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Initial Boundary Value Problem of an Equation from Mathematical Finance*

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Abstract Consider the initial boundary value problem of the strong degenerate parabolic equation

$$\partial_{xx}u + u\partial_y u - \partial_t u = f(x,y,t,u), \quad (x,y,t) \in Q_T = \Omega \times (0,T)$$

with a homogeneous boundary condition. By introducing a new kind of entropy solution, according to Oleinik rules, the partial boundary condition is given to assure the well-posedness of the problem. By the parabolic regularization method, the uniform estimate of the gradient is obtained, and by using Kolmogoroff's theorem, the solvability of the equation is obtained in $BV(Q_T)$ sense. The stability of the solutions is obtained by Kruzkov's double variables method.

Keywords Mathematical finance, Oleinik rules, Partial boundary condition, Entropy solution, Kruzkov's double variables method
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1 Introduction

In this paper, we consider the initial boundary value problem of the following equation:

$$\partial_{xx}u + u\partial_y u - \partial_t u = f(x,y,t,u), \quad (x,y,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with the suitably smooth boundary $\partial\Omega$. Equation (1.1) arises in mathematical finance (see [1]), and arises when studying nonlinear physical phenomena such as the combined effects of diffusion and convection of matter (see [2]). Antonelli, Barucci and Mancino [1] introduced a new model for the agent's decision under risk, in which the utility function is the solution of Equation (1.1), and in particular, $0 \le u \le 1$. Under the assumption that f is a uniformly Lipschitz continuous function, Crandall, Ishii and Lions [3], Citti, Pascucci and Polidoro [4], Antonelli and Pascucci [5], step by step, proved that there is a local classical solution of Cauchy problem of Equation (1.1).

Clearly, Equation (1.1) is a strong degenerate parabolic equation since it lacks the second-order partial derivative term $\partial_{yy}u$. There are some different ways to deal with the existence and uniqueness of the global weak solution of the Cauchy problem of Equation (1.1). For example, Equation (1.1) is the special case of the degenerate parabolic equations discussed in [6–7], etc.,

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and one can refer to [8–12] for the related results. However, Zhan [13] showed that the global weak solution of Equation (1.1) can not be classical generally. In other words, some blow-up phenomena happen in finite time. Based on these facts, we are interested in the initial boundary value problem of Equation (1.1).

It is well known that there are some rules on how to quote the initial boundary value problem of a linear degenerate parabolic equation, for which one can refer to Oleinik's books [14–15] etc., and we call these rules as Oleinik rules for simplicity. If $\Omega = (0, R) \times (0, N) \subset \mathbb{R}^2$, considering the nonnegative solutions, according to Oleinik rules, and using Oleinik's line method (see [16]), the local classical solution of Equation (1.1) has been discussed in [17]. The main aim of this paper is to discuss the initial boundary value problem of Equation (1.1), provided that the spatial variables (x, y) lie in a general domain $\Omega \subset \mathbb{R}^2$, and the boundary $\partial\Omega$ is suitably smooth. Certainly, the initial value condition is always required, i.e.,

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega. \tag{1.2}$$

To assure the well-posedness of Equation (1.1), according to Oleinik rules, we should impose

$$u(x, y, t) = 0, \quad (x, y, t) \in \Sigma_3 \tag{1.3}$$

as the homogeneous boundary value condition, where

$$\Sigma_3 = \{(x, y, t) \in \Sigma = \partial\Omega \times [0, T) : n_1(x, y, t) \neq 0\}, \tag{1.4}$$

and $\vec{n} = \{n_1, n_2, 0\}$ is the outer unit normal vector of Σ. We shall investigate the solvability of Equation (1.1) with the initial value (1.2) and the partial boundary value condition (1.3). The most important innovation of the paper lies in how to get a suitable entropy solution of (1.1)–(1.3) to arrive at its well-posedness. We shall use the general parabolic regularization method, i.e., considering the initial boundary value problem of the following equation:

$$\varepsilon\Delta u_\varepsilon + \partial_{xx}u_\varepsilon + u_\varepsilon\partial_y u_\varepsilon - \partial_t u_\varepsilon = f(x, y, t, u_\varepsilon), \quad (x, y, t) \in \Omega \times (0, T), \tag{1.5}$$

to prove the existence of the solution. In order to prove the compactness of $\{u_\varepsilon\}$, we need some estimates on $\{u_\varepsilon\}$. Based on the estimates, by Kolmogoroff's theorem, and using some ideas of [6–7] and [18], the existence of the solution is proved.

Theorem 1.1 *Suppose that $u_0(x) \in L^\infty(\Omega)$ is suitably smooth. If f_x, f_y, f_t are bounded functions, and f_u is bounded too when u is bounded, then Equation* (1.1) *with the initial boundary value conditions* (1.2)–(1.3) *has an entropy solution.*

The entropy solution in Theorem 1.1 is in the BV sense, which is defined in the following Definition 2.1. Moreover, we shall use Kruzkov's double variables method (see [19]), to discuss the stability of the solutions. Due to the complicated formula of the entropy solution defined, some special techniques are used. Beyond one's imagination, if we consider the special domain, such as the half space $\Omega = \mathbb{R}^2_+$, or the unit disc $\Omega = \{(x, y) : x^2 + y^2 < 1\}$, then the stability of the solution may be free from the limitation of the boundary value condition.

Kobayasi K. and Ohwa H. [25] studied the well-posedness of anisotropic degenerate parabolic equations

$$\partial_t u + \mathrm{div} f(u) = \nabla \cdot (A(u)\nabla u) \tag{1.6}$$

with the inhomogeneous boundary condition on a bounded rectangle by using the kinetic formulation which was introduced in [26]. Li Y. and Wang Q. [27] considered the entropy solutions of the homogeneous Dirichlet boundary value problem of (1.6) in an arbitrary bounded domain. Since the entropy solutions defined in [25, 27] are only in the L^∞ space, the existence of the trace (defined in the traditional way, which was called the strong trace in [27]) on the boundary is not guaranteed, the appropriate definition of entropy solutions are quoted, and the trace of the solution on the boundary is defined in an integral formula sense, which was called the weak trace in [27]. So, not only Definition 2.1 in our paper is different from the definitions of entropy solutions in [25, 27], but also the trace of the solution in our paper is in the traditional way.

2 Definition of the Solution

Following references [20–21], $u \in BV(Q_T)$, $Q_T = \Omega \times (0,T)$, if and only if $u \in L^1_{\mathrm{loc}}(Q_T)$ and

$$\int_0^T \int_{B_\rho} |u(x+h_1, y+h_2, t+h_3) - u(x,y,t)| \mathrm{d}x\mathrm{d}t \le K|h|,$$

where

$$B_\rho = \{(x,y) \in \mathbb{R}^2; |X| < \rho\}, \quad h = (h_1, h_2, h_3)$$

and K is a positive constant. This is equivalent to that the generalized derivatives of every function in $BV(Q_T)$ are regular Radon measures on Q_T.

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, v be the normal of Γ_u at $X = (x,y,t)$, and $u^+(X)$ and $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(v, Y - X) > 0$ and $(v, Y - X) < 0$ respectively. For the continuous function $p(x,y,t,u)$ and $u \in BV(Q_T)$, define

$$\widehat{p}(x,y,t,u) = \int_0^1 p(x,y,t,\tau u^+ + (1-\tau)u^-)\mathrm{d}\tau, \tag{2.1}$$

which is called the composite mean value of p. For a given t, we denote Γ_u^t, H^t, $(v_1^t, \cdots, v_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, the Housdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$ respectively. Moreover, if $f(s) \in C^1(\mathbb{R})$ and $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \quad i = 1,2,\ x_1 = x,\ x_2 = y. \tag{2.2}$$

Let $S_\eta(s) = \int_0^s h_\eta(\tau)\mathrm{d}\tau$ for small $\eta > 0$, where $h_\eta(s) = \frac{2}{\eta}\big(1 - \frac{|s|}{\eta}\big)_+$. Obviously $h_\eta(s) \in C(\mathbb{R})$ and

$$\begin{aligned} &h_\eta(s) \ge 0, \quad |sh_\eta(s)| \le 1, \quad |S_\eta(s)| \le 1; \\ &\lim_{\eta\to 0} S_\eta(s) = \mathrm{sgn}(s), \quad \lim_{\eta\to 0} sS'_\eta(s) = 0, \end{aligned} \tag{2.3}$$

where sgn represents the sign function.

Definition 2.1 *A function u is said to be the entropy solution of* (1.1)–(1.3), *if*

(1) $u \in BV(Q_T) \cap L^\infty(Q_T)$, *and there exists a function $g^1 \in L^2(Q_T)$, such that*

$$\iint_{Q_T} g^1(x,y,t)\varphi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t = \iint_{Q_T} \frac{\partial u}{\partial x}\varphi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{2.4}$$

for any $\varphi(x,y,t) \in L^2(Q_T)$.

(2) *For any $0 \le \varphi \in C_0^2(Q_T)$ any $k \in \mathbb{R}$, and any small $\eta > 0$, u satisfies*

$$\begin{aligned}&\iint_{Q_T} [I_\eta(u-k)\varphi_t - B_\eta(u,k)\varphi_y + I_\eta(u-k)\varphi_{xx} \\ &- f(\cdot,u)S_\eta(u-k)\varphi - S'_\eta(u-k)(\partial_x u)^2\varphi]\mathrm{d}x\mathrm{d}y\mathrm{d}t \ge 0.\end{aligned} \tag{2.5}$$

For any $k \in \mathbb{R}$, $\eta > 0$, here

$$B_\eta(u,k) = \int_k^u sS_\eta(s-k)\mathrm{d}s, \quad I_\eta(u-k) = \int_0^{u-k} S_\eta(s)\mathrm{d}s.$$

(3) *The trace on the boundary*

$$\gamma u \,|_{\sum_3} = 0. \tag{2.6}$$

(4) *The initial value condition is true in the sense that*

$$\lim_{t\to 0}\int_\Omega |u(x,y,t) - u_0(x,y)|\mathrm{d}x\mathrm{d}y = 0, \quad \text{a.e. } (x,y) \in \Omega. \tag{2.7}$$

Clearly, by (2.5), we have

$$\iint_{Q_T} [I_\eta(u-k)\varphi_t - B_\eta(u,k)\varphi_y + I_\eta(u-k)\varphi_{xx} - f(\cdot,u)S_\eta(u-k)\varphi]\mathrm{d}x\mathrm{d}y\mathrm{d}t \ge 0.$$

Let $\eta \to 0$ in this inequality. We have

$$\iint_{Q_T} \Big[|u-k|\varphi_t - \frac{1}{2}\mathrm{sgn}(u-k)(u^2-k^2)\varphi_y + |u-k|\varphi_{xx} - f(\cdot,u)\mathrm{sgn}(u-k)\varphi\Big]\mathrm{d}x\mathrm{d}t \ge 0.$$

This is just the entropy solution defined in [23–24]. Thus if u is the entropy solution in Definition 2.1, then u is an entropy solution defined in general cases.

3 Proof of Theorem 1.1

Lemma 3.1 (see [28]) *Assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set and let $g_k, f \in L^q(\Omega)$, as $k \to \infty$, $g_k \rightharpoonup f$ weakly in $L^q(\Omega)$, $1 \le q < \infty$. Then*

$$\lim_{k\to\infty} \inf \|g_k\|^q_{L^q(\Omega)} \ge \|g\|^q_{L^q(\Omega)}.$$

We now consider the following regularized problem:

$$\varepsilon\Delta u + \partial_{xx}u + u\partial_y u - \partial_t u = f(\cdot,u), \quad (x,y,t) \in \Omega \times (0,T), \tag{3.1}$$

with the initial value (1.2) and the homogeneous boundary value condition

$$u(x,y,t) = 0, \quad (x,y,t) \in \Sigma = \partial\Omega \times [0,T). \tag{3.2}$$

Under the assumptions of Theorem 1.1, it is well known that there is a classical solution u_ε of the initial boundary value problem of (3.1) with (1.2) and (3.2), and e.g., one can refer to the chapter 8 of [29].

We need to make some estimates for u_ε of (3.1). Firstly, since $u_0(x) \in L^\infty(\Omega)$ is suitably smooth, by the maximum principle, we have

$$|u_\varepsilon| \leq \|u_0\|_{L^\infty} \leq c. \tag{3.3}$$

Secondly, let's make the BV estimates of u_ε.

Lemma 3.2 (see [18]) *Let u_ε be the solution of* (3.1) *with* (1.2) *and* (3.2). *If the assumptions of Theorem* 2.2 *are true, then*

$$\varepsilon \int_{\partial\Omega} \Big|\frac{\partial u_\varepsilon}{\partial n}\Big| \mathrm{d}\sigma \leq c_1 + c_2 \Big(|\nabla u_\varepsilon|_{L^1(\Omega)} + \Big|\frac{\partial u_\varepsilon}{\partial t}\Big|_{L^1(\Omega)}\Big)$$

with constants $c_i, i = 1, 2$ independent of ε, where $\vec{n} = \{n_1, n_2\}$ is the outer normal vector of Ω, and $\nabla u_\varepsilon = \{\partial_x u_\varepsilon, \partial_y u_\varepsilon\}$.

Theorem 3.1 *Let u_ε be the solution of* (3.1) *with* (1.2) *and* (3.2). *If the assumptions of Theorem* 1.1 *are true, then*

$$|\mathrm{grad}\, u_\varepsilon|_{L^1(\Omega)} \leq c, \tag{3.4}$$

where $|\mathrm{grad}\, u|^2 = \sum\limits_{i=1}^{2} |\frac{\partial u}{\partial x_i}|^2 + |\frac{\partial u}{\partial t}|^2$, c is independent of ε, and $x_1 = x,\ x_2 = y$.

Proof In what follows, we simply denote the solution of (3.1) with (1.2) and (3.2), u_ε, as $u,\ x_1 = x,\ x_2 = y,\ x_3 = t$ sometimes from the context, and the dual index of i represents the sum from 1 to 2, while the dual index of s or p represents the sum from 1 to 3. Differentiate (3.1) with respect to $x_s,\ s = 1, 2, 3$, and sum up for s after multiplying the resulting relation by $u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|}$. Then integrating over Ω yields

$$\begin{aligned}
&\int_\Omega \frac{\partial u_{x_s}}{\partial t} u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{\partial}{\partial t} \int_0^{|\mathrm{grad}\, u|} S_\eta(\tau) \mathrm{d}\tau \mathrm{d}x\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega I_\eta(|\mathrm{grad}\, u|) \mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_s} \frac{\partial}{\partial x} \Big(\frac{\partial u}{\partial x}\Big) u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{\partial}{\partial x}(u_{xx_s}) u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{\partial}{\partial x}(u_{xx_s}) \frac{\partial}{\partial \xi_s} I_\eta(|\mathrm{grad}\, u|) \mathrm{d}x\mathrm{d}y \\
&= \int_{\partial\Omega} u_{xx_s} n_1 \frac{\partial}{\partial \xi_s} I_\eta(|\mathrm{grad}\, u|) \mathrm{d}\sigma - \int_\Omega \frac{\partial^2 I_\eta(|\mathrm{grad}\, u|)}{\partial \xi_s \partial \xi_p} u_{x_s x} u_{x_p x} \mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{3.6}$$

where $\xi_s = u_{x_s}$, and $\mathrm{d}\sigma$ is the surface integrable unit.

$$\varepsilon \int_\Omega \triangle u_{x_s} u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y$$

$$= \varepsilon \int_{\partial\Omega} \frac{\partial I_\eta(|\mathrm{grad}\, u|)}{\partial x_i} n_i \mathrm{d}\sigma - \varepsilon \int_\Omega \frac{\partial^2 I_\eta(|\mathrm{grad}\, u|)}{\partial \xi_s \partial \xi_p} u_{x_s x_i} u_{x_p x_i} \mathrm{d}x\mathrm{d}y, \tag{3.7}$$

$$\int_\Omega \frac{\partial(u u_y)}{\partial x_s} u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega \Big(u \frac{\partial(u_y)}{\partial x_s} u_{x_s} + \frac{\partial u}{\partial y} |\mathrm{grad}\, u|^2 \Big) \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega u \frac{\partial I_\eta(|\mathrm{grad}\, u|)}{\partial y} \mathrm{d}x\mathrm{d}y + \int_\Omega \frac{\partial u}{\partial y} |\mathrm{grad}\, u| S_\eta(|\mathrm{grad}\, u|) \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega \frac{\partial u}{\partial y} [|\mathrm{grad}\, u| S_\eta(|\mathrm{grad}\, u|) - I_\eta(|\mathrm{grad}\, u|)] \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} u I_\eta(|\mathrm{grad}\, u|) n_2 \mathrm{d}\sigma$$
$$= - \int_\Omega \frac{\partial u}{\partial y} [|\mathrm{grad}\, u| S_\eta(|\mathrm{grad}\, u|) - I_\eta(|\mathrm{grad}\, u|)] \mathrm{d}x\mathrm{d}y. \tag{3.8}$$

By the assumption that f_t, f_x, f_y are bounded, and f_u is bounded due to $|u| \leq c$, then

$$\int_\Omega \frac{\partial f(x,y,t,u)}{\partial x_s} u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y \leq c \int_\Omega |\mathrm{grad}\, u| \mathrm{d}x\mathrm{d}y. \tag{3.9}$$

From (3.5)–(3.9), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega I_\eta(|\mathrm{grad}\, u|) \mathrm{d}x\mathrm{d}y$$
$$= - \int_\Omega \frac{\partial^2 I_\eta(|\mathrm{grad}\, u|)}{\partial \xi_s \partial \xi_p} u_{x_s x} u_{x_p x} \mathrm{d}x\mathrm{d}y - \varepsilon \int_\Omega \frac{\partial^2 I_\eta(|\mathrm{grad}\, u|)}{\partial \xi_s \partial \xi_p} u_{x_s x_i} u_{x_p x_i} \mathrm{d}x\mathrm{d}y$$
$$+ \int_\Omega \frac{\partial u}{\partial y} [|\mathrm{grad}\, u| S_\eta(|\mathrm{grad}\, u|) - I_\eta(|\mathrm{grad}\, u|)] \mathrm{d}x\mathrm{d}y - \int_\Omega \frac{\partial f(t,x,y,u)}{\partial x_s} u_{x_s} \frac{S_\eta(|\mathrm{grad}\, u|)}{|\mathrm{grad}\, u|} \mathrm{d}x\mathrm{d}y$$
$$+ \int_{\partial\Omega} \frac{\partial I_\eta(|\mathrm{grad}\, u|)}{\partial x} n_1 \mathrm{d}\sigma + \varepsilon \int_{\partial\Omega} \frac{\partial I_\eta(|\mathrm{grad}\, u|)}{\partial x_i} n_i \mathrm{d}\sigma. \tag{3.10}$$

Observe that on $\Sigma = \partial\Omega \times [0,T)$,

$$u = 0, \quad u_{x_3}|_\Sigma = u_t|_\Sigma = 0,$$

which implies that

$$\partial_{xx} u|_\Sigma + \varepsilon \triangle u|_\Sigma = f(x,y,t,0). \tag{3.11}$$

Let the surface integral in (3.10) be

$$S = \int_{\partial\Omega} \frac{\partial I_\eta(|\mathrm{grad}\, u|)}{\partial x} n_1 \mathrm{d}\sigma + \varepsilon \int_{\partial\Omega} \frac{\partial I_\eta(|\mathrm{grad}\, u|)}{\partial x_i} n_i \mathrm{d}\sigma.$$

Then, by Lemma 3.2, using (3.11), $\lim_{\eta\to 0} S$ can be estimated by $|\mathrm{grad}\, u|_{L_1(\Omega)}$, and one can refer to [18] for details.

Thus, by (3.10), letting $\eta \to 0$, and noticing that

$$\lim_{\eta\to 0} [|\mathrm{grad}\, u| S_\eta(|\mathrm{grad}\, u|) - I_\eta(|\mathrm{grad}\, u|)] = 0,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\mathrm{grad}\, u| \mathrm{d}x\mathrm{d}y \leq c_1 + c_2 \int_\Omega |\mathrm{grad}\, u| \mathrm{d}x\mathrm{d}y, \tag{3.12}$$

and by the well known Gronwall lemma, we have

$$\int_{\Omega} |\mathrm{grad}\, u| \mathrm{d}x\mathrm{d}y \leq c, \tag{3.13}$$

where c is a constant independent of t. By (3.13), using Equation (3.1), it is easy to show that

$$\iint_{Q_T} |u_{x_1}|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}t = \iint_{Q_T} |u_x|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t \leq c. \tag{3.14}$$

Now, we denote back that u_ε is the solution of (3.1). Thus by Kolmogoroff's theorem, there exists a subsequence $\{u_{\varepsilon_n}\}$ of u_ε and a function $u \in BV(Q_T) \cap L^\infty(Q_T)$ such that u_{ε_n} is strongly convergent to u, so $u_{\varepsilon_n} \to u$ a.e. on Q_T. By (3.14), there exist functions $g^1 \in L^2(Q_T)$ and a subsequence of $\{\varepsilon\}$, and we can simply denote this subsequence as ε, such that when $\varepsilon \to 0$,

$$\frac{\partial u_\varepsilon}{\partial x} \rightharpoonup g^1 \quad \text{in } L^2(Q_T).$$

We now prove that u is a generalized solution of (1.1)–(1.3). Let $\varphi \in C_0^2(Q_T)$, $\varphi \geq 0$. Multiplying (3.1) by $\varphi S_\eta(u_\varepsilon - k)$, and integrating over Q_T, we obtain

$$\begin{aligned}\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \varphi S_\eta(u_\varepsilon - k) \mathrm{d}x\mathrm{d}y\mathrm{d}t = &\iint_{Q_T} \frac{\partial}{\partial x}\Big(\frac{\partial u_\varepsilon}{\partial x}\Big) \varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &+ \varepsilon \iint_{Q_T} \Delta u_\varepsilon \varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &+ \iint_{Q_T} u_\varepsilon u_{\varepsilon y} \varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &- \iint_{Q_T} f(x,y,t,u_\varepsilon)\varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t. \end{aligned} \tag{3.15}$$

Let's calculate every term in (3.15) by the part integral method:

$$\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t = -\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_t \mathrm{d}x\mathrm{d}y\mathrm{d}t, \tag{3.16}$$

$$\begin{aligned}\varepsilon \iint_{Q_T} \Delta u_\varepsilon \varphi S_\eta(u_\epsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t &= -\varepsilon \iint_{Q_T} \nabla u_\varepsilon (S_\eta(u_\varepsilon - k)\nabla\varphi + \varphi S'_\eta(u_\varepsilon - k)\nabla u_\varepsilon)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= -\varepsilon \iint_{Q_T} \nabla u_\varepsilon S_\eta(u_\varepsilon - k)\nabla\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\quad - \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^2 S'_\eta(u_\varepsilon - k)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t, \end{aligned} \tag{3.17}$$

$$\begin{aligned}\iint_{Q_T} \partial_{xx} u_\varepsilon \varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t &= -\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial x}(S_\eta(u_\varepsilon - k)\varphi_x + \varphi S'_\eta(u_\varepsilon - k)u_{\varepsilon x})\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= -\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial x} S_\eta(u_\varepsilon - k)\varphi_x \mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\quad - \iint_{Q_T} (u_{\varepsilon x})^2 S'_\eta(u_\varepsilon - k)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t, \end{aligned} \tag{3.18}$$

$$-\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial x} S_\eta(u_\varepsilon - k)\varphi_x \mathrm{d}x\mathrm{d}y\mathrm{d}t = \iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{xx}\mathrm{d}x\mathrm{d}y\mathrm{d}t, \tag{3.19}$$

$$\iint_{Q_T} u_\varepsilon \frac{\partial u_\varepsilon}{\partial y}\varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t = -\iint_{Q_T} B_\eta(u_\varepsilon, k)\varphi_y \mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{3.20}$$

From (3.15)–(3.20), we have

$$\begin{aligned}&\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_t \mathrm{d}x\mathrm{d}y\mathrm{d}t + \iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{xx}\mathrm{d}x\mathrm{d}y\mathrm{d}t - \iint_{Q_T} B_\eta(u_\varepsilon, k)\varphi_y \mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &- \varepsilon\iint_{Q_T} \nabla u_\varepsilon\cdot\nabla\varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t - \varepsilon\iint_{Q_T} |\nabla u_\varepsilon|^2\, S_\eta'(u_\varepsilon - k)\varphi\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &- \iint_{Q_T}(u_{\varepsilon x})^2 S_\eta'(u_\varepsilon - k)\varphi\mathrm{d}x\mathrm{d}y\mathrm{d}t - \iint_{Q_T} f(x,y,t,u_\varepsilon)\varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t = 0.\end{aligned} \tag{3.21}$$

By (3.21), we have

$$\begin{aligned}&\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_t \mathrm{d}x\mathrm{d}y\mathrm{d}t + \iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{xx}\mathrm{d}x\mathrm{d}y\mathrm{d}t - \iint_{Q_T} B_\eta(u_\varepsilon, k)\varphi_y \mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &- \varepsilon\iint_{Q_T} \nabla u_\varepsilon\cdot\nabla\varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}y\mathrm{d}t - \iint_{Q_T}(u_{\varepsilon x})^2 S_\eta'(u_\varepsilon - k)\varphi\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &- \iint_{Q_T} f(x,y,t,u_\varepsilon)\varphi S_\eta(u_\varepsilon - k)\mathrm{d}x\mathrm{d}t \geq 0.\end{aligned} \tag{3.22}$$

By Lemma 3.1,

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\iint_{Q_T} S_\eta'(u_\varepsilon - k)\frac{\partial u_\varepsilon}{\partial x}\frac{\partial u_\varepsilon}{\partial x}\varphi\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ \geq &\iint_{Q_T} |g^1|^2 S_\eta'(u - k)\varphi\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned} \tag{3.23}$$

Let $\varepsilon\to 0$ in (3.22). By (3.23), we get (2.5). At the same time, (2.6) is naturally concealed in the limiting process.

The proof of (2.7) is similar to that in [2, 6], so we omit the details here.

4 Double Variables Method

Lemma 4.1 (see [6]) *Let u be a solution of* (1.1). *Then*

$$(u^+ - u^-)v_1 = 0, \quad \text{a.e. } (x,y,t) \text{ on } \Gamma_u, \tag{4.1}$$

which is true in the sense of the Hausdorff measure $H_2(\Gamma_u)$.

In this section, we shall prove how to use the double variables method to consider the stability of the solutions. Let u, v be two entropy solutions of (1.1) with initial values

$$u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \tag{4.2}$$

and with the homogeneous boundary value $u(x,y,t) = v(x,y,t) = 0$ when $(x,y,t)\in\Sigma_3$. For simplicity, we denote the spatial variables (x,y) as (x_1,x_2) or (y_1,y_2) in what follows, and correspondingly, $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2,\ \mathrm{d}y = \mathrm{d}y_1\mathrm{d}y_2$.

By Definition 2.1, we have

$$\iint_{Q_T}[I_\eta(u-k)\varphi_t - B_\eta(u,k)\varphi_{x_2} + I_\eta(u-k)\varphi_{x_1x_1} \\ - S'_\eta(u-k)|g^1(u)|^2\varphi - f(\cdot,u)S_\eta(u-k)\varphi]\mathrm{d}x\mathrm{d}t \geq 0, \tag{4.3}$$

$$\iint_{Q_T}[I_\eta(v-l)\varphi_\tau - B_\eta(v,l)\varphi_{y_2} + I_\eta(v-l)\varphi_{y_1y_1} \\ - S'_\eta(v-l)|g^1(v)|^2\varphi - f(\cdot,v)S_\eta(v-l)\varphi]\mathrm{d}y\mathrm{d}\tau \geq 0. \tag{4.4}$$

Let $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$. Here $\phi(x,t) \geq 0$, $\phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\prod_{i=1}^{2}\omega_h(x_i-y_i), \tag{4.5}$$

$$\omega_h(s) = \frac{1}{h}\omega\Big(\frac{s}{h}\Big), \quad \omega(s) \in C_0^\infty(R), \quad \omega(s) \geq 0, \quad \omega(s) = 0 \text{ if } |s| > 1, \int_{-\infty}^{\infty}\omega(s)\mathrm{d}s = 1. \tag{4.6}$$

We choose $k = v(y,\tau)$, $l = u(x,t)$, $\varphi = \psi(x,t,y,\tau)$ in (4.3)–(4.4), integrate over Q_T, add them together, and then we get

$$\iint_{Q_T}\iint_{Q_T}[I_\eta(u-v)(\psi_t+\psi_\tau) - (B_\eta(u,v)\psi_{x_2} + B_\eta(v,u)\psi_{y_2}) \\ + I_\eta(u-v)\psi_{x_1x_1} + I_\eta(v-u)\psi_{y_1y_1}] \\ - \{S'_\eta(u-v)(|g^1(u)|^2 + |g^1(v)|^2) - [f(\cdot,u)S_\eta(u-v) + f(\cdot,v)S_\eta(v-u)]\}\varphi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau. \tag{4.7}$$

Clearly,

$$\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0, \quad \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0, \quad i = 1,\cdots,N;$$
$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t}j_h, \quad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i}j_h.$$

Noticing that

$$\lim_{\eta\to 0}B_\eta(u,v) = \lim_{\eta\to 0}B_\eta(v,u) = -\frac{1}{2}\mathrm{sgn}(u-v)(u^2-v^2), \tag{4.8}$$

as $\eta \to 0$, we have

$$\iint_{Q_T}\iint_{Q_T}[B_\eta(u,v)\psi_{x_2} + B_\eta(v,u)\psi_{y_2}]\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\ \to -\frac{1}{2}\iint_{Q_T}\iint_{Q_T}\mathrm{sgn}(u-v)(u^2-v^2)\phi_{x_2}j_h\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau,$$

and as $h \to 0$, we have

$$-\frac{1}{2}\iint_{Q_T}\iint_{Q_T}\mathrm{sgn}(u-v)(u^2-v^2)\phi_{x_2}j_h\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\ \to -\frac{1}{2}\iint_{Q_T}\mathrm{sgn}(u-v)(u^2-v^2)\phi_{x_2}\mathrm{d}x\mathrm{d}t. \tag{4.9}$$

For the third term and the forth term in the bracket of (4.7), we have

$$\iint_{Q_T}\iint_{Q_T}[I_\eta(u-v)\psi_{x_1x_1} + I_\eta(v-u)\psi_{y_1y_1}]\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau$$

$$
\begin{aligned}
&= \iint_{Q_T}\iint_{Q_T} I_\eta(u-v)(\phi_{x_1x_1}j_h + 2\phi_{x_1}j_{hx_1} + \phi j_{hx_1x_1}) + I_\eta(v-u)\phi j_{hy_1y_1}\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} \{I_\eta(u-v)\phi_{x_1x_1}j_h + I_\eta(u-v)\phi_{x_1}j_{hx_1} + I_\eta(v-u)\phi_{x_1}j_{hx_1}\}\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad - \iint_{Q_T}\iint_{Q_T} \Big\{\Big[\int_0^1 S_\eta(su^+ + (1-s)u^- - v)\mathrm{d}s \\
&\quad - \int_0^1 S_\eta(v - su^+ - (1-s)u^-)\mathrm{d}s\Big]\frac{\partial u}{\partial x_1}\phi j_{hx_1}\Big\}\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau. \qquad (4.10)
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)(|g^1(u)|^2 + |g^1(v)|^2)\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)(|g^1(u)| - |g^1(v)|)^2\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad + 2\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)g^1(u)g^1(v)\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)[|g^1(u)| - |g^1(v)|]^2\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&\quad + 2\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\partial_{x_1}u\partial_{y_1}v\psi\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}\tau, \qquad (4.11)
\end{aligned}
$$

and by the properties of the BV function (the equality (2.2) and Lemma 4.1)

$$
\begin{aligned}
&\iint_{Q_T}\iint_{Q_T} \partial_{x_1}\partial_{y_1}\int_v^u\int_\delta^v S'_\eta(\sigma-\delta)\mathrm{d}\sigma\mathrm{d}\delta\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} \psi\partial_{y_1}\int_0^1\int_{su^++(1-s)u^-}^v S'_\eta(\sigma - su^+ - (1-s)u^-)\mathrm{d}\sigma\mathrm{d}s\frac{\partial u}{\partial x_1}\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} \psi\partial_{x_1}\int_0^u \mathrm{d}\delta\cdot\partial_{y_1}\int_0^v \mathrm{d}\delta S'_\eta(v-u)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau, \\
&\iint_{Q_T}\iint_{Q_T} \partial_{x_1}\partial_{y_1}\int_v^u\int_\delta^v S'_\eta(\sigma-\delta)\mathrm{d}\sigma\mathrm{d}\delta\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} \psi\partial_{y_1}\Big(\int_0^1\int_{su^++(1-s)u^-}^v S'_\eta(\sigma - su^+ - (1-s)u^-)\frac{\partial u}{\partial x_1}\mathrm{d}\sigma\mathrm{d}s\Big)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= \iint_{Q_T}\iint_{Q_T} \phi j_{hx_1}\int_0^1\int_{su^++(1-s)u^-}^v S'_\eta(\sigma - su^+ - (1-s)u^-)\mathrm{d}\sigma\mathrm{d}s\frac{\partial u}{\partial x_1}\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau.
\end{aligned}
$$

We have

$$
\begin{aligned}
&- \iint_{Q_T}\iint_{Q_T} \Big[\int_0^1 S_\eta(su^+ + (1-s)u^- - v)\mathrm{d}s \\
&- \int_0^1 S_\eta(v - su^+ - (1-s)u^-)\mathrm{d}s\Big]\frac{\partial u}{\partial x_1}j_{hx_1}\phi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&+ 2\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\partial x_1\int_0^u \mathrm{d}s\cdot\partial y_1\int_0^v \mathrm{d}s\psi\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau \\
&= - \iint_{Q_T}\iint_{Q_T} \Big[\int_0^1 S_\eta(su^+ + (1-s)u^- - v)\mathrm{d}s
\end{aligned}
$$

$$- \int_0^1 S_\eta(v - su^+ - (1-s)u^-)\mathrm{d}s\Big] \frac{\partial u}{\partial x_1} j_{hx_1}\phi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau$$
$$+ 2 \iint_{Q_T} \iint_{Q_T} \int_0^1 \int_{su^+ + (1-s)u^-}^{v} S'_\eta(\sigma - su^+ - (1-s)u^-)\mathrm{d}\sigma\mathrm{d}s \frac{\partial u}{\partial x_1} j_{hx_1}\phi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau$$
$$= \iint_{Q_T} \iint_{Q_T} \Big[\int_0^1 S_\eta(su^+ + (1-s)u^- - v)\mathrm{d}s$$
$$+ \int_0^1 S_\eta(v - su^+ - (1-s)u^-)\Big] \frac{\partial u}{\partial x_1} j_{hx_1}\phi \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}\tau = 0. \tag{4.12}$$

Noticing that $\lim\limits_{\eta\to 0} I_\eta(u-v) = \lim\limits_{\eta\to 0} I_\eta(v-u) = |u-v|$, we have

$$\lim_{\eta\to 0}[I_\eta(u-v)\phi_{x_1} j_{hx_1} + I_\eta(v-u)\phi_{x_1} j_{hy_1}] = 0. \tag{4.13}$$

Combining (4.7)–(4.13), and letting $\eta \to 0,\ h \to 0$, we get

$$\iint_{Q_T} \Big\{ |u(x,t) - v(x,t)|\phi_t + |u-v|\phi_{x_1x_1} - \frac{1}{2}\mathrm{sgn}(u-v)(u^2 - v^2)\phi_{x_2}$$
$$- [f(\cdot,u) - f(\cdot,v)]\mathrm{sgn}(u-v)\phi \Big\} \mathrm{d}x\mathrm{d}t \ge 0. \tag{4.14}$$

5 Stability of the Solutions

In the last section of this paper, we shall discuss the stability of the solutions.

Theorem 5.1 *Let* $0 \le u \le 1,\ 0 \le v \le 1$ *be two solutions of Equation* (1.1) *with the homogeneous boundary value* $\gamma u \mid_{\Sigma_3} = \gamma v \mid_{\Sigma_3} = 0$, *and with the different initial values* $u_0(x_1, x_2),\ v_0(x_1,\ x_2) \in L^\infty(\Omega)$ *respectively. Suppose that* $|f_u(\cdot, u)| \le c$, *and that the distance function* $d(x) = \mathrm{dist}(x, \partial\Omega)$ *satisfies*

$$|d_{x_1x_1}| \le c, \tag{5.1}$$

so then for any $t \in (0, T)$,

$$\int_\Omega |u(x_1, x_2, t) - v(x_1, x_2, t)|\mathrm{d}x_1\mathrm{d}x_2$$
$$\le \int_\Omega |u_0(x_1, x_2) - v_0(x_1, x_2)|\mathrm{d}x_1\mathrm{d}x_2 + c\,\mathrm{ess\,sup}\,|u-v|_{(x,t)\in\Sigma'_3\times(0,T)}, \tag{5.2}$$

where $\Sigma'_3 = \partial\Omega \setminus \Sigma_3$.

Proof Let δ_ε be the mollifier as usual. In detail, for the known function

$$\delta(s) = \begin{cases} \dfrac{1}{A}\mathrm{e}^{\frac{1}{|s|^2-1}}, & \text{if } |s| < 1, \\ 0, & \text{if } |s| \ge 1, \end{cases}$$

where

$$A = \int_{B_1(0)} \mathrm{e}^{\frac{1}{|s|^2-1}}\mathrm{d}x,$$

and for any given $\varepsilon > 0,\ \delta_\varepsilon(s)$ is defined as

$$\delta_\varepsilon(s) = \frac{1}{\varepsilon}\delta\Big(\frac{s}{\varepsilon}\Big).$$

Now, we can choose ϕ in (4.14) by

$$\phi(x,t)=\omega_{\lambda\varepsilon}(x)\eta(t),$$

where $\eta(t)\in C_0^\infty(0,T)$, and $\omega_{\lambda\varepsilon}(x)\in C_0^2(\Omega)$ is defined as follows. For any given small enough $0<\lambda,\ 0\le\omega_\lambda\le 1,\ \omega|_{\partial\Omega}=0$ and

$$\omega_\lambda(d)=1,\quad \text{if } d(x)=\text{dist}(x,\partial\Omega)\ge\lambda.$$

When $0\le d(x)\le\lambda$,

$$\omega_\lambda(d(x))=1-\frac{(d(x)-\lambda)^2}{\lambda^2},$$

and when $d<0$, we let $\omega_\lambda(d)=0$. Then

$$\begin{aligned}
\omega_{\lambda\varepsilon}&=\omega_\lambda*\delta_\varepsilon(d)=\int_{-\infty}^{\infty}\omega_\lambda(d-s)\delta_\varepsilon(s)\mathrm{d}s,\\
\omega'_{\lambda\varepsilon}(d)&=\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}}\omega'_\lambda(d-s)\delta_\varepsilon(s)\mathrm{d}s\\
&=-\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}}\frac{2(d-s-\lambda)}{\lambda^2}\delta_\varepsilon(s)\mathrm{d}s,\\
|\omega'_{\lambda\varepsilon}(d)|&\le\frac{c}{\lambda}, &(5.3)\\
\omega''_{\lambda\varepsilon}(d)&=\omega''_\lambda*\delta_\varepsilon(d)=-\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}}\frac{2}{\lambda^2}\delta_\varepsilon(s)\mathrm{d}s\\
&=-\frac{2}{\lambda^2}\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}}\delta_\varepsilon(s)\mathrm{d}s. &(5.4)
\end{aligned}$$

Now,

$$\begin{aligned}
\phi_{x_1x_1}&=\eta(t)(\omega_{\lambda\varepsilon}(d(x)))_{x_1x_1}\\
&=\eta(t)(\omega'_{\lambda\varepsilon}(d)d_{x_1})_{x_1}\\
&=\eta(t)[\omega''_{\lambda\varepsilon}(d)d_{x_1}^2+\omega'_{\lambda\varepsilon}(d)d_{x_1x_1}]\\
&=\eta(t)\Big[-\frac{2}{\lambda^2}d_{x_1}^2\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}}\delta_\varepsilon(s)\mathrm{d}s+\omega'_{\lambda\varepsilon}(d)d_{x_1x_1}\Big].
\end{aligned}$$

Using the condition (5.1), and $|d_{x_1x_1}|\le c$, and with the fact that $|d_{x_i}|\le|\nabla d|=1,\ i=1,2,\ 0\le f_u(\cdot,u)\le c,\ 0\le u,v\le 1$, and from (4.14), we have

$$\begin{aligned}
&\iint_{Q_T}|u(x,t)-v(x,t)|\phi_t\mathrm{d}x\mathrm{d}t+c\int_0^T\int_{\Omega_{\lambda+\varepsilon}}\eta(t)|\omega'_{\lambda\varepsilon}(d)\ |\ |u-v|\mathrm{d}x\mathrm{d}t\\
&+c\iint_{Q_T}|u(x,t)-v(x,t)|\phi\mathrm{d}x\mathrm{d}t\ge 0. \qquad (5.5)
\end{aligned}$$

Here $\Omega_\lambda=\{x:d(x)=\text{dist}(x,\partial\Omega)<\lambda\}$. By (5.3),

$$0\le\iint_{Q_T}|u(x,t)-v(x,t)|\eta'(t)|\omega_{\lambda\varepsilon}(d)\mathrm{d}x\mathrm{d}t+c\int_0^T\int_{\Omega_{\lambda+\varepsilon}}\eta(t)|\omega'_{\lambda\varepsilon}(d)\ |\ |u-v|\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&+c\iint_{Q_T}|u(x,t)-v(x,t)|\eta(t)\omega_{\lambda\varepsilon}(d)\mathrm{d}x\mathrm{d}t\\
\le&\iint_{Q_T}|u(x,t)-v(x,t)|\eta'(t)|\omega_{\lambda\varepsilon}(d)\mathrm{d}x\mathrm{d}t+c\int_0^T\eta(t)\mathrm{d}t\frac{1}{\lambda}\int_{\Omega_{\lambda+\varepsilon}}|u-v|\mathrm{d}x\\
&+c\iint_{Q_T}|u(x,t)-v(x,t)|\eta(t)\omega_{\lambda\varepsilon}(d)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Let $\varepsilon\to 0$. Then

$$
\begin{aligned}
0\le&\iint_{Q_T}|u(x,t)-v(x,t)|\eta'(t)|\omega_{\lambda}(d)\mathrm{d}x\mathrm{d}t+c\int_0^T\eta(t)\mathrm{d}t\frac{1}{\lambda}\int_{\Omega_{\lambda}}|u-v|\mathrm{d}x\\
&+c\iint_{Q_T}|u(x,t)-v(x,t)|\eta(t)\omega_{\lambda}(d)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

As $\lambda\to 0$, according to the definition of the trace, by $\gamma u\mid_{\Sigma_3}=\gamma v\mid_{\Sigma_3}=0$, we have

$$
\begin{aligned}
0\le&\iint_{Q_T}|u(x,t)-v(x,t)|\eta'(t)\mathrm{d}x\mathrm{d}t+c\int_0^T\eta(t)||u-v|_{\partial\Omega}\mathrm{d}t\\
&+c\iint_{Q_T}|u(x,t)-v(x,t)|\eta(t)\mathrm{d}x\mathrm{d}t\\
=&\iint_{Q_T}|u(x,t)-v(x,t)|\eta'_t\mathrm{d}x\mathrm{d}t+c\int_0^T\eta(t)||u-v|_{\Sigma_3'\times(0,T)}\mathrm{d}t\\
&+c\iint_{Q_T}|u(x,t)-v(x,t)|\eta(t)\mathrm{d}x\mathrm{d}t. \qquad (5.6)
\end{aligned}
$$

Let $0<s<\tau<T$, and

$$
\eta(t)=\int_{\tau-t}^{s-t}\alpha_\epsilon(\sigma)\mathrm{d}\sigma,\quad \epsilon<\min\{\tau,T-s\}.
$$

Here $\alpha_\epsilon(t)$ is the kernel of the mollifier with $\alpha_\epsilon(t)=0$ for $t\notin(-\epsilon,\epsilon)$. Let $\epsilon\to 0$. Then

$$
\begin{aligned}
\int_\Omega|u(x,s)-v(x,s)|\mathrm{d}x\le&\int_\Omega|u(x,\tau)-v(x,\tau)|\mathrm{d}x+c\,\mathrm{ess\,sup}\mid|u-v|_{\Sigma_3'\times(0,T)}\\
&+\int_s^\tau\int_\Omega|u(x,t)-v(x,t)|\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

By the Gronwall lemma, the desired result follows by letting $s\to 0$, i.e.,

$$
|u(x,\tau)-v(x,\tau)|_{L^1(\Omega)}\le|u(x,0)-v(x,0)|_{L^1(\Omega)}+c\,\mathrm{ess\,sup}\,|u-v|_{\Sigma_3'\times(0,T)}.
$$

So we have the conclusion.

Theorem 5.2 *Let u,v be two solutions of Equation* (1.1) *with the different initial values $u_0(x_1,x_2),v_0(x_1,x_2)\in L^\infty(\Omega)$ respectively. $0\le u\le 1,\ 0\le v\le 1$. Suppose that the domain Ω is just the unit disc $\{(x_1,x_2):x_1^2+x_2^2<1\}$, and suppose $|f_u(\cdot,u)|\le c$. Then*

$$
\int_\Omega|u(x,t)-v(x,t)|\varphi(x)\mathrm{d}x\mathrm{d}y\le\int_\Omega|u_0-v_0|\mathrm{d}x, \qquad (5.7)
$$

where

$$
\varphi(x)=1-|x|^2=1-(x_1^2+x_2^2). \qquad (5.8)
$$

Proof We can choose $\phi=\varphi(x)\eta(t)$ in (4.14), where $\eta(t)\in C_0^\infty(0,T)$. Then we have

$$\iint_{Q_T}\{|u(x,t)-v(x,t)|\eta_t'\varphi(x)-2|u-v|\eta(t)+|u-v|^2x_2\eta(t)+c|u-v|\varphi(x)\eta(t)\}\mathrm{d}x\mathrm{d}t\geq 0.$$

Due to that $0\leq u\leq 1,\ 0\leq v\leq 1,\ |x_2|\leq 1$,

$$-2|u-v|+|u-v|^2x_2=-|u-v|(2-|u+v|x_2)\leq 0.$$

We have

$$c\iint_{Q_T}|u(x,t)-v(x,t)|\eta_t\varphi(x)\mathrm{d}x\mathrm{d}t+\iint_{Q_T}|u(x,t)-v(x,t)|\eta_t'\varphi(x)\mathrm{d}x\mathrm{d}t\geq 0,$$

and as the proof of Theorem 5.1, we have

$$\int_\Omega|[u(x,t)-v(x,t)]\varphi(x)|\mathrm{d}x\mathrm{d}y\leq\int_\Omega|u_0-v_0|\varphi(x)\mathrm{d}x\leq\int_\Omega|u_0-v_0|\mathrm{d}x.$$

Theorem 5.3 *It is supposed that the domain $\Omega=\mathbb{R}^2_+=\{(x_1,x_2):x_2>0\}$. Let u,v be two solutions of Equation* (1.1) *with the different initial values $u_0(x_1,x_2),v_0(x_1,x_2)\in L^\infty(\Omega)\cap L^1(\Omega)$ respectively. If $|f_u(\cdot,u)|\leq c$, then*

$$\int_\Omega|[u(x,t)-v(x,t)]\omega_\lambda(x)|\mathrm{d}x\mathrm{d}y\leq\int_\Omega|u_0-v_0|\mathrm{d}x, \tag{5.9}$$

where

$$\omega_\lambda(x)=\frac{1}{2}\Big[1+\sin\frac{1}{\lambda}\Big(d-\frac{\lambda\pi}{2}\Big)\Big],\quad 0\leq d(x)\leq\pi\lambda,$$
$$\omega_\lambda(x)=1,\quad d(x)\geq\pi\lambda,$$

and $d(x)=\mathrm{dist}(x,\partial\Omega)=x_2$.

Remark 5.1 The condition of the initial values $u_0(x_1,x_2)$, $v_0(x_1,x_2)\in L^\infty(\Omega)\cap L^1(\Omega)$ in the theorem is stronger than the solutions obtained in Theorem 1.1. At the same time, due to $\Omega=\mathbb{R}^2_+=\{(x_1,x_2):x_2>0\}$, it implies that

$$\Sigma_3=\{(x,t)\in\Sigma=\partial\Omega\times[0,T):n_1(x,t)\neq 0\}$$

is an empty set. It means that the solution of the equation is free from the limitation of the boundary value in this case.

Proof We can choose ϕ in (4.14) by

$$\phi(x,t)=\omega_\lambda(x)\eta(t),$$

where $\eta(t)\in C_0^\infty(0,T)$, and then

$$\frac{\partial\omega_\lambda(x)}{\partial x_2}=\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d-\frac{\lambda\pi}{2}\Big)d_{x_2}=\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d-\frac{\lambda\pi}{2}\Big),$$
$$\frac{\partial^2\omega_\lambda(x)}{\partial x_1^2}=-\frac{1}{2\lambda^2}\sin\frac{1}{\lambda}\Big(d-\frac{\lambda\pi}{2}\Big)d_{x_1}^2+\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d-\frac{\lambda\pi}{2}\Big)d_{x_1x_1}=0.$$

From (4.14), we have

$$\iint_{Q_T} |u(x,t)-v(x,t)|\omega_\lambda(x)\eta_t'\mathrm{d}x\mathrm{d}t - \frac{1}{4\lambda}\int_0^T\int_{\Omega_{\pi\lambda}} |u^2-v^2|\cos\frac{1}{\lambda}\Big(d-\frac{\lambda\pi}{2}\Big)\mathrm{d}x\mathrm{d}t$$
$$-\iint_{Q_T}[f(\cdot,u)-f(\cdot,v)]\mathrm{sgn}(u-v)\phi\mathrm{d}x\mathrm{d}t \geq 0, \tag{5.10}$$

where $\Omega_{\pi\lambda}=\{x: d(x,\partial\mathbb{R}^2_+)=x_2<\pi\lambda\}$. Then $\cos\frac{1}{\lambda}\big(d-\frac{\lambda\pi}{2}\big)\geq 0$, and by (5.10), we have

$$c\iint_{Q_T}|u(x,t)-v(x,t)|\omega_\lambda(x)\eta_t\mathrm{d}x\mathrm{d}t+\iint_{Q_T}|u(x,t)-v(x,t)|\omega_\lambda(x)\eta_t'\mathrm{d}x\mathrm{d}t\geq 0. \tag{5.11}$$

As the proof of Theorem 5.1, we have

$$\int_\Omega |[u(x,t)-v(x,t)]\omega_\lambda(x)|\mathrm{d}x \leq \int_\Omega |u_0-v_0|\omega_\lambda(x)\mathrm{d}x \leq \int_\Omega |u_0-v_0|\mathrm{d}x.$$

The proof of Theorem 5.3 is complete.

At the end of this paper, let us consider a special domain

$$\Omega_R=\{(x_1,x_2): 1>x_1>0, 1>x_2>0\}.$$

Then $\Sigma_3=\{(x,t): x_1=0 \text{ or } x_1=1\}\subset\partial\Omega_R\times(0,T)$. We denote $\Sigma_3'=\partial\Omega_R\setminus\Sigma_3$, and let

$$d_1(x)=2(x_1-x_1^2);$$
$$d_2(x)=\begin{cases} x_2, & \text{if } \ 0<x_2\leq\dfrac{1}{2},\\ 1-x_2, & \text{if } \ \dfrac{1}{2}<x_2<1.\end{cases}$$

For small enough λ, we set

$$\omega_\lambda(x)=\begin{cases}\dfrac{1}{2}\Big[1+\sin\dfrac{1}{\lambda}\Big(d_2(x)-\dfrac{\lambda\pi}{2}\Big)\Big], & \text{if } \ 0\leq d_2(x)\leq\pi\lambda,\\ 1, & \text{if } d_2(x)\geq\pi\lambda.\end{cases}$$

Theorem 5.4 *It is supposed that the domain $\Omega=\Omega_R$. Let u,v be two solutions of Equation* (1.1) *with the homogeneous boundary value*

$$\gamma u\mid_{\Sigma_3}=\gamma v\mid_{\Sigma_3}=0,$$

and with the different initial values $u_0(x_1,x_2), v_0(x_1,x_2)\in L^\infty(\Omega)\in L^\infty(\Omega)$ respectively. Then

$$\int_\Omega |u(x,t)-v(x,t)|\omega_\lambda(x)d_1(x)\mathrm{d}x\mathrm{d}y\leq\int_\Omega|u_0-v_0|\mathrm{d}x+c\,\mathrm{ess\,sup}\,|u-v|_{(x,t)\in\Sigma_3'\times(0,T)}. \tag{5.12}$$

Proof Through a limit process, we can choose ϕ in (4.14) by

$$\phi(x,t)=\omega_\lambda(x)d_1(x)\eta(t),$$

where $\eta(t)\in C_0^\infty(0,T)$.

When $0 < x_2 = d_2(x) < \pi\lambda$,

$$\phi_{x_2}(x,t) = d_1(x)\eta(t)\omega_{\lambda x_2}(x) = d_1(x)\eta(t)\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d - \frac{\lambda\pi}{2}\Big)d_{x_2}$$
$$= d_1(x)\eta(t)\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d - \frac{\lambda\pi}{2}\Big).$$

When $x_2 > \frac{1}{2}$ and $0 < 1 - x_2 = d_2(x) < \pi\lambda$,

$$\phi_{x_2}(x,t) = d_1(x)\eta(t)\omega_{\lambda x_2}(x) = d_1(x)\eta(t)\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d - \frac{\lambda\pi}{2}\Big)d_{x_2}$$
$$= -d_1(x)\eta(t)\frac{1}{2\lambda}\cos\frac{1}{\lambda}\Big(d - \frac{\lambda\pi}{2}\Big).$$

Certainly, when $\pi\lambda \le x_2 \le 1 - \pi\lambda$,

$$\phi_{x_2}(x,t) = 0.$$

At the same time, it is clear that

$$\phi_{x_1x_1}(x,t) = -2.$$

From (4.14), we have

$$\begin{aligned}
&\iint_{Q_T} |u(x,t) - v(x,t)|\omega_\lambda(x)d_1(x)\eta_t'\mathrm{d}x\mathrm{d}t \\
&- 2\iint_{Q_T} |u(x,t) - v(x,t)|\omega_\lambda(x)\eta_t\mathrm{d}x\mathrm{d}t \\
&- \frac{1}{4\lambda}\int_0^T \int_{\Omega_{0\pi\lambda}} |u^2 - v^2| d_1(x)\cos\frac{1}{\lambda}\Big(d_2 - \frac{\lambda\pi}{2}\Big)\mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{4\lambda}\int_0^T \int_{\Omega_{1\pi\lambda}} |u^2 - v^2| d_1(x)\cos\frac{1}{\lambda}\Big(d_2 - \frac{\lambda\pi}{2}\Big)\mathrm{d}x\mathrm{d}t \\
&- \iint_{Q_T} [f(\cdot,u) - f(\cdot,v)]\mathrm{sgn}(u-v)\phi\mathrm{d}x\mathrm{d}t \ge 0.
\end{aligned} \tag{5.13}$$

Here

$$\Omega_{0\pi\lambda} = \{x : d_2(x) = x_2 < \pi\lambda\}$$

implies $\cos\frac{1}{\lambda}\big(d - \frac{\lambda\pi}{2}\big) \ge 0$, and

$$\Omega_{1\pi\lambda} = \{x : d_2(x) = 1 - x_2 < \pi\lambda\}$$

implies $\cos\frac{1}{\lambda}\big(d - \frac{\lambda\pi}{2}\big) \le 0$, so then

$$\begin{aligned}
&\iint_{Q_T} |u(x,t) - v(x,t)|\omega_\lambda(x)d_1(x)\eta_t'\mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{4\lambda}\int_0^T \int_{\Omega_{1\pi\lambda}} |u^2 - v^2| d_1(x)\cos\frac{1}{\lambda}\Big(d_2 - \frac{\lambda\pi}{2}\Big)\mathrm{d}x\mathrm{d}t \\
&+ c\iint_{Q_T} |u - v|\omega_\lambda(x)d_1(x)\eta(t)\mathrm{d}x\mathrm{d}t \ge 0.
\end{aligned} \tag{5.14}$$

Due to

$$\Big|\cos\frac{1}{\lambda}\Big(d_2-\frac{\lambda\pi}{2}\Big)\Big|\le 1,$$

we have

$$\begin{aligned}&\iint_{Q_T}|u(x,t)-v(x,t)|\omega_\lambda(x)d_1(x)\eta_t'\mathrm{d}x\mathrm{d}t\\&+\frac{c}{4\lambda}\int_0^T\int_{\Omega_{1\pi\lambda}}|u-v|d_1(x)\mathrm{d}x\mathrm{d}t+c\iint_{Q_T}|u-v|\omega_\lambda(x)d_1(x)\eta(t)\mathrm{d}x\mathrm{d}t\ge 0.\end{aligned}\tag{5.15}$$

Let $\lambda\to 0$ in (5.15). As the proof of Theorem 5.1, we have the conclusion.

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References

[1] Antonelli, F., Barucci, E. and Mancino, M. E., A comparison result for FBSDE with applications to decisions theory, *Math. Meth. Oper. Res.*, **54**, 2001, 407–423.

[2] Escobedo, M., Vazquez, J. L. and Zuazua, E. Entropy solutions for diffusion-convection equations with partial diffusivity, *Trans. Amer. Math. Soc.*, **343**, 1994, 829–842.

[3] Crandall, M. G., Ishii, H. and Lions, P. L., User's guide to viscosity solutions of second partial differential equations, *Bull. Amer. Math. Soc.* (*N.S.*), **27**, 1992, 1–67.

[4] Antonelli, F. and Pascucci, A., On the viscosity solutions of a stochastic differential utility problem, *J. Diff. Equ.*, **186**, 2002, 69–87.

[5] Citti, G., Pascucci, A. and Polidoro, S., Regularity properties of viscosity solutions of a non-Hormander degenerate equation, *J. Math. Pures Appl.*, **80**(9), 2001, 901–918.

[6] Zhan, H. S., The study of the Cauchy problem of a second order quasilinear degenerate parabolic equation and the parallelism of a Riemannian manifold, Doctor Thesis, Xiamen University, 2004.

[7] Zhan, H. S. and Zhao, J. N., The stability of solutions for second quasilinear degenerate parabolic equation, Acta Math. Sinica, Chinese Ser., **50**, 2007, 615–628.

[8] Chen, G. Q. and Perthame, B., Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations, *Ann. I. H. Poincare-AN*, **20**, 2003, 645–668.

[9] Chen, G. Q. and Dibendetto, E., Stability of entropy solutions to Cauchy problem for a class of nonlinear hyperbolic-parabolic equations, *SIAM J. Math. Anal.*, **33**, 2001, 751–762.

[10] Karlsen, K. H. and Risebro, N. H., On the uniqueness of entropy solutions of nonlinear degenerate parabolic equations with rough coefficient, *Discrete Contain. Dye. Sys.*, **9**, 2003, 1081–1104.

[11] Bendahamane, M. and Karlsen, K. H., Reharmonized entropy solutions for quasilinear anisotropic degenerate parabolic equations, *SIAM J. Math. Anal.*, **36**, 2004, 405–422.

[12] Carrillo, J., Entropy solutions for nonlinear degenerate problems, *Arch. Rational Mech. Anal.*, **147**, 1999, 269–361.

[13] Zhan, H. S., The solution of the Cauchy problem for a quasilinear degenerate parabolic equation (in Chinese), *Chin. Ann. Math. Ser. A*, **33**(4), 2012, 449–460.

[14] Oleiňik, O. A., A problem of Fichera, *Dokl. Akad. Nauk SSSR*, **154**, 1964, 1297–1300. *Soviet Math. Dokl.*, **5**, 1964, 1129–1133.

[15] Oleiňik, O. A., Linear equations of second order with nonnegative characteristic form, *Math. Sb.*, **69**, 1966, 111–140; English transl.: *Amer. Math. Soc. Tranl.*, **65**, 1967, 167–199.

[16] Oleiňik, O. A. and Samokhin, V. N., Mathematical Models in Boundary Layer Theorem, Chapman and Hall/CRC, Boca Raton, London, New York, Washington, 1999.

[17] Zhan, H. S., On a quasilinear degenerate parabolic equation (in Chinese), *Chin. Ann. Math. Ser. A*, **27**(6), 2006, 731–740.

[18] Wu, Z. and Zhao, J., The first boundary value problem for quasilinear degenerate parabolic equations of second order in several variables, *Chin. Ann. Math. Ser. B*, **4**, 1983, 319–358.

[19] Kružkov, S. N., First order quasilinear equations in several independent variables, *Math. USSR-Sb.*, **10**, 1970, 217–243.

[20] Vol'pert, A. I. and Hudjaev, S. I., On the problem for quasilinear degenerate parabolic equations of second order (in Russian), *Mat. Sb.*, **3**, 1967, 374–396.

[21] Wu, Z., Zhao, J., Yin, J. and Li, H., Nonlinear Diffusion Equations, World Scientific Publishing, Singapore, 2001.

[22] Zhan, H. S., Harnack estimates for weak solutions to a singular parabolic equation, *Chin. Ann. Math. Ser. B*, **32**(3), 2011, 397–416.

[23] Volpert, A. I. and Hudjave, S. I., Analysis of Class of Discontinuous Functions and the Equations of Mathematical Physics, Izda. Nauka Moskwa, Martinus Nijhoff Publishers, Moscow, 1975 (in Russian).

[24] Volpert, A. I., BV space and quasilinear equations, *Mat. Sb.*, **73**, 1967, 255–302.

[25] Kobayasi, K. and Ohwa, H., Homogeneous Dirichlet problems for quasilinear anisotropic degenerate parabolic-hyperbolic equations, *J. Diff. Equ.*, **252**, 2012, 4719–4741.

[26] Lions, P. L., Perthame, B. and Tadmor, E., A kinetic formulation of multidimensional conservation laws and related equations, *J. Amer. Math. Soc.*, **7**, 1994, 169–191.

[27] Li, Y. and Wang, Q., Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle, *J. Diff. Equ.*, **252**, 2012, 137–167.

[28] Evans, L. C., Weak convergence methods for nonlinear partial differential equations, Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics, 74, 1998.

[29] Gu, L. K., Second Order Parabolic Partial Differential Equations, The Publishing Company of Xiamen University, Xiamen, 2002.