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Cohen-Fischman-Westreich's Double Centralizer Theorem for Almost-Triangular Hopf Algebras*

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Abstract In this paper, the authors study the Cohen-Fischman-Westreich's double centralizer theorem for triangular Hopf algebras in the setting of almost-triangular Hopf algebras.

Keywords Schur's double centralizer theorem, Cohen-Fischman-Westreich's double centralizer theorem, Almost-triangular Hopf algebra, R-Lie algebra
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1 Introduction

Let V be a finite-dimensional vector space over a field $\Bbbk$ of characteristic 0. Then for any positive integer m, the symmetric group S_m acts on $V^{\otimes m}$ via the twist map and the Lie algebra $gl(V)$ acts on $V^{\otimes m}$ via its comultiplication. Schur's double centralizer theorem originally established a correspondence between the above representations, which stated that S_m and $U(gl(V))$ are mutual centralizers in $\mathrm{End}_{\Bbbk}V^{\otimes m}$. Berele and Regev [1] generalized this result to the Lie superalgebra $pl(V)$, where V is a $\mathbb{Z}_2$-graded vector space, Jimbo [2] stated a similar result for $U_q(sl(2))$, and Kirillov and Reshetikhin [3] for $U_q(su(2))$. Fischman [4] used purely Hopf algebraic methods to give a short proof of both these situations. In 1994, Cohen, Fischman and Westreich [5] considered the situation of triangular Hopf algebras.

In [6], the authors introduced and studied almost-triangular Hopf algebras as a generalization of triangular Hopf algebras. Naturally, this paper is devoted to establishing the Cohen-Fischman-Westreich's double centralizer theorem for triangular Hopf algebras (see [5]) in the setting of almost-triangular Hopf algebras.

This paper is organized as follows. In Section 2, we recall some definitions and results about quasi-triangular Hopf algebras and R-Lie algebras. In Section 3, we introduce the definition of the R-universal enveloping algebra of an R-Lie algebra in the setting of almost-triangular Hopf algebras, which generalizes the corresponding results in the setting of triangular Hopf algebras.

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In the final section, we establish the Cohen-Fischman-Westreich's double centralizer theorem for almost-triangular Hopf algebras (see Theorem 4.2).

2 Preliminaries

Throughout this paper, $\Bbbk$ is a fixed field. Unless otherwise stated, all vector spaces, algebras, coalgebras, maps and unadorned tensor products are over $\Bbbk$. For a coalgebra C, we denote its comultiplication by $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $\forall c \in C$ and for a left C-comodule (M, φ), we denote its coaction by $\varphi(m) = m_{[-1]} \otimes m_{[0]}$, $\forall m \in M$, where the summation symbols are omitted. We refer to [7] for the Hopf algebras theory.

Let H be a bialgebra and A be a left H-module algebra. The smash product $A\sharp H$ of A and H is defined as follows: For all $a, b \in A$ and $h, g \in H$,

(i) as $\Bbbk$-spaces, $A\sharp H = A \otimes H$,

(ii) multiplication is given by

$$(a\sharp h)(b\sharp g) = a(h_{(1)} \cdot b)\sharp h_{(2)}g.$$

Note that $A\sharp H$ is an algebra with the unit $1_A\sharp 1_H$.

Recall from [8] that a quasi-triangular Hopf algebra is a pair (H, R), where H is a Hopf algebra and $R = R^1 \otimes R^2 \in H \otimes H$ (where the summation symbols are also omitted) satisfying the following conditions (with $r = R$):

(1) R is invertible,

(2) $R\Delta(h) = \Delta^{\mathrm{op}}(h)R$ for all $h \in H$,

(3) $(\Delta \otimes \mathrm{id})(R) = R^1 \otimes r^1 \otimes R^2 r^2$,

(4) $(\mathrm{id} \otimes \Delta)(R) = R^1 r^1 \otimes r^2 \otimes R^2$.

It is easy to get that

$$R^{-1} = S(R^1) \otimes R^2, \quad \varepsilon(R^1)R^2 = 1.$$

Remark 2.1 (1) If the antipode of H is bijective, then $(S \otimes S)(R) = R$.

(2) (H, R) is triangular if $R^1 r^2 \otimes R^2 r^1 = 1 \otimes 1$.

(3) (H, R) is almost-triangular if $R^1 r^2 \otimes R^2 r^1 \in C(H \otimes H) = C(H) \otimes C(H)$, the center of $H \otimes H$ (see [6]).

Let ${}_H\mathcal{M}$ denote the category of the left H module category. For each $V \in {}_H\mathcal{M}$, $\mathrm{End}_{\Bbbk} V \in {}_H\mathcal{M}$, where for each $f \in \mathrm{End}_{\Bbbk} V$ and $h \in H$,

$$(h \cdot f)(v) = h_{(1)} \cdot (f(S(h_{(2)}) \cdot v)).$$

Moreover, if V, $W \in {}_H\mathcal{M}$, then $V \otimes W \in M$, where for each $v \otimes w \in V \otimes W$ and $h \in H$,

$$h \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w.$$

The tensor algebra of V, $T(V)$ is an H module algebra. Then ${}_H\mathcal{M}$ is a monoidal category. When (H, R) is quasi-triangular, the category ${}_H\mathcal{M}$ is a braided category with the braiding $\psi_{V,W} : V \otimes W \to W \otimes V$ given by

$$\psi(v \otimes w) = R^2 \cdot w \otimes R^1 \cdot v$$

for any $V, W \in {}_H\mathcal{M}$, and $v \in V$, $w \in W$.

Let (H, R) be a quasi-triangular Hopf algebra. Recall from [5] that an R-Lie algebra is an object $L \in {}_H\mathcal{M}$ together with an H-morphism $[\,,\,]_R : L \otimes L \to L$ satisfying

(i) R-anticommutativity: $[l_1, l_2]_R = -[R^2 \cdot l_2, R^1 \cdot l_1]_R$;

(ii) R-Jacobi identity:

$$0 = \{l_1 \otimes l_2 \otimes l_3\}_R + \{S_{312}(l_1 \otimes l_2 \otimes l_3)\}_R + \{S_{231}(l_1 \otimes l_2 \otimes l_3)\}$$

for all $l_1, l_2, l_3 \in L$, where $\{l_1 \otimes l_2 \otimes l_3\}_R = [l_1, [l_2, l_3]_R]_R$, $S_{312} = \psi_{12} \circ \psi_{23}$, $S_{231} = \psi_{23} \circ \psi_{12}$, $\psi_{23}(l_1 \otimes l_2 \otimes l_3) = l_1 \otimes (R^2 \cdot l_3) \otimes (R^1 \cdot l_2)$ and $\psi_{12}(l_1 \otimes l_2 \otimes l_3) = R^2 \cdot l_2 \otimes R^1 \cdot l_1 \otimes l_3$.

Note that $[\,,\,]_R$ being an H-module homomorphism means that for all $h \in H$ and $l_1, l_2 \in L$,

$$h \cdot [l_1, l_2]_R = [(h_{(1)} \cdot l_1), (h_{(2)} \cdot l_2)]_R.$$

3 *R*-universal Enveloping Hopf Algebras

In this section, we introduce the definition of the R-universal enveloping algebra of an R-Lie algebra (see [5]) in the setting of almost-triangular Hopf algebras, which generalizes the corresponding results in the setting of triangular Hopf algebras.

Let (H, R) be a triangular Hopf algebra and A be any left H-module algebra. In [5], the authors derived an R-Lie algebra denoted by A^- from A by defining an inner R-Lie product

$$[\,,\,]_R : A \otimes A \to A, \quad [a, b]_R = ab - (R^2 \cdot b)(R^1 \cdot a)$$

for any $a, b \in A$.

However, if (H, R) is an almost-triangular Hopf algebra, A^- is not necessarily an R-Lie algebra. In the following, we will discuss the condition under which A^- is an R-Lie algebra. Unless otherwise stated, we always let (H, R) denote an almost-triangular Hopf algebra.

Proposition 3.1 *Let $(A, \cdot)$ be a left H-module algebra satisfying $R^1 r^2 \cdot a \otimes R^2 r^1 \cdot b = a \otimes b$ for all $a, b \in A$. Then $(A, [\,,\,]_R)$ is an R-Lie algebra.*

Proof It is easy to get that $[\,,\,]_R$ satisfies the R-anticommutativity. Indeed, for any $a, b \in A$,

$$\begin{aligned} -[R^2 \cdot b, R^1 \cdot a]_R &= (r^2 R^1 \cdot a)(r^1 R^2 \cdot b) - (R^2 \cdot b)(R^1 \cdot a) \\ &= ab - (R^2 \cdot b)(R^1 \cdot a) \\ &= [a, b]_R. \end{aligned}$$

In order to check the R-Jacobi identity, we have the following computations: For any $a, b, c \in A$,

$$\begin{aligned} &\{a \otimes b \otimes c\}_R \\ &= [a, [b, c]_R]_R \\ &= [a, bc - (R^2 \cdot c)(R^1 \cdot b)]_R \\ &= a(bc - (R^2 \cdot c)(R^1 \cdot b)) - (r^2 \cdot (bc - (R^2 \cdot c)(R^1 \cdot b)))(r^1 \cdot a) \end{aligned}$$

$$
\begin{aligned}
&= abc - a(R^2\cdot c)(R^1\cdot b) - (r^2\cdot bc)(r^1\cdot a) + (r^2\cdot(R^2\cdot c)(R^1\cdot b))(r^1\cdot a)\\
&= abc - a(R^2\cdot c)(R^1\cdot b) - (r^2_{(1)}\cdot b)(r^2_{(2)}\cdot c)(r^1\cdot a) + (r^2_{(1)}R^2\cdot c)(r^2_{(2)}R^1\cdot b)(r^1\cdot a)\\
&= abc - a(R^2\cdot c)(R^1\cdot b) - (r^2\cdot b)(R^2\cdot c)(R^1r^1\cdot a) + (r^2R^2\cdot c)(P^2R^1\cdot b)(P^1r^1\cdot a),\\
&\quad \{S_{312}(a\otimes b\otimes c)\}_R\\
&= \{\psi_{12}\circ\psi_{23}(a\otimes b\otimes c)\}_R = \{\psi_{12}(a\otimes R^2\cdot c\otimes R^1\cdot b)\}_R\\
&= \{r^2R^2\cdot c\otimes r^1\cdot a\otimes R^1\cdot b\}_R = [r^2R^2\cdot c, [(r^1\cdot a),(R^1\cdot b)]_R]_R\\
&= [r^2R^2\cdot c, (r^1\cdot a)(R^1\cdot b) - (P^2R^1\cdot b)(P^1r^1\cdot a)]_R\\
&= (r^2R^2\cdot c)(r^1\cdot a)(R^1\cdot b) - (r^2R^2\cdot c)(P^2R^1\cdot b)(P^1r^1\cdot a)\\
&\quad - (Q^2_{(1)}r^1\cdot a)(Q^2_{(2)}R^1\cdot b)(Q^1r^2R^2\cdot c) + (Q^2_{(1)}P^2R^1\cdot b)(Q^2_{(2)}P^1r^1\cdot a)(Q^1r^2R^2\cdot c)\\
&= (r^2R^2\cdot c)(r^1\cdot a)(R^1\cdot b) - (r^2R^2\cdot c)(P^2R^1\cdot b)(P^1r^1\cdot a)\\
&\quad - (Q^2r^1\cdot a)(P^2R^1\cdot b)(P^1Q^1r^2R^2\cdot c) + (Q^2P^2R^1\cdot b)(U^2P^1r^1\cdot a)(U^1Q^1r^2R^2\cdot c)\\
&= (r^2R^2\cdot c)(r^1\cdot a)(R^1\cdot b) - (r^2R^2\cdot c)(P^2R^1\cdot b)(P^1r^1\cdot a)\\
&\quad - abc + (Q^2P^2R^1\cdot b)(U^2P^1r^1\cdot a)(U^1Q^1r^2R^2\cdot c)\\
&= (r^2R^2\cdot c)(r^1\cdot a)(R^1\cdot b) - (r^2R^2\cdot c)(P^2R^1\cdot b)(P^1r^1\cdot a)\\
&\quad - abc + (P^2U^2R^1\cdot b)(P^1Q^2r^1\cdot a)(U^1Q^1r^2R^2\cdot c)\\
&= (r^2R^2\cdot c)(r^1\cdot a)(R^1\cdot b) - (r^2R^2\cdot c)(P^2R^1\cdot b)(P^1r^1\cdot a)\\
&\quad - abc + (P^2\cdot b)(P^1\cdot a)c,
\end{aligned}
$$

and

$$
\begin{aligned}
&\quad \{S_{231}(a\otimes b\otimes c)\}_R\\
&= \{\psi_{23}\circ\psi_{12}(a\otimes b\otimes c)\}_R\\
&= \{R^2\cdot b\otimes r^2\cdot c\otimes r^1R^1\cdot a\}_R\\
&= [R^2\cdot b, (r^2\cdot c)(r^1R^1\cdot a) - (P^2r^1R^1\cdot a)(P^1r^2\cdot c)]_R\\
&= (R^2\cdot b)(r^2\cdot c)(r^1R^1\cdot a) - (R^2\cdot b)(P^2r^1R^1\cdot a)(P^1r^2\cdot c)\\
&\quad - (Q^2_{(1)}r^2\cdot c)(Q^2_{(2)}r^1R^1\cdot a)(Q^1R^2\cdot b) + (Q^2_{(1)}P^2r^1R^1\cdot a)(Q^2_{(2)}P^1r^2\cdot c)(Q^1R^2\cdot b)\\
&= (R^2\cdot b)(r^2\cdot c)(r^1R^1\cdot a) - (R^2\cdot b)(P^2r^1R^1\cdot a)(P^1r^2\cdot c)\\
&\quad - (Q^2r^2\cdot c)(P^2r^1R^1\cdot a)(P^1Q^1R^2\cdot b) + (Q^2P^2r^1R^1\cdot a)(U^2P^1r^2\cdot c)(U^1Q^1R^2\cdot b)\\
&= (R^2\cdot b)(r^2\cdot c)(r^1R^1\cdot a) - (R^2\cdot b)(R^1\cdot a)c - (r^2P^2\cdot c)(r^1Q^2R^1\cdot a)(P^1Q^1R^2\cdot b)\\
&\quad + a(U^2\cdot c)(U^1\cdot b)\\
&= (R^2\cdot b)(r^2\cdot c)(r^1R^1\cdot a) - (R^2\cdot b)(R^1\cdot a)c - (r^2P^2\cdot c)(r^1\cdot a)(P^1\cdot b)\\
&\quad + a(U^2\cdot c)(U^1\cdot b).
\end{aligned}
$$

Hence

$$\{a\otimes b\otimes c\}_R + \{S_{312}(a\otimes b\otimes c)\}_R + \{S_{231}(a\otimes b\otimes c)\}_R = 0.$$

Example 3.1 Let (H,R) be a triangular Hopf algebra, and then any left H-module algebra A satisfies $R^1r^2\cdot a\otimes R^2r^1\cdot b = a\otimes b$ for any $a,b\in A$. So A^- is an R-Lie algebra with $[\,,\,]_R$.

Example 3.2 For any Hopf algebra H, H is a left H-module algebra via the adjoint action, i.e., $h \triangleright g = h_{(1)} g S(h_{(2)})$ for all $h, g \in H$. If (H, R) is almost-triangular, then H^- is an R-Lie algebra with $[\,,\,]_R$.

Proof By Proposition 3.1, we just need to show that

$$R^1 r^2 \triangleright h \otimes R^2 r^1 \triangleright g = h \otimes g$$

for all $h, g \in H$. For this, we compute

$$\begin{aligned}
& R^1 r^2 \triangleright h \otimes R^2 r^1 \triangleright g \\
&= R^1_{(1)} r^2_{(1)} h S(R^1_{(2)} r^2_{(2)}) \otimes R^2_{(1)} r^1_{(1)} g S(R^2_{(2)} r^1_{(2)}) \\
&= R^1 U^2 h S(P^1 r^2) \otimes R^2_{(1)} P^2_{(1)} r^1_{(1)} U^1_{(1)} g S(R^2_{(2)} P^2_{(2)} r^1_{(2)} U^1_{(2)}) \\
&= R^1 Q^1 U^2 V^2 S(P^1 W^1 r^2 X^2 S^{-1}(h)) \otimes Q^2 W^2 r^1 U^1 S(R^2 P^2 X^1 V^1 S^{-1}(g)) \\
&= R^1 Q^1 U^2 V^2 S(W^1 r^2 P^1 X^2 S^{-1}(h)) \otimes W^2 r^1 Q^2 U^1 S(R^2 P^2 X^1 V^1 S^{-1}(g)) \\
&= R^1 Q^1 U^2 V^2 S(W^1 r^2 S^{-1}(h) P^1 X^2) \otimes W^2 r^1 Q^2 U^1 S(R^2 V^1 S^{-1}(g) P^2 X^1) \\
&= R^1 V^2 Q^1 U^2 S(S^{-1}(h) W^1 r^2 P^1 X^2) \otimes Q^2 U^1 W^2 r^1 S(R^2 V^1 S^{-1}(g) P^2 X^1) \\
&= R^1 V^2 Q^1 U^2 S(W^1 r^2 P^1 X^2) h \otimes Q^2 U^1 W^2 r^1 S(R^2 V^1 P^2 X^1) g \\
&= R^1 V^2 S(r^2 P^1 X^2) S(W^1) Q^1 U^2 h \otimes W^2 Q^2 U^1 r^1 S(R^2 V^1 P^2 X^1) g \\
&= R^1 V^2 S(P^1 X^2) S(r^2_{(1)}) r^2_{(2)} h \otimes r^1 S(R^2 P^2 X^1 V^1) g \\
&= R^1 V^2_{(1)} S(V^2_{(2)}) S(P^1) h \otimes S(R^2 P^2 V^1) g \\
&= R^1_{(1)} S(R^1_{(2)}) h \otimes S(R^2) g \\
&= h \otimes g.
\end{aligned}$$

Example 3.3 Let (H, R) be an almost-triangular Hopf algebra with a bijective antipode. If V is a finite-dimensional left H-module satisfying $R^1 r^2 \cdot v_1 \otimes R^2 r^1 \cdot v_2 = v_1 \otimes v_2$ for any $v_1, v_2 \in V$, then $\mathrm{End}_{\Bbbk} V$ is a left H-module algebra satisfying

$$R^1 r^2 \cdot f_1 \otimes R^2 r^1 \cdot f_2 = f_1 \otimes f_2$$

for any $f_1, f_2 \in \mathrm{End}_{\Bbbk} V$, where $(h \cdot f)(v) = h_{(1)} \cdot f(S(h_{(2)}) \cdot v)$ for any $h \in H,\ f \in \mathrm{End}_{\Bbbk} V$ and $v \in V$. Therefore, $\mathrm{End}_{\Bbbk} V^-$ is an R-Lie algebra with $[\,,\,]_R$.

Proof It is easy to check that $\mathrm{End}_{\Bbbk} V$ is a left H-module algebra. We just prove the identity

$$R^1 r^2 \cdot f_1 \otimes R^2 r^1 \cdot f_2 = f_1 \otimes f_2$$

for any $f_1, f_2 \in \mathrm{End}_{\Bbbk} V$. Indeed, for any $v_1, v_2 \in V$, we have the following computations:

$$\begin{aligned}
& (R^1 r^2 \triangleright f_1)(v_1) \otimes (R^2 r^1 \triangleright f_2)(v_2) \\
&= R^1_{(1)} r^2_{(1)} \cdot f_1(S(R^1_{(2)} r^2_{(2)}) \cdot v_1) \otimes R^2_{(1)} r^1_{(1)} \cdot f_2(S(R^2_{(2)} r^1_{(2)}) \cdot v_2) \\
&= R^1 Q^1 U^2 V^2 \cdot f_1(S(P^1 W^1 r^2 X^2) \cdot v_1) \otimes Q^2 W^2 r^1 U^1 \cdot f_2(S(R^2 P^2 X^1 V^1) \cdot v_2) \\
&= R^1 Q^1 U^2 V^2 \cdot f_1(S(P^1 X^2 W^1 r^2) \cdot v_1) \otimes Q^2 U^1 W^2 r^1 \cdot f_2(S(R^2 P^2 X^1 V^1) \cdot v_2)
\end{aligned}$$

$$
\begin{aligned}
&= R^1V^2\cdot f_1(S(r^2)S(W^1)X^2P^1\cdot v_1)\otimes W^2r^1\cdot f_2(S(V^1)X^1P^2S(R^2)\cdot v_2)\\
&= R^1V^1\cdot f_1(S(r^2)W^2\cdot v_1)\otimes W^1r^1\cdot f_2(V^2S(R^2)\cdot v_2)\\
&= R^1\cdot f_1(S(r^2_{(1)})r^2_{(2)}\cdot v_1)\otimes r^1\cdot f_2(R^2_{(1)}S(R^2_{(2)})\cdot v_2)\\
&= f_1(v_1)\otimes f_2(v_2),
\end{aligned}
$$

where the fourth identity holds because $(S\otimes S)(R)=R$ and the fifth holds because $S(R^1)\cdot v_1\otimes R^2\cdot v_2 = R^2\cdot v_1\otimes R^1\cdot v_2$. So from Proposition 3.1, $\mathrm{End}_{\Bbbk}V^-$ is an R-Lie algebra with $[\,,\,]_R$.

Example 3.4 Let (H,R) be an almost-triangular Hopf algebra and V be a finite-dimensional left H-module such that the representation of H on V,

$$\pi_V : H\to \mathrm{End}_{\Bbbk}V,\quad \pi_V(h)(x)=h\cdot x\quad \text{for all } h\in H, x\in V$$

is a surjection. Then $\mathrm{End}_{\Bbbk}V$ is a left H-module algebra satisfying

$$R^1r^2\cdot f\otimes R^2r^1\cdot f' = f\otimes f'\quad \text{for all } f,f'\in \mathrm{End}_{\Bbbk}V,$$

where $h\cdot f=\pi_V(h_{(1)})f\pi_V(S(h_{(2)}))$, $\pi_V : H\to \mathrm{End}_{\Bbbk}V$, defined by

$$\pi_V(h)(x)=h\cdot x\quad \text{for any } h\in H, f\in \mathrm{End}_{\Bbbk}V \text{ and } x\in V$$

is an algebra homomorphism. Hence $\mathrm{End}_{\Bbbk}V^-$ is an R-Lie algebra with $[\,,\,]_R$.

Proof For any $f,f'\in \mathrm{End}_{\Bbbk}V$, since π_V is a surjection, there exist $h,h'\in H$ such that $\pi_V(h)=f$ and $\pi_V(h')=f'$. Then we have

$$
\begin{aligned}
&R^1r^2\triangleright f\otimes R^2r^1\triangleright f'\\
&=\pi_V(R^1_{(1)}r^2_{(1)})\pi_V(h)\pi_V(S(R^1_{(2)}r^2_{(2)}))\otimes \pi_V(R^2_{(1)}r^1_{(1)})\pi_V(h')\pi_V(S(R^2_{(2)}r^1_{(2)}))\\
&=\pi_V(R^1U^2hS(P^1r^2))\otimes \pi_V(R^2_{(1)}P^2_{(1)}r^1_{(1)}U^1_{(1)}h'S(R^2_{(2)}P^2_{(2)}r^1_{(2)}U^1_{(2)}))\\
&=\pi_V(R^1Q^1U^2V^2S(P^1W^1r^2X^2S^{-1}(h)))\otimes \pi_V(Q^2W^2r^1U^1S(R^2P^2X^1V^1S^{-1}(h')))\\
&=\pi_V(R^1Q^1U^2V^2S(W^1r^2P^1X^2S^{-1}(h)))\otimes \pi_V(W^2r^1Q^2U^1S(R^2P^2X^1V^1S^{-1}(h')))\\
&=\pi_V(R^1Q^1U^2V^2S(W^1r^2S^{-1}(h)P^1X^2))\otimes \pi_V(W^2r^1Q^2U^1S(R^2V^1S^{-1}(h')P^2X^1))\\
&=\pi_V(R^1V^2Q^1U^2S(S^{-1}(h)W^1r^2P^1X^2))\otimes \pi_V(Q^2U^1W^2r^1S(R^2V^1S^{-1}(h')P^2X^1))\\
&=\pi_V(R^1V^2Q^1U^2S(W^1r^2P^1X^2)h)\otimes \pi_V(Q^2U^1W^2r^1S(R^2V^1P^2X^1)h')\\
&=\pi_V(R^1V^2S(r^2P^1X^2)S(W^1)Q^1U^2h)\otimes \pi_V(W^2Q^2U^1r^1S(R^2V^1P^2X^1)h')\\
&=\pi_V(R^1V^2S(P^1X^2)S(r^2_{(1)})r^2_{(2)}h)\otimes \pi_V(r^1S(R^2P^2X^1V^1)h')\\
&=\pi_V(R^1V^2_{(1)}S(V^2_{(2)})S(P^1)h)\otimes \pi_V(S(R^2P^2V^1)h')\\
&=\pi_V(R^1_{(1)}S(R^1_{(2)})h)\otimes \pi_V(S(R^2)h')\\
&=f\otimes f'.
\end{aligned}
$$

Hence $\mathrm{End}_{\Bbbk}V^-$ is an R-Lie algebra with $[\,,\,]_R$.

Remark 3.1 (i) If V is a simple H-module, then π_V is a surjection.

(ii) Let V be a semi-simple H-module, i.e., $V = V_1^{k_1} \oplus \cdots \oplus V_s^{k_s}$, where for any $i, j = 1, \cdots, s$, $V_i \ncong V_j$ as H-modules when $i \neq j$, and V_i are simple H-modules. If $k_1 = \cdots = k_s = 1$, then π_V is a surjection.

Definition 3.1 *Let $(L, [\,,\,]_R)$ be an R-Lie algebra satisfying*

$$R^1 r^2 \cdot l_1 \otimes R^2 r^1 \cdot l_2 = l_1 \otimes l_2 \quad \textit{for all } l_1, l_2 \in L.$$

An R-universal enveloping algebra of L is a pair (U, u), where U is an associative left H-module algebra such that

$$R^1 r^2 \cdot u_1 \otimes R^2 r^1 \cdot u_2 = u_1 \otimes u_2 \quad \textit{for all } u_1, u_2 \in U,$$

$u : L \to U^-$ is an R-Lie homomorphism, and the following holds: For any associative H-module algebra A satisfying $R^1 r^2 \cdot a \otimes R^2 r^1 \cdot b = a \otimes b \ \forall a, b \in A$, and any R-Lie homomorphism $f : L \to A^-$, there exists a unique H-module algebra homomorphism $g : U \to A$, such that $g \circ u = f$.

Proposition 3.2 *Let A be a left H-module algebra such that for all $a, b \in A$, $R^1 r^2 \cdot a \otimes R^2 r^1 \cdot b = a \otimes b$. Then the map $u : A^- \to U(A^-)$ is an injection.*

Proof Clearly the identity map from A^- to A^- is an R-Lie map. By the universality of $U(A^-)$, there exists a unique H-module algebra homomorphism $g : U(A^-) \to A^-$ such that $g \circ u = \mathrm{id}$. Hence u is an injection.

Proposition 3.3 *Let $(L, [\,,\,]_R)$ be an R-Lie algebra satisfying $R^1 r^2 \cdot l_1 \otimes R^2 r^1 \cdot l_2 = l_1 \otimes l_2$ for all $l_1, l_2 \in L$. Then L has an R-universal enveloping algebra $(U(L) = T(L)/I, u)$, where I is the ideal of $T(L)$ generated by*

$$\{l_1 \otimes l_2 - R^2 \cdot l_1 \otimes R^1 \cdot l_2 - [l_1, l_2]_R \ \textit{for all } l_1, l_2 \in L\},$$

and $u : L \to T(L)/I$ is the canonical map: $l \mapsto l + I = \overline{l}$.

Proof The proof is similar to the one in the setting of triangular Hopf algebras in [5].

From now on, we write $[\,,\,]$ for $[\,,\,]_R$.

Proposition 3.4 *Let L be an R-Lie algebra satisfying $R^1 r^2 \cdot l_1 \otimes R^2 r^1 \cdot l_2 = l_1 \otimes l_2$ for all $l_1, l_2 \in L$. Then there exists an H-module algebra homomorphism*

$$g : U(L \oplus L) \to U(L) \otimes U(L).$$

Proof Define $f : L \oplus L \to U(L) \otimes U(L)$ by

$$(l_1, l_2) \mapsto \overline{l_1} \otimes 1 + 1 \otimes \overline{l_2} \quad \text{for all } l_1, l_2 \in L.$$

Next we show that f is an R-Lie homomorphism. Obviously, f is an H-module homomorphism. It suffices to show that

$$f([(l, s), (l', s')]^{L \oplus L}) = [f(l, s), f(l', s')]^{(U(L) \otimes U(L))^-}.$$

Recall that the multiplication in $U(L)\otimes U(L)$ is

$$(l\otimes s)(l'\otimes s')=l(R^2\cdot l')\otimes(R^1\cdot s)s' \quad \text{for all } l,s,l',s'\in L.$$

Then we have

$$\begin{aligned}
&[f(l,s),f(l',s')]^{(U(L)\otimes U(L))^-}\\
&=[(\overline{l}\otimes 1+1\otimes\overline{s}),(\overline{l'}\otimes 1+1\otimes\overline{s'})]\\
&=(\overline{l}\otimes 1+1\otimes\overline{s})(\overline{l'}\otimes 1+1\otimes\overline{s'})-(R^2\cdot(\overline{l'}\otimes 1+1\otimes\overline{s'}))(R^1\cdot(\overline{l}\otimes 1+1\otimes\overline{s}))\\
&=(\overline{l}\otimes 1+1\otimes\overline{s})(\overline{l'}\otimes 1+1\otimes\overline{s'})-(R^2\cdot\overline{l'}\otimes 1+1\otimes R^2\cdot\overline{s'})(R^1\cdot\overline{l}\otimes 1+1\otimes R^1\cdot\overline{s})\\
&=\overline{ll'}\otimes 1+\overline{l}\otimes\overline{s'}+(R^2\cdot\overline{l'})\otimes(R^1\cdot\overline{s})+1\otimes\overline{s}\overline{s'}-((R^2\cdot\overline{l'})(R^1\cdot\overline{l})\otimes 1\\
&\quad+(R^2\cdot\overline{l'})\otimes(R^1\cdot\overline{s})+(r^2R^1\cdot\overline{l})\otimes(r^1R^2\cdot\overline{s'})+1\otimes(R^2\cdot\overline{s'})(R^1\cdot\overline{s}))\\
&=\overline{ll'}\otimes 1-(R^2\cdot\overline{l'})(R^1\cdot\overline{l})\otimes 1+1\otimes\overline{s}\overline{s'}-1\otimes(R^2\cdot\overline{s'})(R^1\cdot\overline{s}))\\
&=[\overline{l},\overline{l'}]\otimes 1+1\otimes[\overline{s},\overline{s'}]=\overline{[l,l']}\otimes 1+1\otimes\overline{[s,s']}\\
&=f([l,l'],[s,s'])=f([(l,s),(l',s')]^{L\oplus L}).
\end{aligned}$$

So f is an R-Lie map.

Now by the universal property of $U(L\oplus L)$, there exists an H-module algebra homomorphism $g:U(L\oplus L)\to U(L)\otimes U(L)$.

Theorem 3.1 *Let L be an R-Lie algebra satisfying $R^1r^2\cdot l_1\otimes R^2r^1\cdot l_2=l_1\otimes l_2$ for all $l_1,l_2\in L$. Then $U(L)$ in Proposition 3.4 is a Hopf algebra in the category ${}_H\mathcal{M}$ with*

$$\begin{aligned}
&\Delta(\overline{l})=\overline{l}\otimes 1+1\otimes\overline{l},\\
&S(\overline{l})=-\overline{l},\quad S(\overline{s}\overline{t})=(R^2\cdot S(\overline{t})(R^1\cdot S(\overline{s})),\\
&\varepsilon(\overline{l})=0,\quad \varepsilon(1)=1
\end{aligned}$$

for all $l\in L$ and $\overline{s},\overline{t}\in U(L)$.

Proof Analogous to the proof in the case of triangular Hopf algebras ([5, Theorem 2.6]).

Next we give an application of Theorem 3.1. Let V be a finite-dimensional left H-module such that for any $v_1,v_2\in V$, $R^1r^2\cdot v_1\otimes R^2r^1\cdot v_2=v_1\otimes v_2$. So from Example 3.4 and Theorem 3.1, $U(\mathrm{End}_{\Bbbk}V^-)$ is a Hopf algebra in the category ${}_H\mathcal{M}$, which implies that $U(\mathrm{End}_{\Bbbk}V^-)\sharp H$ is a Radford's biproduct. In the following, we will discuss when $U(\mathrm{End}_{\Bbbk}V^-)\sharp H$ is an almost-triangular Hopf algebra.

Theorem 3.2 *Let (H,R) be an almost-triangular Hopf algebra and V be a finite-dimensional left H-module such that for any $v_1,v_2\in V$, $R^1r^2\cdot v_1\otimes R^2r^1\cdot v_2=v_1\otimes v_2$. Then $(U(\mathrm{End}_{\Bbbk}V^-)\sharp H,\ (1\sharp R^1)\otimes(1\sharp R^2))$ is an almost-triangular Hopf algebra if and only if $R^1r^2\otimes R^2r^1\cdot f=1\otimes f$, and $R^1r^2\cdot f\otimes R^2r^1=f\otimes 1$ for any $f\in\mathrm{End}_{\Bbbk}V$.*

Proof Denote $U(\mathrm{End}_{\Bbbk}V^-)\sharp H$ by B and $(1\sharp R^1)\otimes(1\sharp R^2)$ by $\overline{R}$. It is easy to get that $(\Delta_B\otimes\mathrm{id})(\overline{R})=\overline{R}^{13}\overline{R}^{23}$ and $(\mathrm{id}\otimes\Delta_B)(\overline{R})=\overline{R}^{13}\overline{R}^{12}$. Next we show that $(B,\overline{R})$ is almost cocommutative if and only if $R^1r^2 \otimes R^2r^1 \cdot f = 1 \otimes f$ for any $f \in \mathrm{End}_{\Bbbk}V$. For this, on the one hand, for any $f \otimes h \in B$, we have

$$\begin{aligned}
&\Delta^{\mathrm{cop}}(f \otimes h)((1\sharp R^1) \otimes (1\sharp R^2))\\
&= ((f_{(2)(0)}\sharp h_{(2)}) \otimes (f_{(1)}\sharp f_{(2)(-1)}h_{(1)}))(1\sharp R^1) \otimes (1\sharp R^2))\\
&= (f_{(2)(0)}\sharp h_{(2)}R^1) \otimes (f_{(1)}\sharp f_{(2)(-1)}h_{(1)}R^2)\\
&= (r^1 \cdot f_{(2)}\sharp h_{(2)}R^1) \otimes (f_{(1)}\sharp r^2 h_{(1)}R^2)\\
&= (r^1 \cdot f_{(2)}\sharp R^1 h_{(1)}) \otimes (f_{(1)}\sharp r^2 R^2 h_{(2)})\\
&= (r^1 \cdot 1\sharp R^1 h_{(1)}) \otimes (f\sharp r^2 R^2 h_{(2)}) + (r^1 \cdot f\sharp R^1 h_{(1)}) \otimes (1\sharp r^2 R^2 h_{(2)})\\
&= (1\sharp R^1 h_{(1)}) \otimes (f\sharp R^2 h_{(2)}) + (r^1 \cdot f\sharp R^1 h_{(1)}) \otimes (1\sharp r^2 R^2 h_{(2)}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&((1\sharp R^1) \otimes (1\sharp R^2))\Delta(f \otimes h)\\
&= (1\sharp R^1)(f_{(1)}\sharp f_{(2)(-1)}h_{(1)}) \otimes (1\sharp R^2)(f_{(2)(0)}\sharp h_{(2)})\\
&= (R^1_{(1)} \cdot f_{(1)}\sharp R^1_{(2)} r^2 h_{(1)}) \otimes (R^2_{(1)} r^1 \cdot f_{(2)}\sharp R^2_{(2)} h_{(2)})\\
&= (R^1 Q^1 \cdot f_{(1)}\sharp P^1 V^1 r^2 h_{(1)}) \otimes (Q^2 V^2 r^1 \cdot f_{(2)}\sharp R^2 P^2 h_{(2)})\\
&= (R^1 Q^1 \cdot f\sharp P^1 V^1 r^2 h_{(1)}) \otimes (Q^2 V^2 r^1 \cdot 1\sharp R^2 P^2 h_{(2)})\\
&\quad + (R^1 Q^1 \cdot 1\sharp P^1 V^1 r^2 h_{(1)}) \otimes (Q^2 V^2 r^1 \cdot f\sharp R^2 P^2 h_{(2)})\\
&= (R^1 \cdot f\sharp P^1 h_{(1)}) \otimes (1\sharp R^2 P^2 h_{(2)}) + (1\sharp P^1 V^1 r^2 h_{(1)}) \otimes (V^2 r^1 \cdot f\sharp P^2 h_{(2)})\\
&= (R^1 \cdot f\sharp P^1 h_{(1)}) \otimes (1\sharp R^2 P^2 h_{(2)}) + (1\sharp P^1 h_{(1)}) \otimes (f\sharp P^2 h_{(2)}).
\end{aligned}$$

Then from the above computations, we get that $(B, \overline{R})$ is almost cocommutative if and only if

$$(1\sharp R^1 h_{(1)}) \otimes (f\sharp R^2 h_{(2)}) = (1\sharp P^1 V^1 r^2 h_{(1)}) \otimes (V^2 r^1 \cdot f\sharp P^2 h_{(2)}),$$

which is equivalent to $R^1r^2 \otimes R^2r^1 \cdot f = 1 \otimes f$.

Finally, we check that $\overline{R}^{21}\overline{R}$ belongs to the center of $B\otimes B$ if and only if $R^1r^2\otimes R^2r^1\cdot f = 1\otimes f$, and $R^1r^2 \cdot f \otimes R^2r^1 = f \otimes 1$ for any $f \in \mathrm{End}_{\Bbbk}V$. Indeed, we have the following computations:

$$\begin{aligned}
&(1\sharp R^1)(1\sharp r^2)(f\sharp h) \otimes (1\sharp R^2)(1\sharp r^1)(f'\sharp h')\\
&= (R^1_{(1)} r^2_{(1)} \cdot f\sharp R^1_{(2)} r^2_{(2)} h) \otimes (R^2_{(1)} r^1_{(1)} \cdot f'\sharp R^2_{(2)} r^1_{(2)} h')\\
&= (R^1 Q^1 U^2 V^2 \cdot f\sharp P^1 W^1 r^2 X^2 h) \otimes (Q^2 W^2 r^1 U^1 \cdot f'\sharp R^2 P^2 X^1 V^1 h')\\
&= (R^1 V^2 \cdot f\sharp P^1 W^1 r^2 X^2 h) \otimes (W^2 r^1 \cdot f'\sharp R^2 P^2 X^1 V^1 h')\\
&= (R^1 V^2 \cdot f\sharp h P^1 X^2 W^1 r^2) \otimes (W^2 r^1 \cdot f'\sharp h' R^2 V^1 P^2 X^1)
\end{aligned}$$

and

$$(f\sharp h)(1\sharp R^1)(1\sharp r^2) \otimes (f'\sharp h')(1\sharp R^2)(1\sharp r^1) = (f\sharp h R^1 r^2) \otimes (f'\sharp h' R^2 r^1).$$

Thus it is not hard to get the conclusion. So we complete the proof.

Remark 3.2 If V is a finite-dimensional left H-module such that the representation of H on V is a surjection, then $R^1r^2\otimes R^2r^1\cdot f = 1\otimes f$, and $R^1r^2\cdot f\otimes R^2r^1 = f\otimes 1$ for any $f \in \mathrm{End}_{\Bbbk}V$.

4 Cohen-Fischman-Westreich's Double Centralizer Theorem in the Setting of Almost-Triangular Hopf Algebras

In this section, we always let (H,R) be an almost-triangular Hopf algebra and V be a finite-dimensional left H-module such that for any $v_1,v_2\in V$, $R^1r^2\cdot v_1\otimes R^2r^1\cdot v_2=v_1\otimes v_2$. In Section 3, we have already showed that $U(\mathrm{End}_{\Bbbk}V^-)\sharp H$ is a Radford's biproduct.

In the following, we always denote $U(\mathrm{End}_{\Bbbk}V^-)\sharp H$ by B. Obviously, V is a left B-module via $(f\sharp h)\cdot v=f(h\cdot v)$ for any $f\in \mathrm{End}_{\Bbbk}V,\ h\in H$ and $v\in V$. So we have a representation of B on V $\rho: B\to \mathrm{End}_{\Bbbk}V$ given by

$$\rho(f\sharp h)(v)=f(h\cdot v).$$

Clearly, ρ is a surjection. The representation ρ induces a representation ρ^m on V^m as follows:

$$\rho^m(b)(v_1\otimes\cdots\otimes v_m)=\rho(b_{(1)})(v_1)\otimes\cdots\rho(b_{(m)})(v_m)$$

for any $b\in B$.

Notation 4.1 (i) For any $b\in B$, denote $\rho(b)$ by $\underline{b}$. So $\rho^m(b)=\underline{b_{(1)}}\otimes\cdots\underline{b_{(m)}}$.

(ii) For any $h\in H$, it is easy to get that $\rho(\mathrm{id}\sharp h)=\rho(1\sharp h)$. So denote $\rho(\mathrm{id}\sharp h)$ and $\rho(1\sharp h)$ by $\underline{h}$.

Now we consider the symmetric group S_m. Define a representation $\phi: \Bbbk S_m\to \mathrm{End}_{\Bbbk}V^{\otimes m}$ by

$$(i,i+1)\cdot(v_1\otimes\cdots\otimes v_m)=v_1\otimes\cdots\otimes R^2\cdot v_{i+1}\otimes R^1\cdot v_i\cdots\otimes v_m.$$

The action of $\Bbbk S_m$ on $\mathrm{End}_{\Bbbk}V^{\otimes m}$ is given by

$$\begin{aligned}&((i,i+1)\cdot(f_1\otimes\cdots\otimes f_m))(v_1\otimes\cdots\otimes v_m)\\&=(i,i+1)\cdot(f_1\otimes\cdots\otimes f_m)((i,i+1)\cdot(v_1\otimes\cdots\otimes v_m))\\&=(i,i+1)\cdot(f_1(v_1)\otimes\cdots\otimes f_i(R^2\cdot v_{i+1})\otimes f_{i+1}(R^1\cdot v_i)\cdots\otimes f_m(v_m))\\&=f_1(v_1)\otimes\cdots\otimes r^2\cdot f_{i+1}(R^1\cdot v_i)\otimes r^1\cdot f_i(R^2\cdot v_{i+1})\cdots\otimes f_m(v_m)\\&=(f_1\otimes\cdots\otimes \underline{r^2}f_{i+1}\underline{R^1}\otimes \underline{r^1}f_i\underline{R^2}\otimes\cdots\otimes f_m)(v_1\otimes\cdots\otimes v_m).\end{aligned}$$

In the following lemma, we have repeated occurrences of R denoted by $R_1,\cdots,R_j$, where $R=R_i$ for all i. For convenience, we shall write R_0^2 for an empty word and R_0^1 for 1.

Lemma 4.1 *Let (H,R) be an almost-triangular Hopf algebra and V be a finite-dimensional left H-module such that for any $v_1,v_2\in V$, $R^1r^2\cdot v_1\otimes R^2r^1\cdot v_2=v_1\otimes v_2$. Then for any $f\in(\mathrm{End}_{\Bbbk}V)^-$, we have*

(i) $\Delta^m(f\sharp 1)=\sum\limits_{j=0}^{m}(1\sharp R_1^2)\otimes\cdots(1\sharp R_j^2)\otimes(R_j^1\cdots R_1^1\cdot f\sharp 1)\otimes 1^{\otimes m-j}$;

(ii) $\rho^{m+1}(f\sharp 1)=(1+(2,1)+\cdots(m+1,m)\cdots(2,1))\cdot(f\otimes \mathrm{id}^{\otimes m})$.

Proof

$$(i,i+1)\cdot(f_1\otimes\cdots\otimes f_i\otimes\mathrm{id}\otimes\cdots\otimes f_m)=f_1\otimes\cdots\otimes f_{i-1}\otimes\underline{R^2}\otimes R^1\cdot f_i\otimes\cdots\otimes f_m.$$

Since

$$(i, i+1)\cdot(f_1\otimes\cdots\otimes f_i\otimes \mathrm{id}\otimes\cdots\otimes f_m) = f_1\otimes\cdots\otimes f_{i-1}\otimes \underline{r^2}\ \underline{R^1}\otimes \underline{r^1} f_i \underline{R^2}\otimes\cdots\otimes f_m,$$

it suffices to check that $\underline{r^2}\ \underline{R^1}\otimes\underline{r^1} f\underline{R^2} = \underline{R^2}\otimes R^1\cdot f$. Indeed, we have

$$\begin{aligned}(\underline{r^2}\ \underline{R^1}\otimes\underline{r^1} f\underline{R^2})(v_1\otimes v_2) &= r^2R^1\cdot v_1\otimes r^1\cdot f(R^2\cdot v_2)\\ &= r^2P^2Q^2R^1\cdot v_1\otimes r^1\cdot f(S(P^1)Q^1R^2\cdot v_2)\\ &= r^2P^2\cdot v_1\otimes r^1\cdot f(S(P^1)\cdot v_2)\\ &= (\underline{R^2}\otimes R^1\cdot f)(v_1\otimes v_2).\end{aligned}$$

Hence with the same idea of Lemma 3.7 in [5], we can obtain our lemma.

Theorem 4.1 *Let (H,R) be an almost Hopf algebra and V a finite-dimensional vector space over a field $\Bbbk$ of characteristic 0. If V is a left H-module such that for any $v_1, v_2\in V$ and $f\in \mathrm{End}_{\Bbbk}V$, $R^1r^2\cdot v_1\otimes R^2r^1\cdot v_2 = v_1\otimes v_2$, $R^1r^2\otimes R^2r^1\cdot f = 1\otimes f$, and $R^1r^2\cdot f\otimes R^2r^1 = f\otimes 1$, then we have*

(i) $\mathrm{End}_{\phi(\Bbbk S_m)}V^{\otimes m} = \rho^m(B)$;

(ii) $\mathrm{End}_{\rho^m(B)}V^{\otimes m} = \phi(\Bbbk S_m)$.

Proof (i) First we show that $\rho^m(B)\subset(\mathrm{End}_{\Bbbk}V^{\otimes m})^{\Bbbk S_m}$. Indeed, for any $b\in B$, we have

$$\begin{aligned}&((i,i+1)\cdot\rho^m(b))(v_1\otimes\cdots\otimes v_m)\\ &= (i,i+1)\cdot\rho^m(b)(v_1\otimes\cdots\otimes R^2\cdot v_{i+1}\otimes R^1\cdot v_i\otimes\cdots\otimes v_m)\\ &= b_{(1)}\cdot v_1\otimes\cdots\otimes(1\sharp r^2)b_{(i+1)}(1\sharp R^1)\cdot v_i\otimes(1\sharp r^1)b_{(i)}(1\sharp R^2)\cdot v_{i+1}\otimes\cdots\otimes b_{(m)}\cdot v_m\\ &= b_{(1)}\cdot v_1\otimes\cdots\otimes b_{(i)}(1\sharp r^2)(1\sharp R^1)\cdot v_i\otimes b_{(i+1)}(1\sharp r^1)(1\sharp R^2)\cdot v_{i+1}\otimes\cdots\otimes b_{(m)}\cdot v_m\\ &= \rho^m(b)(v_1\otimes\cdots\otimes v_m).\end{aligned}$$

Next we claim that $\mathrm{End}_{\phi(\Bbbk S_m)}V^{\otimes m} = (\mathrm{End}_{\Bbbk}V^{\otimes m})^{\Bbbk S_m}$. On the one hand, for any $f_1\otimes\cdots\otimes f_m\in\mathrm{End}_{\phi(\Bbbk S_m)}V^{\otimes m}$ and $v_1,\cdots,v_m\in V$, we have

$$\begin{aligned}&((i,i+1)\cdot(f_1\otimes\cdots\otimes f_m))(v_1\otimes\cdots\otimes v_m)\\ &= (i,i+1)\cdot(f_1\otimes\cdots\otimes f_m)((i,i+1)\cdot(v_1\otimes\cdots\otimes v_m))\\ &= (i,i+1)\cdot((i,i+1)\cdot(f_1\otimes\cdots\otimes f_m)(v_1\otimes\cdots\otimes v_m))\\ &= (f_1\otimes\cdots\otimes f_m)(v_1\otimes\cdots\otimes v_m),\end{aligned}$$

which means $f_1\otimes\cdots\otimes f_m\in(\mathrm{End}_{\Bbbk}V^{\otimes m})^{\Bbbk S_m}$. So $\mathrm{End}_{\phi(\Bbbk S_m)}V^{\otimes m}\subset(\mathrm{End}_{\Bbbk}V^{\otimes m})^{\Bbbk S_m}$. On the other hand, for any $f_1\otimes\cdots\otimes f_m\in(\mathrm{End}_{\Bbbk}V^{\otimes m})^{\Bbbk S_m}$, we compute

$$\begin{aligned}&(f_1\otimes\cdots\otimes f_m)((i,i+1)\cdot(v_1\otimes\cdots\otimes v_m))\\ &= ((i,i+1)\cdot(f_1\otimes\cdots\otimes f_m))(v_1\otimes\cdots\otimes R^2\cdot v_{i+1}\otimes R^1\cdot v_i\otimes\cdots\otimes v_m)\\ &= f_1(v_1)\otimes\cdots\otimes r^2\cdot f_{i+1}(P^1R^2\cdot v_{i+1})\otimes r^1\cdot f_i(P^2R^1\cdot v_i)\otimes\cdots f_m(v_m)\\ &= (i,i+1)\cdot(f_1\otimes\cdots\otimes f_m)(v_1\otimes\cdots\otimes v_m),\end{aligned}$$

which means $f_1 \otimes \cdots \otimes f_m \in \mathrm{End}_{\phi(\Bbbk S_m)} V^{\otimes m}$. So $(\mathrm{End}_{\Bbbk} V^{\otimes m})^{\Bbbk S_m} \subset \mathrm{End}_{\phi(\Bbbk S_m)} V^{\otimes m}$. Therefore $\mathrm{End}_{\phi(\Bbbk S_m)} V^{\otimes m} = (\mathrm{End}_{\Bbbk} V^{\otimes m})^{\Bbbk S_m}$.

Since there exists a trace 1 element in $\mathrm{End}_{\Bbbk} V^{\otimes m}$, we have $(\mathrm{End}_{\Bbbk} V^{\otimes m})^{\Bbbk S_m} = t \cdot (\mathrm{End}_{\Bbbk} V^{\otimes m})$, where $t = \sum_{\sigma \in S_m} \sigma$. Thus, to show (i), it suffices to show that $\rho^m(B) \subset t \cdot (\mathrm{End}_{\Bbbk} V^{\otimes m})$ which follows as in [5].

(ii) Follows, as in [4].

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