leftrightarrow f - g = \sum \alpha_i a_i s_i b_i,$$

where $\alpha_i \in k,\ a_i, b_i \in X^*,\ s_i \in S$, and $a_i\overline{s_i}b_i < \omega$ for all i. If this is the case, then we say that f is congruent to g modulo S and ω, and denote it by $f \equiv g$ mod $(S;\omega)$. If an element f is congruent to 0 modulo S for some ω, then we say f is trivial modulo S. If for any elements $f, g \in S$ and $\omega \in X^*$, the composition $(f, g)_\omega$, whenever it is defined, is trivial modulo S, then we say S is closed under composition. If S is not closed under composition, then we will need to expand S by attaching all nontrivial compositions (inductively) to S to obtain a completion S^c. We call S^c (if it is closed under composition, then $S = S^c$) a Gröbner-Shirshov basis for the ideal $\langle S\rangle$ of $k\langle X\rangle$. Often, by abusing language, we call S^c a Gröbner-Shirshov basis of $k\langle X\rangle$.

Now we recall the definition of the double free module.

Definition 2.1 (see [8]) *Let X, Y be two sets, and $\text{mod}_{k\langle X\rangle}\langle\, Y\rangle$ a free left $k\langle X\rangle$-module with the basis Y. Then $\text{mod}_{k\langle X\rangle}\langle Y\rangle = \bigoplus_{y\in Y} k\langle X\rangle y$ is called a double free module.*

Let X, Y be two sets with well orderings and $X^*Y = \{uy \mid u \in X^*, y \in Y\}$. For any $\omega \in X^*Y$, we have a unique expression $\omega = x_1 \cdots x_n y$, where $x_i \in X,\ i = 1, \cdots, n,\ y \in Y,\ n \geq 0$. Set

$$\mathrm{wt}(\omega) = (n, y, x_1, \cdot \cdot \cdot, x_n).$$

We define an ordering " $\prec$ " on X^*Y as follows: For any $\omega, \acute{\omega} \in X^*Y$,

$$\omega \prec \acute{\omega} \Leftrightarrow \mathrm{wt}(\omega) < \mathrm{wt}(\acute{\omega}),$$

where $<$ is a lexicographical ordering. Clearly, the ordering $\prec$ satisfies

$$\omega \prec \acute{\omega} \Rightarrow a\omega \prec a\acute{\omega} \quad \text{for all } a \in X^*.$$

So the ordering " $\prec$ " is admissible.

Definition 2.2 (see [7]) *Let $S \subset \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ be a non-empty subset generated by some monic elements, and " $\prec$ " the admissible ordering defined above. We say that S is a Gröbner-Shirshov basis in the free module $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$, if all compositions in S are trivial modulo S.*

The following is the composition-diamond lemma for the double free module, the central result about the Gröbner-Shirshov bases theory of the double free module.

Lemma 2.1 (see [8]) *Let $S \subset \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ be a non-empty subset generated by some monic elements, and " $\prec$ " the admissible ordering defined above. The following statements are equivalent:*

(1) *S is a Gröbner-Shirshov basis of $\mathrm{mod}_{k\langle X\rangle}Y$;*

(2) *If $0 \neq f \in k\langle X\rangle S$, then $\overline{f} = a\overline{s}$ for some $a \in X^*$ and $s \in S$;*

(2′) *If $0 \neq f \in k\langle X\rangle S$, then $f = \sum \alpha_i a_i s_i$ with $a_1\overline{s_1} > a_2\overline{s_2} \cdots$, where $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$;*

(3) *$\mathrm{Irr}(S) = \{\omega \in X^*Y \mid \omega \neq a\overline{s},\ a \in X^*,\ s \in S\}$ is a k-linear basis for the factor module $\mathrm{mod}_{k\langle X\rangle}\langle Y \mid S\rangle = \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle / k\langle X\rangle S$.*

The following theorem explains the relation between the Gröbner-Shirshov bases of the associative algebra and the double free module.

Theorem 2.1 (see [7]) *Let X, Y be two sets with well orderings, " $<$ " a monomial ordering on X^* and " $\prec$ " the admissible ordering defined above. Let $S \subset k\langle X\rangle$ be a subset generated by some monic elements. Then, $S \subset k\langle X\rangle$ is a Gröbner-Shirshov basis of $k\langle X\rangle$ if and only if $SX^*Y \subset \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ is a Gröbner-Shirshov basis of $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ with respect to the ordering $\prec$.*

Next, we recall some notions about quantum groups from [9] and [11].

Let k be a field and $A = (a_{ij})$ a symmetrizable $n \times n$ Cartan matrix, that is, an integer matrix with $a_{ii} = 2,\ a_{ij} \leq 0\ (i \neq j)$ and there is an integral diagonal matrix $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ such that DA is a symmetric matrix, where $d_1, \cdots, d_n$ are non-negative integers. Let q be a nonzero element of k so that it is not a root of unity. The quantum group $U_q(A)$ is a free k-algebra with generators $\{E_i, K_i^{\pm 1}, F_i \mid 1 \leq i, j \leq n\}$, subject to the relations

$$\begin{aligned} K = \{&K_iK_j - K_jK_i,\ K_iK_i^{-1} - 1,\ K_i^{-1}K_i - 1,\ E_jK_i^{\pm 1} - q^{\mp d_i a_{ij}}K_i^{\pm 1}E_j, \\ &K_i^{\pm 1}F_j - q^{\mp d_i a_{ij}}F_jK_i^{\pm 1}\}, \\ T = &\left\{E_iF_j - F_jE_i - \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}\right\}, \end{aligned}$$

$$S^+ = \Big\{ \sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_t E_i^{1-a_{ij}-v} E_j E_i^v \,\Big|\, i \neq j,\ t = q^{2d_i} \Big\},$$
$$S^- = \Big\{ \sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_t F_i^{1-a_{ij}-v} F_j F_i^v \,\Big|\, i \neq j,\ t = q^{2d_i} \Big\}$$

for all $1 \leq i, j \leq n$ and

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \prod_{i=1}^{n} \dfrac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & \text{for } m > n > 0, \\ 1 & \text{for } n = 0 \text{ or } n = m. \end{cases}$$

Let $U_q^0(A)$, $U_q^+(A)$ and $U_q^-(A)$ be the subalgebras of $U_q(A)$ generated by $\{K_i^{\pm 1} \mid 1 \leq i \leq n\}$, $\{E_i \mid 1 \leq i \leq n\}$ and $\{F_i \mid 1 \leq i \leq n\}$, respectively. Then we have the following triangular decomposition of the quantum group $U_q(A)$:

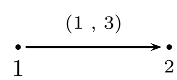
$$U_q(A) \cong U_q^+(A) \otimes U_q^0(A) \otimes U_q^-(A).$$

The following is the main result in [5].

Theorem 2.2 *If the sets S^{+c} and S^{-c} are the Gröbner-Shirshov bases of $U_q^+(A)$ and $U_q^-(A)$, respectively, then the set $S^{+c} \cup K \cup T \cup S^{-c}$ is a Gröbner-Shirshov basis of the quantum group $U_q(A)$.*

3 Gröbner-Shirshov Bases of Irreducible Modules over the Quantum Group G_2

From now on, we consider the quantum group $U_q(G_2)$. We choose the following orientation for G_2



Then the corresponding Cartan matrix A and its minimal symmetrizer D are

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Let

$$X = \{E_1, E_{12}, E_{122}, E_{1222}, E_{11222}, E_2, K_1^{\pm}, K_2^{\pm}, F_1, F_{12}, F_{122}, F_{1222}, F_{11222}, F_2\}$$

be the generating set of $U_q(G_2)$, where $E_1, E_{12}, E_{122}, E_{1222}, E_{11222}, E_2$ are the modified images of the isomorphism classes of indecomposable representations of the species of type G_2 under the canonical isomorphism of Ringel between the corresponding Ringel-Hall algebra $\mathcal{H}(G_2)$ and the positive part of the quantum group $U_q^+(G_2)$, and $F_1, F_{12}, F_{122}, F_{1222}, F_{11222}, F_2$ are the images of the $E_1, E_{12}, E_{122}, E_{1222}, E_{11222}, E_2$ under the convolution automorphism of the quantum group $U_q(G_2)$ (for details, see [16]). The following skew-commutator relations are computed in [16]:

$$E_{1222}E_2 = q^3 E_2 E_{1222},$$
$$E_{122}E_{1222} = q^3 E_{1222}E_{122},$$

$$E_{11222}E_{122} = q^3 E_{122}E_{11222},$$
$$E_{12}E_{11222} = q^3 E_{11222}E_{12},$$
$$E_1E_{12} = q^3 E_{12}E_1,$$
$$E_{122}E_2 = qE_2E_{122} - (q^2 + q^{-2} + 1)E_{1222},$$
$$E_{11222}E_{1222} = q^3 E_{1222}E_{11222} - \frac{q^6 - q^4 - q^2 + 1}{(q + q^{-1})(q^2 + q^{-2} + 1)} E_{122}^3,$$
$$E_{12}E_{122} = qE_{122}E_{12} - (q^2 + q^{-2} + 1)E_{11222},$$
$$E_1E_{11222} = q^3 E_{11222}E_1 - \frac{q^6 - q^4 - q^2 + 1}{(q + q^{-1})(q^2 + q^{-2} + 1)} E_{12}^3,$$
$$E_{11222}E_2 = E_2E_{11222} - \frac{q^3 - q^{-1}}{q + q^{-1}} E_{122}^2,$$
$$E_{12}E_{1222} = E_{1222}E_{12} - \frac{q^3 - q^{-1}}{q + q^{-1}} E_{122}^2,$$
$$E_1E_{122} = E_{122}E_1 - \frac{q^3 - q^{-1}}{q + q^{-1}} E_{12}^2,$$
$$E_{12}E_2 = q^{-1}E_2E_{12} - (q + q^{-1})E_{122},$$
$$E_1E_{1222} = q^{-3}E_{1222}E_1 - (q^2 - 1)E_{122}E_{12} - (q^3 - q - q^{-1})E_{11222},$$
$$E_1E_2 = q^{-3}E_2E_1 - E_{12},$$
$$K_iK_j = K_jK_i,$$
$$K_iK_i^{-1} = 1,$$
$$K_i^{-1}K_i = 1,$$
$$E_jK_i^{\pm 1} = q^{\mp a_{ij}} K_i^{\pm 1}E_j,$$
$$K_i^{\pm 1}F_j = q^{\mp a_{ij}} F_jK_i^{\pm 1},$$
$$E_iF_j = F_jE_i - \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}},$$
$$F_{1222}F_2 = q^3 F_2F_{1222},$$
$$F_{122}F_{1222} = q^3 F_{1222}F_{122},$$
$$F_{11222}F_{122} = q^3 F_{122}F_{11222},$$
$$F_{12}F_{11222} = q^3 F_{11222}F_{12},$$
$$F_1F_{12} = q^3 F_{12}F_1,$$
$$F_{122}F_2 = qF_2F_{122} - (q^2 + q^{-2} + 1)F_{1222},$$
$$F_{11222}F_{1222} = q^3 F_{1222}F_{11222} - \frac{q^6 - q^4 - q^2 + 1}{(q + q^{-1})(q^2 + q^{-2} + 1)} F_{122}^3,$$
$$F_{12}F_{122} = qF_{122}F_{12} - (q^2 + q^{-2} + 1)F_{11222},$$
$$F_1F_{11222} = q^3 F_{11222}F_1 - \frac{q^6 - q^4 - q^2 + 1}{(q + q^{-1})(q^2 + q^{-2} + 1)} F_{12}^3,$$
$$F_{11222}F_2 = F_2F_{11222} - \frac{q^3 - q^{-1}}{q + q^{-1}} F_{122}^2,$$
$$F_{12}F_{1222} = F_{1222}F_{12} - \frac{q^3 - q^{-1}}{q + q^{-1}} F_{122}^2,$$

$$
\begin{aligned}
&F_1F_{122} = F_{122}F_1 - \frac{q^3-q^{-1}}{q+q^{-1}}F_{12}^2,\\
&F_{12}F_2 = q^{-1}F_2F_{12} - (q+q^{-1})F_{122},\\
&F_1F_{1222} = q^{-3}F_{1222}F_1 - (q^2-1)F_{122}F_{12} - (q^3-q-q^{-1})F_{11222},\\
&F_1F_2 = q^{-3}F_2F_1 - F_{12},
\end{aligned}
$$

where $i, j = 1, 2$.

The main result in [16] says that the set S of the above skew-commutator relations is a minimal Gröbner-Shirshov basis of the quantum goup $U_q(G_2)$. Note that the ordering

$$
\begin{aligned}
E_1 &> E_{12} > E_{11222} > E_{122} > E_{1222} > E_2 > K_1 > K_1^{-1} > K_2 > K_2^{-1} > F_1\\
&> F_{12} > F_{11222} > F_{122} > F_{1222} > F_2
\end{aligned}
$$

induces a lexicographic ordering on the monomials of these generators.

Now we are ready to construct a Gröbner-Shirshov basis for the irreducible modules of the quantum group $U_q(G_2)$. Let X^* be a free monoid generated by X, and Λ_1, Λ_2 be fundamental weights. Let υ_λ be the highest weight vector with the highest weight λ, where $\lambda = m_1\Lambda_1 + m_2\Lambda_2$ and m_1, m_2 are non-negative integers. The finite-dimensional highest weight $U_q(G_2)$-module $V_q(\lambda)$ with the highest weight λ generated by υ_λ is defined to be (see Definition 2.1):

$$
V_q(\lambda) = \mathrm{mod}_{k\langle X\rangle}\langle \upsilon_\lambda \mid E_i\upsilon_\lambda = 0,\ K_i\upsilon_\lambda = q^{(\lambda,i)}\upsilon_\lambda,\ F_i^{m_i+1}\upsilon_\lambda = 0,\ 1 \le i \le 2,\ SX^*\upsilon_\lambda = 0\rangle,
$$

where S is a Gröbner-Shirshov basis of $U_q(G_2)$ and $(-,-)$ is the symmetrization of the Euler form (see [16]). From [10] we know that $V_q(\lambda)$ is a finite-dimensional irreducible module, and any irreducible finite-dimensional module on $U_q(G_2)$ can be obtained in this way. Our main result is the following theorem.

Theorem 3.1 *The set*

$$
S_1 = \{E_i\upsilon_\lambda, K_i\upsilon_\lambda - q^{(\lambda,i)}\upsilon_\lambda, F_i^{m_i+1}\upsilon_\lambda\ (1 \le i \le 2)\} \cup SX^*\upsilon_\lambda
$$

is a Gröbner-Shirshov basis of the finite-dimensional irreducible $U_q(G_2)$-module $V_q(\lambda)$.

Proof For convenience, we let

$$
g_i = E_i\upsilon_\lambda, \quad h_i = K_i\upsilon_\lambda - q^{(\lambda,i)}\upsilon_\lambda, \quad p_i = F_i^{m_i+1}\upsilon_\lambda,
$$

where $i = 1, 2$.

Now we prove that S_1 is closed under composition. Since S is a Gröbner-Shirshov basis of $U_q(G_2)$, we know from [7] that $SX^*\upsilon_\lambda$ is closed under composition, and there is no composition between the elements of $\{E_i\upsilon_\lambda, K_i\upsilon_\lambda - q^{(\lambda,i)}\upsilon_\lambda, F_i^{m_i+1}\upsilon_\lambda\}$. So we only need to prove that the compositions between the elements of $\{E_i\upsilon_\lambda, K_i\upsilon_\lambda - q^{(\lambda,i)}\upsilon_\lambda, F_i^{m_i+1}\upsilon_\lambda\}$ and $SX^*\upsilon_\lambda$ are trivial.

For any $u = sa\upsilon_\lambda \in SX^*\upsilon_\lambda,\ s \in S,\ a \in X^*$,

(I) if $a \neq 1$, then we consider the following three cases.

(i) If $SX^*\upsilon_\lambda \ni u = sa_1E_i\upsilon_\lambda$, where $s \in S,\ a_1 \in X^*,\ i = 1, 2$, then $\omega = \overline{s}a_1E_i\upsilon_\lambda$, so

$$
\begin{aligned}
(u, g_i)_\omega &= sa_1E_i\upsilon_\lambda - \overline{s}a_1E_i\upsilon_\lambda\\
&= (s - \overline{s})a_1E_i\upsilon_\lambda\\
&\equiv 0 \mod (S_1, \omega).
\end{aligned}
$$

(ii) If $SX^*\upsilon_\lambda \ni u = sa_1F_i^l\upsilon_\lambda,\ s \in S,\ a_1 \in X^*,\ i = 1, 2$, then there is no composition when $0 < l < m_i + 1$, and when $l \geq m_i + 1$, we have $\omega = \overline{u} = \overline{s}a_1F_i^l\upsilon_\lambda = \overline{s}a_1F_i^{l-m_i-1}p_i$. So

$$\begin{aligned}(u, p_i)_\omega &= sa_1F_i^l\upsilon_\lambda - \overline{s}a_1F_i^{l-m_i-1}p_i \\ &= (s - \overline{s})a_1F_i^{l-m_i-1}p_i \\ &\equiv 0 \mod (S_1, \omega).\end{aligned}$$

(iii) If $SX^*\upsilon_\lambda \ni u = sa_1K_i\upsilon_\lambda$, where $s \in S,\ a_1 \in X^*,\ i = 1, 2$, and $s = \overline{s} + t,\ t < \overline{s}$, we have $\omega = \overline{u} = \overline{s}a_1K_i\upsilon_\lambda$. So

$$\begin{aligned}(u, h_i)_\omega &= sa_1K_i\upsilon_\lambda - \overline{s}a_1h_i \\ &= ta_1K_i\upsilon_\lambda + \overline{s}a_1K_i\upsilon_\lambda - \overline{s}a_1K_i\upsilon_\lambda + \overline{s}a_1q^{(\lambda,i)}\upsilon_\lambda \\ &\equiv ta_1q^{(\lambda,i)}\upsilon_\lambda + \overline{s}a_1q^{(\lambda,i)}\upsilon_\lambda \mod (S_1, \omega) \\ &\equiv sa_1q^{(\lambda,i)}\upsilon_\lambda \mod (S_1, \omega) \\ &\equiv 0 \mod (S_1, \omega).\end{aligned}$$

(II) if $a = 1$, that is, $u = s\upsilon_\lambda \in SX^*\upsilon_\lambda$, where $s \in S = S^+ \cup K \cup T \cup S^-$, then we consider the following four cases.

(i) If $s \in S^+$, then $\overline{s} = E_xE_y$, where $E_x, E_y \in A,\ A = \{E_{12}, E_{122}, E_{1222}, E_{11222}\}$. Since we know from [17] that each $E_{12}, E_{122}, E_{1222}$ and E_{11222} is polynomial of E_1 and E_2 without constant term, the proof is the same as (i) in (I).

(ii) If $s \in S^-$, then by using the convolution automorphism (see [10]) we convert this case to the case (i).

(iii) If $s \in K$, then we have the following three compositions:

If $u = (K_lK_p - K_pK_l)\upsilon_\lambda$, where $(l, p) > (p, l)$, $\overline{u} = K_lK_p\upsilon_\lambda$, then $\overline{u} = K_l\overline{h}_i = \omega$, when $p = i$, where $i = 1, 2$. So

$$\begin{aligned}(u, h_i)_\omega &= K_lK_i\upsilon_\lambda - K_iK_l\upsilon_\lambda - K_lK_i\upsilon_\lambda + q^{(\lambda,i)}K_l\upsilon_\lambda \\ &\equiv -q^{(\lambda,i)}q^{(\lambda,l)}\upsilon_\lambda + q^{(\lambda,i)}q^{(\lambda,l)}\upsilon_\lambda \mod (S_1, \omega) \\ &\equiv 0 \mod (S_1, \omega).\end{aligned}$$

If $u = (K_j^{-1}K_j - 1)\upsilon_\lambda$, then $\overline{u} = K_i^{-1}\overline{h}_i$, when $j = i$. Thus

$$\begin{aligned}(u, h_i)_\omega &= K_i^{-1}K_i\upsilon_\lambda - \upsilon_\lambda - K_i^{-1}K_i\upsilon_\lambda + q^{(\lambda,i)}K_i^{-1}\upsilon_\lambda \\ &\equiv -\upsilon_\lambda + q^{(\lambda,i)}K_i^{-1}\upsilon_\lambda \mod (S_1, \omega).\end{aligned}$$

Since $K_i\upsilon_\lambda - q^{(\lambda,i)}\upsilon_\lambda \in S_1$, we have $K_i^{-1}(K_i\upsilon_\lambda - q^{(\lambda-i)}\upsilon_\lambda) \in \langle S_1\rangle$. Again, since $K_i^{-1}K_i\upsilon_\lambda - \upsilon_\lambda \in \langle S_1\rangle$, we have $K_i^{-1}\upsilon_\lambda - q^{-(\lambda,i)}\upsilon_\lambda \in \langle S_1\rangle$. Hence

$$\begin{aligned}(u, h_i)_\omega &\equiv -\upsilon_\lambda + \upsilon_\lambda \mod (S_1, \omega) \\ &\equiv 0 \mod (S_1, \omega).\end{aligned}$$

If $u = (E_jK_l - q^{\mp a_{ij}}K_lE_j)\upsilon_\lambda$, then $\overline{u} = E_j\overline{h_i} = \omega$, when $l = i$. So

$$\begin{aligned}(u, h_i)_\omega &= E_jK_i\upsilon_\lambda - q^{\mp a_{ij}}K_iE_j\upsilon_\lambda - E_jK_i\upsilon_\lambda + q^{(\lambda,i)}E_j\upsilon_\lambda \\ &\equiv 0 \mod (S_1, \omega).\end{aligned}$$

(iv) If $s \in T$, then there is no composition.

The proof is complete.

In order to specialize the quantum group $U_q(G_2)$ at $q=1$, we give another version $U'_q(G_2)$ of $U_q(G_2)$ as follows.

The k-algebra $U'_q(G_2)$ is generated by $\{E_1, E_2, K_1, K_2, K_1^{-1}, K_2^{-1}, F_1, F_2, L_1, L_2\}$ subject to the relations

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = 1, \qquad K_i^{-1}K_i = 1,$$
$$E_jK_i^{\pm 1} = q^{\mp d_i a_{ij}} K_i^{\pm 1} E_j, \qquad K_i^{\pm 1}F_j = q^{\mp d_i a_{ij}} F_j K_i^{\pm 1},$$
$$E_iF_j - F_jE_i = \delta_{ij} L_i, \qquad (q^{2d_i} - q^{-2d_i})L_i = K_i^2 - K_i^{-2}, \qquad [L_i, L_j] = 0,$$

where $1 \le i, j \le 2$, and

$$[L_1, E_1] = q^2(E_1K_1^2 - K_1^{-2}E_1), \qquad [L_1, E_2] = -\frac{1}{q^2+1}(E_2K_1^2 - K_1^{-2}E_2),$$
$$[L_2, E_1] = -\frac{q^{12}+q^6+1}{q^{12}(q^6+1)}(E_1K_2^2 - K_2^{-2}E_1), \qquad [L_2, E_2] = q^6(E_2K_2^2 - K_2^{-2}E_2),$$
$$[L_1, F_1] = -q^{-2}(F_1K_1^2 - K_1^{-2}F_1), \qquad [L_1, F_2] = \frac{q^2}{q^2+1}(F_2K_1^2 - K_1^{-2}F_2),$$
$$[L_2, F_1] = \frac{q^6(q^{12}+q^6+1)}{q^6+1}(F_1K_2^2 - K_2^{-2}F_1), \qquad [L_2, F_2] = -q^{-6}(F_2K_2^2 - K_2^{-2}F_2),$$
$$\sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_t E_i^{1-a_{ij}-v} E_j E_i^v \quad (1 \le i \ne j \le 2),$$
$$\sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_t F_i^{1-a_{ij}-v} F_j F_i^v \quad (1 \le i \ne j \le 2).$$

Then we have the following result.

Theorem 3.2 *The two k-algebras $U_q(G_2)$ and $U'_q(G_2)$ are isomorphic.*

Proof We define two k-algebra homomorphisms ϕ and ψ as follows:

$$\phi : U_q(G_2) \to U'_q(G_2) \quad \text{by} \quad E_i \longmapsto E_i,\ F_i \longmapsto F_i,\ K_i \longmapsto K_i$$

and

$$\psi : U'_q(G_2) \to U_q(G_2) \quad \text{by} \quad E_i \longmapsto E_i,\ F_i \longmapsto F_i,\ K_i \longmapsto K_i,\ L_i \longmapsto [E_i, F_i].$$

Then, we need to verify that these two maps are well-defined, that is, they are compatible with the defining relations for $U_q(G_2)$ and $U'_q(G_2)$. Because of the definitions of ϕ and ψ, we only need to consider the relations relevant to L_i. First, we prove that ϕ is well-defined. Since

$$\phi([E_i, F_j]) = \phi(E_iF_j - F_jE_i) = E_iF_j - F_jE_i = [E_i, F_j] = \delta_{ij}L_i,$$
$$\phi\Big(\delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}\Big) = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} = \delta_{ij}\frac{(q^{2d_i} - q^{-2d_i})L_i}{q^{2d_i} - q^{-2d_i}} = \delta_{ij}L_i,$$

we have

$$\phi([E_i, F_j]) = \phi\Big(\delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}\Big),$$

and ϕ is well-defined.

Next, we prove that ψ is well-defined. Clearly,

$$\psi([E_i,F_j]) = \psi(E_iF_j - F_jE_i) = E_iF_j - F_jE_i = [E_i,F_j] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}},$$
$$\psi(\delta_{ij}L_i) = \delta_{ij}[E_i,F_j] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}.$$

So

$$\psi([E_i,F_j]) = \psi(\delta_{ij}L_i).$$

Since

$$\psi((q^{2d_i} - q^{-2d_i})L_i) = (q^{2d_i} - q^{-2d_i})[E_i,F_j] = (q^{2d_i} - q^{-2d_i})\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} = K_i^2 - K_i^{-2},$$

one can get

$$\psi((q^{2d_i} - q^{-2d_i})L_i) = \psi(K_i^2 - K_i^{-2}).$$

Similarly, we have

$$\begin{aligned}
\psi([L_i,L_j]) &= \psi(L_iL_j - L_jL_i) \\
&= [E_iF_i][E_jF_j] - [E_jF_j][E_iF_i] \\
&= \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \cdot \frac{K_j^2 - K_j^{-2}}{q^{2d_i} - q^{-2d_i}} - \frac{K_j^2 - K_j^{-2}}{q^{2d_i} - q^{-2d_i}} \cdot \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \\
&= 0, \\
\psi([L_i,E_j]) &= \psi(L_iE_j - E_jL_i) \\
&= [E_iF_i]E_j - E_j[E_iF_i] \\
&= \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \cdot E_j - E_j\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \\
&= \frac{q^{2d_ia_{ij}-1}}{q^{2d_i} - q^{-2d_i}}(E_jK_i^2 + K_i^{-2}E_j) \\
&= \begin{cases} q^2(E_1K_1^2 - K_1^{-2}E_1), \\ -\dfrac{1}{q^2+1}(E_2K_1^2 - K_1^{-2}E_2), \\ -\dfrac{q^{12}+q^6+1}{q^{12}(q^6+1)}(E_1K_2^2 - K_2^{-2}E_1), \\ q^6(E_2K_2^2 - K_2^{-2}E_2), \end{cases} \\
\psi([L_i,F_j]) &= \psi(L_iFj - F_jL_i) \\
&= [E_iF_i]F_j - F_j[E_iF_i] \\
&= \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \cdot F_j - F_j\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \\
&= \frac{q^{-2d_ia_{ij}-1}}{q^{2d_i} - q^{-2d_i}}(F_jK_i^2 + K_i^{-2}F_j)
\end{aligned}$$

$$= \begin{cases} -q^{-2}(F_1K_1^2 - K_1^{-2}F_1), \\ \dfrac{q^2}{q^2+1}(F_2K_1^2 - K_1^{-2}F_2), \\ \dfrac{q^6(q^{12}+q^6+1)}{q^6+1}(F_1K_2^2 - K_2^{-2}F_1), \\ -q^{-6}(F_2K_2^2 - K_2^{-2}F_2). \end{cases}$$

So ψ is well-defined. Finally, we note that

$$\begin{aligned} &\phi\psi(E_i) = E_i, \quad \phi\psi(F_i) = F_i, \quad \phi\psi(K_i) = K_i, \\ &\psi\phi(E_i) = E_i, \quad \psi\phi(F_i) = F_i, \quad \psi\phi(K_i) = K_i, \\ &\phi\psi(L_i) = \phi[E_iF_i] = \phi(E_iF_i - F_iEi) = E_iF_i - F_iEi = [E_iF_i] = L_i. \end{aligned}$$

So

$$\psi\phi = 1_{U_q(G_2)} \quad \text{and} \quad \phi\psi = 1_{U_q'(G_2)}.$$

Therefore, $U_q(G_2) \cong U_q'(G_2)$. The proof is complete.

This isomorphism gives the following Gröbner-Shirshov basis for $U_q'(G_2)$:

(1) $E_{1222}E_2 - q^3E_2E_{1222}$,
(2) $E_{122}E_{1222} - q^3E_{1222}E_{122}$,
(3) $E_{11222}E_{122} - q^3E_{122}E_{11222}$,
(4) $E_{12}E_{11222} - q^3E_{11222}E_{12}$,
(5) $E_1E_{12} - q^3E_{12}E_1$,
(6) $E_{122}E_2 - qE_2E_{122} - (q^2+q^{-2}+1)E_{1222}$,
(7) $E_{11222}E_{1222} - q^3E_{1222}E_{11222} - \frac{q^6-q^4-q^2+1}{(q+q^{-1})(q^2+q^{-2}+1)}E_{122}^3$,
(8) $E_{12}E_{122} - qE_{122}E_{12} - (q^2+q^{-2}+1)E_{11222}$,
(9) $E_1E_{11222} - q^3E_{11222}E_1 - \frac{q^6-q^4-q^2+1}{(q+q^{-1})(q^2+q^{-2}+1)}E_{12}^3$,
(10) $E_{11222}E_2 - E_2E_{11222} - \frac{q^3-q^{-1}}{q+q^{-1}}E_{122}^2$,
(11) $E_{12}E_{1222} - E_{1222}E_{12} - \frac{q^3-q^{-1}}{q+q^{-1}}E_{122}^2$,
(12) $E_1E_{122} - E_{122}E_1 - \frac{q^3-q^{-1}}{q+q^{-1}}E_{12}^2$,
(13) $E_{12}E_2 - q^{-1}E_2E_{12} - (q+q^{-1})E_{122}$,
(14) $E_1E_{1222} - q^{-3}E_{1222}E_1 - (q^2-1)E_{122}E_{12} - (q^3-q-q^{-1})E_{11222}$,
(15) $E_1E_2 - q^{-3}E_2E_1 - E_{12}$,
(16) $K_iK_j - K_jK_i$,
(17) $K_iK_i^{-1} - 1$,
(18) $K_i^{-1}K_i - 1$,
(19) $E_jK_i^{\pm 1} = q^{\mp d_ia_{ij}}K_i^{\pm 1}E_j$,
(20) $K_i^{\pm 1}F_j = q^{\mp d_ia_{ij}}F_jK_i^{\pm 1}$,
(21) $E_iF_j - F_jE_i - \delta_{ij}\frac{K_i^2-K_i^{-2}}{q^{2d_i}-q^{-2d_i}}$,
(22) $F_{1222}F_2 - q^3F_2F_{1222}$,
(23) $F_{122}F_{1222} - q^3F_{1222}F_{122}$,
(24) $F_{11222}F_{122} - q^3F_{122}F_{11222}$,
(25) $F_{12}F_{11222} - q^3F_{11222}F_{12}$,
(26) $F_1F_{12} - q^3F_{12}F_1$,
(27) $F_{122}F_2 - qF_2F_{122} - (q^2+q^{-2}+1)F_{1222}$,

(28) $F_{11222}F_{1222} - q^3F_{1222}F_{11222} - \frac{q^6-q^4-q^2+1}{(q+q^{-1})(q^2+q^{-2}+1)}F_{122}^3$,
(29) $F_{12}F_{122} - qF_{122}F_{12} - (q^2+q^{-2}+1)F_{11222}$,
(30) $F_1F_{11222} - q^3F_{11222}F_1 - \frac{q^6-q^4-q^2+1}{(q+q^{-1})(q^2+q^{-2}+1)}F_{12}^3$,
(31) $F_{11222}F_2 - F_2F_{11222} - \frac{q^3-q^{-1}}{q+q^{-1}}F_{122}^2$,
(32) $F_{12}F_{1222} - F_{1222}F_{12} - \frac{q^3-q^{-1}}{q+q^{-1}}F_{122}^2$,
(33) $F_1F_{122} - F_{122}F_1 - \frac{q^3-q^{-1}}{q+q^{-1}}F_{12}^2$,
(34) $F_{12}F_2 - q^{-1}F_2F_{12} - (q+q^{-1})F_{122}$,
(35) $F_1F_{1222} - q^{-3}F_{1222}F_1 - (q^2-1)F_{122}F_{12} - (q^3-q-q^{-1})F_{11222}$,
(36) $F_1F_2 - q^{-3}F_2F_1 - F_{12}$,
(37) $L_iL_j - L_jL_i$,
(38) $L_iE_j - E_jL_i - \frac{q^{2d_ia_{ij}}-1}{q^{2d_i}-q^{-2d_i}}(E_jK_i^2 - K_i^{-2}E_j)$

$$= \begin{cases} L_1E_1 - E_1L_1 = q^2(E_1K_1^2 - K_1^{-2}E_1), \\ L_1E_2 - E_2L_1 = -\frac{1}{q^2+1}(E_2K_1^2 - K_1^{-2}E_2), \\ L_2E_1 - E_1L_2 = -\frac{q^{12}+q^6+1}{q^{12}(q^6+1)}(E_1K_2^2 - K_2^{-2}E_1), \\ L_2E_2 - E_2L_2 = q^6(E_2K_2^2 - K_2^{-2}E_2), \end{cases}$$

(39) $L_iF_j - F_jL_i - \frac{q^{-2d_ia_{ij}}-1}{q^{2d_i}-q^{-2d_i}}(F_jK_i^2 - K_i^{-2}F_j)$

$$= \begin{cases} L_1F_1 - F_1L_1 = -q^{-2}(F_1K_1^2 - K_1^{-2}F_1), \\ L_1F_2 - F_2L_1 = \frac{q^2}{q^2+1}(F_2K_1^2 - K_1^{-2}F_2), \\ L_2F_1 - F_1L_2 = \frac{q^6(q^{12}+q^6+1)}{q^6+1}(F_1K_2^2 - K_2^{-2}F_1), \\ L_2F_2 - F_2L_2 = -q^{-6}(F_2K_2^2 - K_2^{-2}F_2), \end{cases}$$

where $i, j = 1, 2$.

We denote this Gröbner-Shirshov basis of $U_q'(G_2)$ by S'. Moreover, by the isomorphism ϕ above, we define a $U_q'(G_2)$-module structure on $V_q(\lambda)$ as follows:

$$\begin{aligned} E_i \circ \upsilon_\lambda &= \psi(E_i)\upsilon_\lambda = E_i\upsilon_\lambda = 0, \\ K_i \circ \upsilon_\lambda &= \psi(K_i)\upsilon_\lambda = K_i\upsilon_\lambda = q^{(\lambda,i)}\upsilon_\lambda, \\ F_i^{m_i+1} \circ \upsilon_\lambda &= \psi(F_i^{m_i+1})\upsilon_\lambda = F_i^{m_i+1}\upsilon_\lambda = 0, \\ L_i \circ \upsilon_\lambda &= \psi(L_i)\upsilon_\lambda \\ &= [E_iF_i]\upsilon_\lambda \\ &= \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}\upsilon_\lambda \\ &= \frac{q^{2(\lambda,i)} - q^{-2(\lambda,i)}}{q^{2d_i} - q^{-2d_i}}\upsilon_\lambda \\ &= \frac{q^{-2(\lambda,i)}(q^{4(\lambda,i)} - 1)}{q^{-2d_i}(q^{4d_i} - 1)}\upsilon_\lambda \\ &= \begin{cases} q^{2-2(\lambda,1)}((q^4)^{(\lambda,1)-1} + (q^4)^{(\lambda,1)-2} + \cdots + q^4 + 1)\upsilon_\lambda, & \text{if } i = 1, \\ \dfrac{q^{6-2(\lambda,2)}((q^4)^{(\lambda,2)-1} + (q^4)^{(\lambda,2)-2} + \cdots + q^4 + 1)\upsilon_\lambda}{q^8 + q^4 + 1}, & \text{if } i = 2, \end{cases} \end{aligned}$$

and we denote this finite-dimensional irreducible $U_q'(G_2)$-module by $V_q'(\lambda)$. Then we get the following Gröbner-Shirshov basis for $V_q'(\lambda)$:

$$S_1' = \Big\{ E_i \circ \upsilon_\lambda,\ K_i \circ \upsilon_\lambda - q^{(\lambda,i)}\upsilon_\lambda,\ F_i^{m_i+1} \circ \upsilon_\lambda,$$

$$L_1 \circ \upsilon_\lambda - q^{2-2(\lambda,1)}((q^4)^{(\lambda,1)-1} + (q^4)^{(\lambda,1)-2} + \cdots + q^4 + 1)\upsilon_\lambda,$$
$$L_2 \circ \upsilon_\lambda - \frac{q^{6-2(\lambda,2)}((q^4)^{(\lambda,2)-1} + (q^4)^{(\lambda,2)-2} + \cdots + q^4 + 1)\upsilon_\lambda}{q^8 + q^4 + 1} \ (1 \le i \le 2)\Big\} \cup S'X^*\upsilon_\lambda,$$

where X^* is the free monoid of monomials in the letters in $\{E_i, K_i^{\pm 1}, F_i, L_i \mid 1 \le i \le 2\}$. We denote by $U_1'(G_2)$ the specialization of $U_q'(G_2)$ at $q = 1$. By using the Lie bracket and the formulas (6), (8), (13) and (15), we have

$$\begin{aligned}
E_{12} &= [E_1E_2],\\
E_{122} &= \frac{1}{2}[[E_1E_2]E_2],\\
E_{1222} &= \frac{1}{6}[[[E_1E_2]E_2]E_2],\\
E_{11222} &= \frac{1}{6}[[E_1E_2][[E_1E_2]E_2]].
\end{aligned}$$

From the formulas (1)–(5), (7), (9)–(12) and (14), we have

$$\begin{aligned}
6[E_{1222}E_2] &= [[[[E_1E_2]E_2]E_2]E_2],\\
12[E_{122}E_{1222}] &= [[[E_1E_2]E_2][[[E_1E_2]E_2]E_2]],\\
12[E_{1222}E_{122}] &= [[[E_1E_2][[E_1E_2]E_2]][[E_1E_2]E_2]],\\
6[E_{12}E_{11222}] &= [[E_1E_2][[E_1E_2][[E_1E_2]E_2]]],\\
[E_1E_{12}] &= [E_1[E_1E_2]],\\
36[E_{1222}E_{1222}] &= [[[E_1E_2][[E_1E_2]E_2]][[[E_1E_2]E_2]E_2]],\\
6[E_1E_{11222}] &= [E_1[[E_1E_2][[E_1E_2]E_2]]],\\
6[E_{11222}E_2] &= [[[E_1E_2][[E_1E_2]E_2]]E_2],\\
6[E_{12}E_{1222}] &= [[E_1E_2][[[E_1E_2]E_2]E_2]],\\
-2[E_1E_{122}] &= [[[E_1E_2]E_2]E_1],\\
6[E_1E_{1222}] + 6E_{11222} &= [E_1[[[E_1E_2]E_2]E_2]] + [[E_1E_2][[E_1E_2]E_2]].
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
6[F_{1222}F_2] &= [[[[F_1F_2]F_2]F_2]F_2],\\
12[F_{122}F_{1222}] &= [[[F_1F_2]F_2][[[F_1F_2]F_2]F_2]],\\
12[F_{1222}F_{122}] &= [[[F_1F_2][[F_1F_2]F_2]][[F_1F_2]F_2]],\\
6[F_{12}F_{11222}] &= [[F_1F_2][[F_1F_2][[F_1F_2]F_2]]],\\
[F_1F_{12}] &= [F_1[F_1F_2]],\\
36[F_{1222}F_{1222}] &= [[[F_1F_2][[F_1F_2]F_2]][[[F_1F_2]F_2]F_2]],\\
6[F_1F_{11222}] &= [F_1[[F_1F_2][[F_1F_2]F_2]]],\\
6[F_{11222}F_2] &= [[[F_1F_2][[F_1F_2]F_2]]F_2],\\
6[F_{12}F_{1222}] &= [[F_1F_2][[[F_1F_2]F_2]F_2]],\\
-2[F_1F_{122}] &= [[[F_1F_2]F_2]F_1],\\
6[F_1F_{1222}] + 6F_{11222} &= [F_1[[[F_1F_2]F_2]F_2]] + [[F_1F_2][[F_1F_2]F_2]],
\end{aligned}$$

where $[E_{11222}E_2] = [E_{12}E_{1222}]$, $[F_{11222}F_2] = [F_{12}F_{1222}]$. Then there is a surjection

$$f : U_1'(G_2) \to U(G_2)$$

defined by

$$E_1 \longmapsto x_2,\ E_2 \longmapsto x_1,\ F_1 \longmapsto y_2,\ F_2 \longmapsto y_1,\ K_i \longmapsto 1,\ L_i \longmapsto h_i$$

and Ker$f = \langle K_i - 1 \mid 1 \le i \le 2\rangle$. So

$$U(G_2) \cong U_1'(G_2)/\langle K_i - 1 \mid 1 \le i \le 2\rangle,$$

where $U(G_2)$ is the classical universal enveloping algebra of the simple Lie algebra of type G_2. Hence by replacing the q and all K_i's by 1 in the Gröbner-Shirshov basis S' of $U_q'(G_2)$, and using the map f, we get the following Gröbner-Shirshov basis S_0 of $U(G_2)$:

$$\begin{aligned}
&f_1 = [x_i, y_j] - \delta_{ij} h_i, && f_2 = [h_i, h_j],\\
&f_3 = [h_i, x_j] - a_{ij} x_j, && f_4 = [h_i, y_j] + a_{ij} y_j,\\
&f_5 = [x_2 x_2 x_1], && f_6 = x_2 x_1 x_1 x_1 x_1,\\
&f_7 = (x_2 x_1)(x_2 x_1 x_1 x_1), && f_8 = (x_2 x_1 x_1)(x_2 x_1 x_1 x_1),\\
&f_9 = (x_2 x_1)((x_2 x_1)(x_2 x_1 x_1)), && f_{10} = (x_2 x_1)(x_2 x_1 x_1)(x_2 x_1 x_1),\\
&f_{11} = (x_2 x_1)(x_2 x_1 x_1)(x_2 x_1 x_1 x_1), && f_{12} = (x_2 x_1)(x_2 x_1 x_1) x_2,\\
&f_{13} = x_2 x_1 x_1 x_2, && f_{14} = x_2 x_1 x_1 x_2 - (x_2 x_1)(x_2 x_1 x_1),\\
&f_{15} = [y_2 y_2 y_1], && f_{16} = y_2 y_1 y_1 y_1 y_1,\\
&f_{17} = (y_2 y_1)(y_2 y_1 y_1 y_1), && f_{18} = (y_2 y_1 y_1)(y_2 y_1 y_1 y_1),\\
&f_{19} = (y_2 y_1)((y_2 y_1)(y_2 y_1 y_1)), && f_{20} = (y_2 y_1)(y_2 y_1 y_1)(y_2 y_1 y_1),\\
&f_{21} = (y_2 y_1)(y_2 y_1 y_1)(y_2 y_1 y_1 y_1), && f_{22} = (y_2 y_1)(y_2 y_1 y_1) y_2,\\
&f_{23} = y_2 y_1 y_1 y_2, && f_{24} = y_2 y_1 y_1 y_2 - (y_2 y_1)(y_2 y_1 y_1),
\end{aligned}$$

where $i, j = 1, 2$. In this Gröbner-Shirshov basis, we omit brackets for convenience. The Lie product $[ab]$ will be written as (ab) or ab, $[z_1, z_2 \cdots z_m]$ will mean $z_1[z_2 \cdots z_m]$ and $(z_1 z_2 \cdots z_m)$ will mean $(z_1 z_2 \cdots z_{m-1}) z_m$. Thus, we have $[z_1 z_2 \cdots z_m] = (-1)^{m-1} z_m \cdots z_2 z_1$.

In [4] a minimal Gröbner-Shirshov basis of $U(G_2)$ is given and by comparing it with the basis above, we find that the minimal one is contained in the basis above.

Again, by replacing the q and all K_i's by 1 in the Gröbner-Shirshov basis S_1' of the finite-dimensional irreducible $U_q'(G_2)$-module $V_q'(\lambda)$ and using the map f, we get the following Gröbner-Shirshov basis S_0' of $U(G_2)$-module $V(\lambda)$:

$$S_0' = \{x_i \upsilon_\lambda, h_i \upsilon_\lambda - (\lambda, i)\upsilon_\lambda, y_i^{m_i+1}\upsilon\} \cup S_0 X^* \upsilon_\lambda,$$

where X^* is the free monoid of monomials in the letters in $\{x_i, h_i, y_i \mid 1 \le i \le 2\}$. Note that here we have used the fact $(\lambda, i) = m_i,\ 1 \le i \le n$ (see [3]).

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