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Twistor Spinors and Quasi-twistor Spinors*

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Abstract The author studies the properties and applications of quasi-Killing spinors and quasi-twistor spinors and obtains some vanishing theorems. In particular, the author classifies all the types of quasi-twistor spinors on closed Riemannian spin manifolds. As a consequence, it is known that on a locally decomposable closed spin manifold with nonzero Ricci curvature, the space of twistor spinors is trivial. Some integrability condition for twistor spinors is also obtained.

Keywords Dirac operator, Twistor spinor, Scalar curvature
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1 Introduction

It is well known that the spectrum of the Dirac operator on closed spin manifolds detects subtle information on the geometry and the topology of such manifolds (see [1]). The first sharp estimate for the nonzero eigenvalues λ of the Dirac operator is the well-known Friedrich inequality, which says that

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_{M^n} R, \tag{1.1}$$

where R is the scalar curvature of the closed spin manifold (M^n, g). The case of equality in (1.1) occurs if and only if (M^n, g) admits a nontrivial spinor field ψ called a real Killing spinor, satisfying the following overdetermined elliptic equation:

$$\nabla_X \psi = -\frac{\lambda}{n} X \cdot \psi, \tag{1.2}$$

where $X \in \Gamma(TM^n)$ and the dot "." indicates the Clifford multiplication. Obviously, a Killing spinor is a twistor spinor which is also an eigenspinor. The existence of Killing spinors implies severe restrictions on the manifold. The manifold must be a locally irreducible Einstein manifold and the simply connected manifolds admitting real Killing spinors were completely classified (see [2]). Some classification results for manifolds with twistor spinors can be seen in [3–4].

On the other hand, Lichnerowicz [5] and Hijazi [6] noticed that a manifold admitting a non-zero parallel k-form, for $k \neq 0, n$, carries no real Killing spinor. Furthermore, if (M^n, g) possesses a locally product structure, then there is no Killing spinor. Consequently, this shows

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the estimate (1.1) cannot be sharp, for example, on (quaternionic) Kähler manifolds, and on manifolds with a locally product structure. Indeed, better estimates have been proved in these cases by Kirchberg [7], Hijazi [8], Kramer et al [9], and Kim [10], respectively.

In the paper [8], Kählerian twistor spinors are introduced to get lower bounds for the eigenvalues of the Dirac operator on closed spin Kähler manifolds. Hijazi also studied the uniqueness of Kählerian twistor spinors and obtained some vanishing theorems (see [11]). In particular, he proved that on a Kähler spin manifold with $R \not\equiv 0$, the space of twistor spinors is reduced to zero. Motivated by the paper [8], we study the properties and applications of the quasi-Killing spinors and the quasi-twistor spinors which were used to get lower bounds for the eigenvalues of the Dirac operator in [10–12], and obtain some vanishing theorems. Especially, we prove that on a locally decomposable closed spin manifold with nonzero Ricci curvature, the space of twistor spinor is trivial.

The article is organized as follows: In Section 2, some geometric conventions and preliminaries are given. In Section 3, we discuss the quasi-Killing spinor and its application in lower bounds estimation for the eigenvalues of the Dirac operator. In the final section, the quasi-twistor spinor is investigated. Especially, we give an integrability condition for twistor spinors (see Theorem 4.2). More generally, we study the uniqueness of quasi-twistor spinors on complete Riemannian spin manifolds (see Theorems 4.3–4.4). As a corollary, we know that on a locally decomposable closed spin manifold with $\mathrm{Ric} \not\equiv 0$, the space of twistor spinors is trivial.

2 Preliminaries

Let (M^n, g) be an oriented n-Riemannian manifold. Let β be a $(1,1)$-tensor field on (M^n, g) such that $\beta^2 = \sigma \mathrm{Id}$, $\sigma = \pm 1$ and

$$g(\beta(X), \beta(Y)) = g(X, Y)$$

for all vector fields $X, Y \in \Gamma(TM^n)$ (here Id stands for the identity map). We say (M^n, g, β) is an almost Hermitian manifold if $\sigma = -1$ and an almost product Riemannian manifold if $\sigma = 1$, respectively. Moreover, if $\sigma = -1$ and β is parallel, (M^n, g, β) is called a Kähler manifold. Similarly, we have the following definition.

Definition 2.1 (see [10, 13]) *An n-Riemannian manifold (M^n, g) is called locally decomposable if it is an almost product Riemannian manifold (M^n, g, β) and β is parallel.*

In case that (M^n, g, β) is locally decomposable, the tangent bundle TM^n decomposes into $TM^n = T^+M^n \oplus T^-M^n$ under the action of the endomorphism β, where

$$T^{\pm}M^n \triangleq \{X \in TM^n \mid \beta(X) = \pm X\}.$$

Obviously, the distributions $T^{\pm}M^n$ are integrable since β is parallel. If (M^n, g) is simply connected and complete, then the De Rham decomposition theorem implies that there is a global splitting $(M^n, g) = (M_1 \times M_2, g_1 + g_2)$.

Example 2.1 Suppose that an n-Riemannian manifold (M^n, g) possesses a unit vector field $\xi \in \Gamma(TM^n)$. Then the reflection β defined by

$$\beta(X) \triangleq X - 2g(X, \xi)\xi, \quad X \in \Gamma(TM)$$

is an almost product Riemannian structure. Moreover, it is a locally decomposable Riemannian structure if ξ is a parallel vector field.

We now suppose that (M^n, g) is a Riemannian manifold with a fixed spin structure. We understand the spin structure as a reduction $\mathrm{Spin}M^n$ of the $\mathrm{SO}(n)$-principal bundle of M^n to the universal covering $Ad : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ of the special orthogonal group. The spinor bundle $\Sigma M^n = \mathrm{Spin}M^n \times_\rho \Sigma_n$ on M^n is the associated complex $2^{[\frac{n}{2}]}$ dimensional complex vector bundle, where ρ is the complex spinor representation. The tangent bundle TM^n can be regarded as $TM^n = \mathrm{Spin}M^n \times_{Ad} \mathbb{R}^n$. Consequently, the Clifford multiplication on ΣM^n is the fibrewise action given by

$$\begin{aligned} \mu : TM^n \otimes \Sigma M^n &\longrightarrow \Sigma M^n, \\ X \otimes \psi &\longmapsto X \cdot \psi. \end{aligned}$$

On the spinor bundle ΣM^n, one has a natural Hermitian metric, denoted as the Riemannian metric by $\langle \cdot, \cdot \rangle$. The spinorial connection on the spinor bundle induced by the Levi-Civita connection ∇ on M^n will also be denoted by ∇. The Hermitian metric $\langle \cdot, \cdot \rangle$ and spinorial connection ∇ are compatible with the Clifford multiplication μ. That is

$$\begin{aligned} X\langle \phi, \varphi \rangle &= \langle \nabla_X \phi, \varphi \rangle + \langle \phi, \nabla_X \varphi \rangle, \\ \langle X \cdot \phi, X \cdot \varphi \rangle &= |X|^2 \langle \phi, \varphi \rangle, \\ \nabla_X (Y \cdot \phi) &= \nabla_X Y \cdot \phi + Y \nabla_X \phi \end{aligned}$$

$\forall \phi, \varphi \in \Gamma(\Sigma M^n)$ and $\forall X, Y \in \Gamma(TM^n)$. Using a local orthonormal frame field $\{e_1, \cdots, e_n\}$, the spinorial connection ∇, the Dirac operator D and the twistor operator P, are locally expressed as

$$\nabla_{e_k} \psi = e_k(\psi) + \frac{1}{4} e_i \cdot \nabla_{e_k} e_i \cdot \psi, \tag{2.1}$$

$$D\psi \triangleq e_i \cdot \nabla_{e_i} \psi, \tag{2.2}$$

$$P\psi \triangleq e_i \otimes \Big(\nabla_{e_i} \psi + \frac{1}{n} e_i \cdot D\psi \Big), \tag{2.3}$$

respectively, which satisfy the following important relation:

$$|\nabla \psi|^2 = |P\psi|^2 + \frac{1}{n} |D\psi|^2$$

for any $\psi \in \Gamma(\Sigma M^n)$ (throughout this paper, the Einstein summation notation is always adopted). The kernels of the operators D and P are respectively, the twistor spinors and the harmonic spinors, and they are both conformally invariant. If M is closed, $\mathrm{Ker}\, D = \mathrm{Ker}\, D^2$ on $L^2(\Sigma M^n)$.

Let $R_{X,Y} Z \triangleq (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z$ be the Riemannian curvature of (M^n, g) and denote by $\mathcal{R}_{X,Y} \psi \triangleq (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) \psi$ the spin curvature in the spinor bundle ΣM^n. They are related via the formula

$$\mathcal{R}_{X,Y} \psi = \frac{1}{4} g(R_{X,Y} e_i, e_j) e_i \cdot e_j \cdot \psi. \tag{2.4}$$

We also use the notation

$$R_{ijkl} \triangleq g(R_{e_i, e_j} e_k, e_l)$$

and $R_{ij} = \langle \mathrm{Ric}(e_i), e_j \rangle \triangleq R_{ikkj}, R = R_{ii}$. With the help of the Bianchi identity, (2.4) implies

$$e_i \cdot \mathcal{R}_{e_j,e_i}\psi = -\frac{1}{2}\mathrm{Ric}(e_j) \cdot \psi, \tag{2.5}$$

which in turn gives $2e_i \cdot e_j \cdot \mathcal{R}_{e_i,e_j}\psi = R\psi$. Hence one derives the well-known Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}R, \tag{2.6}$$

where ∇^* is the formal adjoint of ∇ with respect to the natural Hermitian scalar product on ΣM^n. The formula shows the close relation between the scalar curvature R and the Dirac operator D.

On almost Hermitian manifolds or almost product Riemannian manifolds, we can also define the following β-twist D_β of the Dirac operator D by

$$D_\beta\psi \triangleq e_i \cdot \nabla_{\beta(e_i)}\psi = \sigma\beta(e_i) \cdot \nabla_{e_i}\psi. \tag{2.7}$$

It is easy to see that D_β is a formally self-adjoint elliptic operator with respect to L^2-product, if M^n is closed and $\mathrm{div}\beta = 0$. As in the Kählerian case, Kim obtained that $D^2 = D_\beta^2$ holds on the locally decomposable Riemannian spin manifold (M^n, g, β) (see Prop. 2.1 in [10]).

3 Quasi-Killing Spinors

Definition 3.1 (see [10]) *A non-trivial solution ψ to the following field equation on the almost product Riemannian manifold (M^n, g, β)*

$$\nabla_X\psi = aX \cdot \psi + b\beta(X) \cdot \psi, \quad a, b \in C^\infty(M, \mathbb{R}) \tag{3.1}$$

is called a quasi-Killing spinor of type (a, b).

Obviously, if ψ is a quasi-Killing spinor of type (a, b), the energy-momentum tensor associated to ψ is given on the complement of its zero set by

$$\begin{aligned} Q_\psi(X, Y) &\triangleq \frac{1}{2}\mathfrak{Re}\langle X \cdot \nabla_Y\psi + Y \cdot \nabla_X\psi/|\psi|^2\rangle \\ &= -a\langle X, Y\rangle - b\beta(X, Y) \end{aligned}$$

for any $X, Y \in \Gamma(TM^n)$. Especially, the quasi-Killing spinor of type $(a, 0)$ (or $\beta = \pm\mathrm{Id}$) is also called the generalized Killing spinor. In fact, in this case one can prove that the function a must be a constant. That is, ψ is in fact a Killing spinor. In addition, Hijazi proved that a manifold admitting a parallel 1-form carries no real Killing spinors (see [6]). Furthermore, we can prove the following theorem.

Theorem 3.1 *Let ψ be a quasi-Killing spinor of type (a, b) on a locally decomposable Riemannian spin manifold (M^n, g, β), where $\beta \neq \pm\mathrm{Id}$. Then $|\psi|^2$ is a positive constant and*

(1) *if $R \not\equiv 0$, ψ is an eigenspinor of D, $0 \neq a = b$ (or $0 \neq a = -b$) is constant, and R is a positive constant;*

(2) *if $R \equiv 0$, then* $\mathrm{Ric} \equiv 0$;

(3) *the real vector field* X_ψ *defined by*

$$g(X_\psi, Y) \triangleq \sqrt{-1}\langle \psi, Q_\psi(Y)\cdot\psi\rangle, \quad \forall Y \in \Gamma(TM)$$

is a Killing field, i.e., $\mathcal{L}_{X_\psi} g = 0$.

Proof (1) First, from

$$\nabla_i\psi \triangleq \nabla_{e_i}\psi = ae_i\cdot\psi + b\beta(e_i)\cdot\psi,$$

we know that $\nabla_i|\psi|^2 = 2\Re e\langle\nabla_i\psi,\psi\rangle = 0$. Hence $|\psi|^2$ is a positive constant. One can also easily check

$$\begin{aligned} D\psi &= e_i\nabla_i\psi = -(na + b\mathrm{tr}\beta)\psi, \\ D^2\psi &= (na + b\mathrm{tr}\beta)^2\psi - (n\nabla a + \mathrm{tr}\beta\nabla b)\psi \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} D_\beta\psi &= \beta(e_i)\nabla_i\psi = -(nb + a\mathrm{tr}\beta)\psi, \\ D_\beta^2\psi &= (nb + a\mathrm{tr}\beta)^2\psi - (n\beta(\nabla b) + \mathrm{tr}\beta\beta(\nabla a))\psi. \end{aligned} \tag{3.3}$$

In particular

$$\begin{cases} na + b\mathrm{tr}\beta = \pm(nb + a\mathrm{tr}\beta), \\ n\nabla a + \mathrm{tr}\beta\nabla b = n\beta(\nabla b) + \mathrm{tr}\beta\beta(\nabla a), \end{cases}$$

since $D_\beta^2 = D^2$. Noting $\beta \neq \pm\mathrm{Id}$, we have $a = \pm b$, which in turn implies that $\beta(\nabla a) = \pm\nabla a$. Hence the quasi-Killing equation can also be written as

$$\nabla_i\psi = a[e_i \pm \beta(e_i)]\cdot\psi \triangleq -Q_\psi(e_i)\cdot\psi, \tag{3.4}$$

where $Q_\psi = -a(\mathrm{Id} \pm \beta)$. Moreover, by (2.5),

$$\begin{aligned} \frac{1}{2}\mathrm{Ric}(e_i)\cdot\psi = \nabla a\cdot(e_i \pm \beta(e_i))\cdot\psi + (n \pm \mathrm{tr}\beta)a_i\psi \\ + 4a^2(n \pm \mathrm{tr}\beta - 2)[e_i \pm \beta(e_i)]\cdot\psi. \end{aligned} \tag{3.5}$$

Hence performing its Clifford multiplication by e_i yields

$$\Big[\frac{R}{4} + |Q_\psi|^2 - (\mathrm{tr}Q_\psi)^2\Big]\psi = (d\mathrm{tr}Q_\psi - \mathrm{div}Q_\psi)\cdot\psi. \tag{3.6}$$

Using $(\mathrm{tr}Q_\psi)^2 = \frac{1}{4}R + |Q_\psi|^2$, it follows that

$$R = 4a^2(n \pm \mathrm{tr}\beta)(n \pm \mathrm{tr}\beta - 2). \tag{3.7}$$

By $d\,\mathrm{tr}Q_\psi = \mathrm{div}Q_\psi$, we infer that

$$(n \pm \mathrm{tr}\beta)\nabla a = -(Q_\psi)_{ij,i}e_j = \nabla a \pm \beta(\nabla a) = 2\nabla a. \tag{3.8}$$

Consequently, $\nabla a = 0$ since R is non-zero. Moreover, R is a positive constant and $a = b = -\frac{\lambda}{2n_1}$ or $a = -b = -\frac{\lambda}{2n_2}$, where $n_1 \triangleq \dim T^+M^n$, $n_2 \triangleq \dim T^-M^n$.

(2) If $R \equiv 0$, (3.7) yields $a = 0$ or $n \pm \mathrm{tr}\beta - 2 \equiv 0$. If $a = 0$, $\nabla\psi = 0$ and $\mathrm{Ric} \equiv 0$. If $n + \mathrm{tr}\beta - 2 \equiv 0$, we see by (3.5) that

$$\frac{1}{2}\mathrm{Ric}(e_i) \cdot \psi = \nabla a \cdot (e_i + \beta(e_i)) \cdot \psi + (n + \mathrm{tr}\beta) a_i \psi,$$

where $\beta(\nabla a) = \nabla a \triangleq a_i e_i$. Obviously

$$\frac{1}{2}\mathrm{Ric}(\nabla a) \cdot \psi = \nabla a \cdot [\nabla a + \beta(\nabla a)] \cdot \psi + 2|\nabla a|^2 \psi = 0, \tag{3.9}$$

$$\begin{aligned}\frac{1}{2}\mathrm{Ric}(X) \cdot \psi &= \nabla a \cdot [X + \beta(X)] \cdot \psi + 2\langle \nabla a, X\rangle \psi \\ &= 0, \quad \forall X \perp \nabla a,\end{aligned} \tag{3.10}$$

from which the result follows.

(3) Since the Clifford multiplication by vector fields is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$, the vector field X_ψ is real. We need the following formula for arbitrary vector fields $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned}(\mathcal{L}_X g)(Y, Z) &= X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X),\end{aligned}$$

since ∇ is metric and torsion-free. On the other hand, by the definition of X_ψ, at the point p with $\nabla e_i|_p = 0$,

$$\begin{aligned}g(\nabla_i X_\psi, e_j) &= e_i(g(X_\psi, e_j)) \\ &= \sqrt{-1}\langle Q_\psi(e_j) \cdot Q_\psi(e_i) \cdot \psi, \psi\rangle - \sqrt{-1}\langle Q_\psi(e_i) \cdot Q_\psi(e_j) \cdot \psi, \psi\rangle\end{aligned}$$

which is clearly skew-symmetric with respect to e_i, e_j.

Remark 3.1 We can compute if $R \not\equiv 0$ and M^n is closed,

$$R_{ij} = 4a^2(n - 2 \pm \mathrm{tr}\beta)(\delta_{ij} \pm \beta_{ij}). \tag{3.11}$$

Hence, $\mathrm{Ric} \geq 0$ and moreover, by the Bochner-Weitzenböck formula, we know that every harmonic 1-form on M^n is parallel.

One application of the quasi-Killing spinor is another simple proof of the following theorem, which is due to Alexandrov, Grantcharov and Ivanov [12]. The other related issues can be seen in [10, 14–15].

Theorem 3.2 *Let $(M^n, g), n \geq 3$ be a closed Riemannian spin manifold of positive scalar curvature admitting a non-trivial parallel vector field of unit length. Then any eigenvalue λ of the Dirac operator D satisfies*

$$\lambda^2 \geq \frac{n-1}{4(n-2)} \inf_{M^n} R. \tag{3.12}$$

The equality in (3.12) *occurs if and only if there exists a quasi-Killing spinor field of type* $\big(-\frac{\lambda}{2(n-1)}, -\frac{\lambda}{2(n-1)}\big)$ *on* (M^n, g).

Proof First suppose that ξ is a unit parallel vector field and let

$$T_i\varphi \triangleq \nabla_i\varphi + \frac{\lambda}{2(n-1)}e_i\cdot\varphi + \frac{\lambda}{2(n-1)}\beta(e_i)\cdot\varphi, \tag{3.13}$$

where $D\varphi=\lambda\varphi,\ \beta(e_i)\triangleq e_i-2\langle e_i,\xi\rangle\xi$. Then, an elementary calculation provides the following

$$|T\varphi|^2 = |\nabla\varphi|^2 + \frac{1}{2(n-1)}(|D_\beta\varphi - D\varphi|^2 - |D_\beta\varphi|^2 - |D\varphi|^2). \tag{3.14}$$

At the same time, $\nabla\xi=0$ yields

$$\int_{M^n}|D_\beta\varphi|^2 = \int_{M^n}|D\varphi|^2 = \lambda^2\int_{M^n}|\varphi|^2.$$

Hence integrating (3.14) and applying the Schrödinger-Lichnerowicz formula (2.6), we find that

$$\int_{M^n}2(n-1)|T\varphi|^2 = \int_{M^n}2(n-2)\lambda^2|\varphi|^2 - \frac{n-1}{2}R|\varphi|^2 + |D_\beta\varphi - D\varphi|^2.$$

Note by the definition,

$$\begin{aligned}\beta(e_i)\cdot T_i\varphi &= D_\beta\varphi-\lambda\varphi,\\ e_i\cdot T_i\varphi &= D\varphi-\lambda\varphi.\end{aligned}$$

So using the Cauchy-Schwarz inequality leads to

$$\begin{aligned}|D_\beta\varphi - D\varphi|^2 &\le \Big(\sum|\beta(e_i)-e_i||T_i\varphi|\Big)^2\\ &= \Big(\sum(2-2\beta_{ii})^{\frac12}|T_i\varphi|\Big)^2\\ &\le \sum(2-2\beta_{ii})\cdot\sum|T_i\varphi|^2\\ &= 4|T\varphi|^2.\end{aligned}$$

From this, it follows immediately that

$$0\le\frac{n-3}{n-2}\int_{M^n}|T\varphi|^2 \le \int_{M^n}\Big[\lambda^2 - \frac{n-1}{4(n-2)}R\Big]|\varphi|^2.$$

If λ^2 achieves its minimum, then $T\varphi\equiv0$, which implies the associated eigenspinor φ is a non-trivial quasi-Killing spinor field on locally decomposable Riemannian spin manifold (M^n,g,β).

Remark 3.2 It follows from $T\varphi\equiv0,\ \beta_{ij}=\delta_{ij}-2\xi_i\xi_j$ that

$$\nabla_\xi\varphi=0,\quad \nabla_X\varphi = -\frac{\lambda}{n-1}X\cdot\varphi,\quad \forall X\perp\xi.$$

By Bär's result in [2], the universal covering space of the manifolds in the limiting case was described in [12].

Remark 3.3 The proof given above also works if ξ is just a harmonic vector field of unit length, and hence the result in [14] is also obtained. In fact, with the help of the Bochner-Weitzenböck formula on 1-forms, it is not difficult to check that for any ϕ,

$$D_\beta^2\phi = D^2\phi - \xi\cdot\nabla^*\nabla(\xi\cdot\phi) - \nabla^*\nabla\phi. \tag{3.15}$$

Note the fact that ξ is a harmonic vector field of unit length implies $\mathrm{div}(\beta)=0$, hence D_β is self-adjoint with respect to L^2-product. Hence

$$\int_{M^n}|D_\beta\phi|^2 \overset{(3.15)}{=} \int_{M^n}\langle D^2\phi-\xi\cdot\nabla^*\nabla(\xi\cdot\phi)-\nabla^*\nabla\phi,\phi\rangle$$
$$=\int_{M^n}|D\phi|^2+|\nabla(\xi\cdot\phi)|^2-|\nabla\phi|^2.$$

So, if $D\phi=\lambda\phi$, one can use the classical Rayleigh inequality and (2.6) to conclude

$$\int_{M^n}|D_\beta\phi|^2\geq\int_{M^n}|D\phi|^2,$$

and the remaining proof is quite similar to that of Theorem 3.2.

4 Quasi-twistor Spinors

Analogous to the Kählerian twistor equation in [8], we have the following definition.

Definition 4.1 (see [10]) *A non-trivial solution ψ to the following field equation on almost product Riemannian manifold (M^n,g,β):*

$$\nabla_X\psi=pX\cdot D\psi+q\beta(X)\cdot D_\beta\psi, \tag{4.1}$$

is called a quasi-twistor spinor of type (p,q), where $p,q\in\mathbb{R}$.

Remark 4.1 Obviously, the quasi-twistor spinor of type $(-\frac{1}{n},0)$ is the familiar twistor spinor or called conformal Killing spinor which lies in the kernel of the twistor operator P (see (2.3)).

On a locally decomposable spin manifold, a straightforward computation using (4.1) gives the following $\frac{1}{2}$-Ric formula (see [10])

$$\frac{1}{2}\mathrm{Ric}(e_i)\cdot\psi=-pe_i\cdot D^2\psi-(2p+1)\nabla_i(D\psi)-2q\nabla_{\beta(e_i)}(D_\beta\psi)-q\beta(e_i)\cdot DD_\beta\psi$$

and the following useful identity

$$4(p+q+1)D^2\psi=R\psi. \tag{4.2}$$

Theorem 4.1 *Suppose that ψ is a quasi-twistor spinor of type (p,q) on a locally decomposable Riemannian spin manifold (M^n,g,β), $\beta\neq\pm\mathrm{Id}$ and $D\psi=\lambda\psi$, where $\lambda\neq0$. Then $p=q=-\frac{1}{2n_1}$ or $-\frac{1}{2n_2}$.*

Proof (1) If $\mathrm{tr}\beta=0$, then from $D^2\psi=D_\beta^2\psi=\lambda^2\psi\neq0$ and

$$(1+np)D\psi+\mathrm{tr}(\beta)qD_\beta\psi=0,$$
$$(1+nq)D_\beta\psi+\mathrm{tr}(\beta)pD\psi=0,$$

one gets $p=q=-\frac{1}{n}$.

(2) If $p=-\frac{1}{n}$, $q=0$, or $q=-\frac{1}{n}$, $p=0$, then the limiting-case in Friedrich's inequality is achieved, and moreover, M^n carries a nontrivial Killing spinor with a real nonzero Killing number. Hence M^n is locally irreducible, which is a contradiction (see [16]).

(3) If $\mathrm{tr}\beta \neq 0$ and $pq \neq 0$, we know

$$\frac{q\mathrm{tr}\beta}{1+np} = \frac{1+nq}{p\mathrm{tr}\beta} = \pm 1. \tag{4.3}$$

Hence $p = q = -\frac{1}{2n_1}$ or $-\frac{1}{2n_2}$. In this case, $D_\beta\psi = \mp D\psi = \mp\lambda\psi$. Consequently, ψ is a quasi-Killing spinor.

Now we turn to discuss the existence of twistor spinors. It is well-known that

$$\dim(\mathrm{Ker}P) \leq 2^{[\frac{n}{2}]+1} = 2\mathrm{rank}(\Sigma M^n)$$

if $n \geq 3$, and the maximal possible dimension is attained only for conformal flat manifolds as in the case of conformal Killing fields. Furthermore, Hijazi proved that on a Kähler spin manifold with $R \not\equiv 0$, the space of twistor spinors is reduced to zero (see [8]). Another proof of this result can also been seen in [11]. Here, we prove the following theorem.

Theorem 4.2 *Suppose that (M^n, g) is a closed Riemannian spin manifold admitting a non-trivial harmonic vector field ξ, ψ is a non-trivial twistor spinor. Then on M^n the following integrability condition holds:*

$$|\xi|^2\nabla R \cdot \psi = n\xi(R)\xi \cdot \psi + 2(n-1)\xi \cdot D(\mathrm{Ric})(\xi) \cdot \psi, \tag{4.4}$$

where $D(\mathrm{Ric})(\xi) \triangleq e_i \cdot (\nabla_i \mathrm{Ric})(\xi)$.

Proof Let ψ be a non-trivial twistor spinor, i.e.,

$$\nabla_i\psi = -\frac{1}{n}e_i \cdot D\psi, \tag{4.5}$$

which implies the following integrability conditions

$$\frac{n}{2}\mathrm{Ric}(e_i) \cdot \psi = e_i \cdot D^2\psi - (n-2)\nabla_i D\psi \tag{4.6}$$

and

$$D^2\psi = \frac{n}{4(n-1)}R\psi. \tag{4.7}$$

Hence from (4.6),

$$D(|\xi|^2 D^2\psi) = (2-n)D(\xi\nabla_\xi D\psi) - \frac{n}{2}D[\xi \cdot \mathrm{Ric}(\xi) \cdot \psi]. \tag{4.8}$$

First, the harmonicity of the vector field $\xi = \xi_i e_i$, together with the compactness of M^n, implies that

$$0 = D\xi = \mathrm{d}\xi + \delta\xi. \tag{4.9}$$

This means that $\xi_{i,j} = \xi_{j,i}$ and $\xi_{i,i} = 0$, respectively. Moreover,

$$\nabla_\xi\xi = \xi_i\xi_{j,i}e_j = \xi_i\xi_{i,j}e_j = \frac{1}{2}d|\xi|^2. \tag{4.10}$$

On the one hand,

$$
\begin{aligned}
(2-n)D(\xi\nabla_\xi D\psi) &\overset{(4.9)}{=} (2-n)e_i\cdot\xi\cdot\nabla_i(\nabla_\xi D\psi)\\
&= (n-2)(\xi\cdot e_i+2\xi_i)\nabla_i(\nabla_\xi D\psi)\\
&= (n-2)\xi\cdot e_i\cdot(\mathcal{R}_{e_i,\xi}+\nabla_\xi\nabla_i+\nabla_{[e_i,\xi]})D\psi+2(n-2)\nabla_\xi\nabla_\xi D\psi\\
&\overset{(2.5)}{=} (n-2)\Big[\frac{1}{2}\xi\cdot\mathrm{Ric}(\xi)\cdot D\psi+\xi\nabla_\xi D^2\psi+\xi\cdot e_i\cdot\nabla_{[e_i,\xi]}D\psi\Big]\\
&\quad +2(n-2)\nabla_\xi\nabla_\xi D\psi\\
&\overset{(4.6)}{=} \frac{n-2}{2}\xi\cdot\mathrm{Ric}(\xi)\cdot D\psi+n\xi\nabla_\xi D^2\psi-\frac{n}{2}\xi\cdot e_i\cdot\mathrm{Ric}(\nabla_i\xi)\cdot\psi\\
&\quad -n\nabla_\xi[\mathrm{Ric}(\xi)\cdot\psi]+2\nabla_\xi\xi\cdot D^2\psi,
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
-\frac{n}{2}D[\xi\cdot\mathrm{Ric}(\xi)\cdot\psi] &= -\frac{n}{2}e_i\cdot\xi\cdot\nabla_i[\mathrm{Ric}(\xi)\cdot\psi]\\
&= \frac{n}{2}(\xi\cdot e_i+2\xi_i)\nabla_i[\mathrm{Ric}(\xi)\cdot\psi]\\
&= \frac{n}{2}\xi\cdot[D(\mathrm{Ric}\xi)\cdot\psi+e_i\cdot\mathrm{Ric}(\xi)\cdot\nabla_i\psi]+n\nabla_\xi[\mathrm{Ric}(\xi)\cdot\psi]\\
&\overset{(4.5)}{=} \frac{n}{2}\xi\cdot D(\mathrm{Ric}\xi)\cdot\psi-\frac{n-2}{2}\xi\cdot\mathrm{Ric}(\xi)\cdot D\psi+n\nabla_\xi[\mathrm{Ric}(\xi)\cdot\psi].
\end{aligned}
$$

Therefore (4.8) turns into

$$
\begin{aligned}
D(|\xi|^2D^2\psi) &= n\xi\nabla_\xi D^2\psi+2\nabla_\xi\xi\cdot D^2\psi+\frac{n}{2}\xi\cdot D(\mathrm{Ric}(\xi))\cdot\psi-\frac{n}{2}\xi\cdot e_i\cdot\mathrm{Ric}(\nabla_i\xi)\cdot\psi\\
&= n\xi\nabla_\xi D^2\psi+2\nabla_\xi\xi\cdot D^2\psi+\frac{n}{2}\xi\cdot D(\mathrm{Ric})(\xi)\cdot\psi. \qquad (4.11)
\end{aligned}
$$

From (4.5), (4.7) and (4.10)–(4.11), it is clear that

$$|\xi|^2\nabla R\cdot\psi = n\xi(R)\xi\cdot\psi+2(n-1)\xi\cdot D(\mathrm{Ric})(\xi)\cdot\psi.$$

Hence the proof of the theorem is completed.

Remark 4.2 Note

$$
\begin{aligned}
D(\mathrm{Ric}(\xi))\cdot\psi &= e_k\cdot\nabla_k(\xi_iR_{ij}e_j)\cdot\psi\\
&= (\xi_{i,k}R_{ij}+\xi_iR_{ij,k})e_k\cdot e_j\cdot\psi\\
&= \sum_{k\neq j}(\xi_{i,k}R_{ij}+\xi_iR_{ij,k})e_k\cdot e_j\cdot\psi-\mathrm{div}(\mathrm{Ric}(\xi))\psi
\end{aligned}
$$

and any non-trivial twistor spinor on a spin manifold vanishes at most at one point (see [5]). So taking the inner product of (4.4) with $\xi\cdot\psi$ and comparing its real part, we obtain on M^n,

$$2\mathrm{div}(\mathrm{Ric}(\xi))-\xi(R) = 2\xi_{i,j}R_{ij},$$

which is also a corollary of the well-known fact that the Einstein tensor $G \triangleq \mathrm{Ric}-\frac{R}{2}g$ is divergence-free.

As an immediate consequence of the preceding theorem, we obtain the following corollary.

Corollary 4.1 *Suppose a closed Riemannian spin manifold admits a non-trivial parallel vector field and $R \not\equiv 0$, and then the space of twistor spinors is trivial.*

Proof Suppose ξ is a unit parallel vector field, and we denote the dual one-form of ξ also by ξ. Since

$$\Delta\xi = \nabla^*\nabla\xi + \mathrm{Ric}(\xi),$$

it follows that $\mathrm{Ric}(\xi) = 0$. Hence the theorem above implies

$$\nabla R \cdot \psi = n\xi(R)\xi \cdot \psi = 0, \tag{4.12}$$

for any non-trivial twistor spinor ψ. Eventually, we find that $R \equiv \text{constant} \geq 0$, since all eigenvalues of D^2 are non-negative on closed spin manifolds. Hence (4.7) implies that the limiting-case in Friedrich's inequality is achieved, and moreover, (M^n, g) is Einstein with $R \geq 0$. In fact, ψ is the sum of two non-parallel real Killing spinors, or ψ is parallel (in this case $R \equiv 0$), which is a contradiction.

Corollary 4.2 *Suppose spin manifold (M^n, g) is a closed Riemannian symmetric space with $b_1(M) \neq 0$ and $R \not\equiv 0$, and then the space of twistor spinors is trivial.*

Remark 4.3 In fact, from the proof of the theorem above one can easily see that if a (not necessarily closed) Riemannian spin manifold admits a non-trivial unit parallel vector field and $R \not\equiv 0$, $\nabla R \cdot \psi = n\xi(R)\xi \cdot \psi$ still holds for any non-trivial twistor spinor ψ. So R must be a constant (≤ 0).

Now we return to studying the uniqueness of quasi-twistor spinors.

Theorem 4.3 *Let ψ be a quasi-twistor spinor of type (p, q) on a locally decomposable complete Riemannian spin manifold (M^n, g, β), $\beta \neq \pm\mathrm{Id}$. Then*
(1) *If $\nabla R \neq 0$, then $p = q = -\frac{1}{2n_1}$ or $-\frac{1}{2n_2}$;*
(2) *If R is a nonzero constant and $\psi \in L^2(\Sigma M^n)$, we also have $p = q = -\frac{1}{2n_1}$ or $-\frac{1}{2n_2}$, here $n_1 \triangleq \mathrm{dim} T^+M^n$, and $n_2 \triangleq \mathrm{dim} T^-M^n$.*

Proof (1) First, assume $\nabla R \neq 0$. From

$$\nabla_i\psi = pe_i \cdot D\psi + q\beta(e_i) \cdot D_\beta\psi,$$

we obtain

$$(1 + np)D\psi + \mathrm{tr}(\beta)qD_\beta\psi = 0, \tag{4.13}$$
$$(1 + nq)D_\beta\psi + \mathrm{tr}(\beta)pD\psi = 0. \tag{4.14}$$

Noting $R \not\equiv 0$, we also have

$$(1 + np)(1 + nq) = pq(\mathrm{tr}\beta)^2 \tag{4.15}$$

and

$$(1 + np)D^3\psi + \mathrm{tr}(\beta)qD^2D_\beta\psi = 0, \tag{4.16}$$
$$(1 + nq)D^3_\beta\psi + \mathrm{tr}(\beta)pD^2_\beta D\psi = 0. \tag{4.17}$$

Note $D^2 = D_\beta^2$ and $4(p+q+1)D^2\psi = R\psi \not\equiv 0$, so $p+q+1 \neq 0$ and

$$(1+np)(\nabla R\cdot\psi + RD\psi) + \mathrm{tr}(\beta)q[\beta(\nabla R)\cdot\psi + RD_\beta\psi] = 0,$$
$$(1+nq)[\beta(\nabla R)\cdot\psi + RD_\beta\psi] + \mathrm{tr}(\beta)p(\nabla R\cdot\psi + RD\psi) = 0.$$

That is

$$(1+np)\nabla R\cdot\psi + \mathrm{tr}(\beta)q\beta(\nabla R)\cdot\psi = 0, \tag{4.18}$$
$$(1+nq)\beta(\nabla R)\cdot\psi + \mathrm{tr}(\beta)p\nabla R\cdot\psi = 0. \tag{4.19}$$

If $\psi(m) \neq 0$, it follows from (4.18) that

$$(1+np)\nabla R(m) + \mathrm{tr}(\beta)q\beta(\nabla R)(m) = 0.$$

Suppose now $\psi(m) = 0$. Since ψ is a solution of the elliptic differential equation

$$D^2\psi = \frac{1}{4(p+q+1)}R\psi,$$

there exists a sequence of points m_i converging to m such that $\psi(m_i) \neq 0$. Then we have $(1+np)\nabla R + \mathrm{tr}(\beta)q\beta(\nabla R) = 0$ at points m_i and with respect to the continuity of ∇R and $\beta(\nabla R)$. We obtain again $(1+np)\nabla R(m) + \mathrm{tr}(\beta)q\beta(\nabla R)(m) = 0$. Hence from (4.18), using $\beta^2 = \mathrm{Id}$, one gets the system

$$\text{(A)}\quad \begin{cases}(1+np+q\mathrm{tr}\beta)[\nabla R + \beta(\nabla R)] = 0,\\ (1+np-q\mathrm{tr}\beta)[\nabla R - \beta(\nabla R)] = 0.\end{cases}$$

Similarly, from (4.19), we obtain

$$\text{(B)}\quad \begin{cases}(1+nq+p\mathrm{tr}\beta)[\nabla R + \beta(\nabla R)] = 0,\\ (1+nq-p\mathrm{tr}\beta)[\nabla R - \beta(\nabla R)] = 0.\end{cases}$$

Hence, if $\nabla R + \beta(\nabla R) \neq 0$ and $\nabla R \neq 0$, then from (A) and (B)

$$1+nq+p\mathrm{tr}\beta = 0,$$
$$1+np+q\mathrm{tr}\beta = 0.$$

Note

$$\beta \neq \pm\mathrm{Id}.$$

So

$$\begin{vmatrix}\mathrm{tr}\beta & n\\ n & \mathrm{tr}\beta\end{vmatrix} \neq 0.$$

Hence solving the linear equations above leads to

$$p = q = -\frac{1}{2n_1}, \quad \beta(\nabla R) = \nabla R, \tag{4.20}$$

and $D\psi = D_\beta\psi$. Moreover, $\nabla_i\psi = -\frac{1}{n_1}e_i\cdot D\psi$ for $i \leq n_1$ and $\nabla_j\psi = 0$ for $j > n_1$.

If $\nabla R + \beta(\nabla R) = 0$ and $\nabla R \neq 0$, then $\nabla R - \beta(\nabla R) \neq 0$. Moreover, from (A) and (B),

$$1+nq-p\mathrm{tr}\beta = 0, \tag{4.21}$$
$$1+np-q\mathrm{tr}\beta = 0. \tag{4.22}$$

A similar argument shows that

$$p = q = -\frac{1}{2n_2}, \quad \beta(\nabla R) = -\nabla R, \tag{4.23}$$

and $D\psi = -D_\beta\psi$.

(2) Now suppose that R is a nonzero constant and $\psi \in L^2(\Sigma M^n)$. Hence $D^2\psi \in L^2(\Sigma M^n)$ by (4.2). Note for the L^2-norm $\|\cdot\|$ and any number $t > 0$, we have (see [17, p. 96])

$$\|D\psi\|^2 \leq t\|D^2\psi\|^2 + \frac{1}{t}\|D\psi\|^2,$$

which implies that $D\psi \in L^2(\Sigma M^n)$. Therefore we know that ψ lies in the domain of the maximal extension of D. Since M^n is complete, D is essentially self-adjoint as an unbounded operator in $L^2(\Sigma M^n)$, so the maximal and the minimal extensions coincide and $\psi \in \mathrm{dom}(\overline{D}) = \mathrm{dom}(D^*)$.

On the other hand, by combining (4.13) and (4.14) we find that

$$p(1 + np)|D\psi|^2 = q(1 + nq)|D_\beta\psi|^2. \tag{4.24}$$

Therefore by integrating (4.24) and using $D_\beta^2 = D^2$, one obtains

$$(p - q)[n(p + q) + 1]\|D\psi\|^2 = 0. \tag{4.25}$$

Case 1 If $\|D\psi\|^2 (= \|D_\beta\psi\|^2) \equiv 0$, then $\nabla\psi = 0$.

Case 2 If $p = q$, then (4.15) implies that

$$p = q = -\frac{1}{2n_1} \text{ or } -\frac{1}{2n_2}.$$

Case 3 Suppose $p + q = -\frac{1}{n}$. Then

$$D^2\psi = \frac{n}{4(n-1)} R\psi.$$

Clearly, R is a positive constant. Hence M^n carries a non-parallel real Killing spinor, which is a contradiction.

Obviously, from the proof of the above theorem, one gets the following theorem.

Theorem 4.4 *Let ψ be a quasi-twistor spinor of type (p, q) on a locally decomposable closed Riemannian spin manifold (M^n, g, β), $\beta \neq \pm\mathrm{Id}$. Then*

(1) $p = q = -\frac{1}{2n_1}$ *or* $-\frac{1}{2n_2}$; *or*

(2) $\nabla\psi = 0$.

Remark 4.4 When $D\psi = \lambda\psi$, Kim and Alexandrov classify all the types of spin manifolds admitting non-trivial quasi-twistor spinors of type $\left(-\frac{1}{2n_1}, -\frac{1}{2n_1}\right)$.

Corollary 4.3 *On a locally decomposable closed Riemannian spin manifold with $\beta \neq \pm\mathrm{Id}$ and $\mathrm{Ric} \not\equiv 0$, the space of twistor spinors is trivial.*

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